ANALYSIS HW 2

CLAY SHONKWILER

1

a) Let c_n be a sequence of real numbers that converge to c. Show that their "average", $\frac{1}{n}[c_1 + \dots c_n]$, also converges to c.

Give an example of a sequence which does *not* converge but whose mean does converge.

Proof. Let $\epsilon > 0$. c_n is convergent, so there exists a bound $M > 0$ on the sequence. Since $c_n \to c$, there exists $N_1 \in \mathbb{N}$ such that, if $n > N_1$,

$$||c_n - c|| < \frac{\epsilon}{2},$$

which is to say

$$c - \frac{\epsilon}{2} < c_n < c + \frac{\epsilon}{2}.$$

Let $N = \frac{2N_1(M+||c||)}{\epsilon} > N_1$. Then, if $n > N$ and $c \geq 0$,

$$\begin{aligned}
||\tfrac{c_1+\dots c_n}{n} - c|| &= ||\left(\tfrac{c_1+\dots+c_{N_1}}{n} - \tfrac{N_1 c}{n}\right) + \left(\tfrac{c_{N_1+1}+\dots+c_n}{n} - \tfrac{(n-N_1)c}{n}\right)|| \\
&\leq ||\tfrac{c_1+\dots+c_{N_1}}{n} - \tfrac{N_1 c}{n}|| + ||\tfrac{c_{N_1+1}+\dots+c_n}{n} - \tfrac{(n-N_1)c}{n}|| \\
&\leq \tfrac{1}{n}(||c_1 - c|| + \dots + ||c_N - c||) + ||\tfrac{c+\epsilon/2}{n} + \dots + \tfrac{c+\epsilon/2}{n} - \tfrac{n-N_1}{n}c|| \\
&\leq \tfrac{N_1}{n}(M + ||c||) + \tfrac{n-N_1}{n}||(c+\epsilon/2) - c|| \\
&< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

On the other hand, if $n > N$ and $c < 0$,

$$\begin{aligned}
||\tfrac{c_1+\dots c_n}{n} - c|| &= ||\left(\tfrac{c_1+\dots+c_{N_1}}{n} - \tfrac{N_1 c}{n}\right) + \left(\tfrac{c_{N_1+1}+\dots+c_n}{n} - \tfrac{(n-N_1)c}{n}\right)|| \\
&\leq ||\tfrac{c_1+\dots+c_{N_1}}{n} - \tfrac{N_1 c}{n}|| + ||\tfrac{c_{N_1+1}+\dots+c_n}{n} - \tfrac{(n-N_1)c}{n}|| \\
&\leq \tfrac{1}{n}(||c_1 - c|| + \dots + ||c_N - c||) + ||\tfrac{c-\epsilon/2}{n} + \dots + \tfrac{c-\epsilon/2}{n} - \tfrac{n-N_1}{n}c|| \\
&\leq \tfrac{N_1}{n}(M + ||c||) + \tfrac{n-N_1}{n}||(c-\epsilon/2) - c|| \\
&< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

In either case, we see that the arithmetic mean converges to c. □

Example: Let $c_n = (-1)^{n+1}$. Clearly, the sequence $\{c_n\}$ does not converge. However,

$$\frac{c_1 + \dots c_n}{n} = \begin{cases} 0 & n \text{ is even} \\ 1/n & n \text{ is odd} \end{cases}$$

Hence, the arithmetic mean converges to 0, though the sequence itself does not converge at all.

b) Let $f(t)$ be a continuous function for $0 \leq t < \infty$. If $\lim_{t\to\infty} f(t) = c$, show that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(t)dt = c.$$

Proof.

$$\frac{1}{T}\int_0^T f(t)dt = \frac{1}{T}\lim_{n\to\infty}\sum_{j=1}^{n} f\left((j-1)\frac{T}{n}\right)\frac{T}{n} = \lim_{n\to\infty}\sum_{j=1} n\frac{f\left((j-1)\frac{T}{n}\right)}{n}.$$

Hence,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(t)dt = \lim_{T\to\infty}\left(\lim_{n\to\infty}\sum_{j=1} n\frac{f((j-1)\frac{T}{n}}{n}\right) = c,$$

where we achieve this second equality by noting that the numerator converges to c and then applying the result proved in (a) above. □

2

Let $f \in C^1([a,b])$ have the properties

$$i).f(a) < 0, f(b) > 0 \quad ii).f'(x) > 0.$$

Use a contracting mapping argument to find a sequence of points x_j that converge to the solution of $f(x) = 0$.

Argument: By the Intermediate Value Theorem, we know there exists $c \in [a,b]$ such that $f(c) = 0$. Let $M = \max_{x\in[a,b]} f'(x)$. Choose α such that

$$\frac{1}{10M} < \alpha < \frac{1}{2M}.$$

Let

$$g(x) = x - \alpha f(x).$$

Then, for $x, y \in [a,b]$, $x > y$,

$$\frac{x-y}{10} < \alpha(f(x) - f(y)) < \frac{x-y}{2}$$

so

$$|g(x)-g(y)| = |x-\alpha f(x)-(y-\alpha f(y))| = |(x-y)-\alpha(f(x)-f(y)| \leq .9|x-y|$$

Hence, g is a contraction, meaning it has exactly one fixed point. Clearly, this fixed point occurs when $f(x) = 0$. Hence, we can form a sequence converging to the point c where $f(c) = 0$ by letting $x_0 = b$ and defining the sequence recursively as follows:

$$x_{n+1} = g(x_n).$$

♣

3

For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $1 \le p < \infty$, let $||x||_p := [\sum |x_j|^p]^{1/p}$. Also, define $||x||_\infty = \max_j |x_j|$. Show that

$$\lim_{p\to\infty} ||x||_p = ||x||_\infty.$$

Proof. Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and let $\epsilon > 0$. Define

$$M > \frac{\log n}{\log\left(\frac{\epsilon}{\max_j |x_j|} + 1\right)}.$$

Then, if $m > M$,

$$\begin{aligned}
|||x||_m - ||x||_\infty| &= |\left(\sum |x_i|^m\right)^{1/m} - \max_j |x_j|| \\
&\le |\left(\sum (\max_i |x_i|)^m\right)^{1/m} - \max_j |x_j|| \\
&= |(n\,(\max_i |x_i|)^m)^{1/m} - \max_j |x_j|| \\
&= |n^{1/m} \max_j |x_j| - \max_j |x_j|| \\
&= |(n^{1/m} - 1) \max_j |x_j|| \\
&= (n^{1/m} - 1) \max_j |x_j| \\
&< (n^{1/M} - 1) \max_j |x_j| \\
&= \left(\left(\frac{\epsilon}{\max_j |x_j|} + 1\right) - 1\right) \max_j |x_j| \\
&= \epsilon.
\end{aligned}$$

So we conclude that $||x||_p \to ||x||_\infty$ as $p \to \infty$. $\square$

4

Let $f(x)$ and $K(x, y)$ be continuous functions for $x, y \in [0, 1]$. Consider the following linear *integral equation* for the function $u(x)$:

$$u(x) = f(x) + \lambda \int_0^1 K(x, y)u(y)dy \tag{1}$$

a) Show that for all $\lambda \in \mathbb{R}$ sufficiently near zero, this equation has a unique solution.

Proof. We use the uniform norm throughout the following, as $C([0, 1])$ is complete under the uniform norm. Since f is continuous on $[0, 1]$, it is bounded by M, such that $||f|| < M$. Also, there exists M' such that

$$|| \int_0^1 K(x, y)dy|| < M'.$$

Let $|\lambda| < \min\{1, \frac{1}{\sqrt{M'}}\}$. Then

$$\sum_{k=1}^\infty |\lambda|^k$$

converges. Which is to say the sequence of partial sums is Cauchy. Hence, there exists $N \in \mathbb{N}$ such that, if $m > n > N$,

$$\frac{\epsilon}{MM'} > \sum_{i=1}^{m} |\lambda|^i - \sum_{j=1}^{n} |\lambda|^j = |\lambda|^m + |\lambda|^{m-1} + \ldots + |\lambda|^n.$$

Define u_j recursively, where $u_0 \equiv 0$ and

$$u_{j+1}(x) = f(x) + \int_0^1 K(x,y)u_j(y)dy$$

for all $j \geq 1$. Clearly, each u_j is continuous, since it is a sum of continuous functions. Furthermore, for all $j \geq 1$,

$$\begin{aligned}
||u_{j+1} - u_j|| \quad &= ||f(x) + \lambda \textstyle\int_0^1 K(x,y)u_j(y)dy - \left(f(x) + \lambda \int_0^1 K(x,y)u_{j-1}(y)dy\right)|| \\
&= ||\lambda \textstyle\int_0^1 K(x,y)\left(u_j(y) - u_{j-1}(y)\right)dy|| \\
&= |\lambda|||\textstyle\int_0^1 \left(K(x,y)\left(\lambda \int_0^1 K(x,y)(u_{j-1}(y) - u_{u-2}(y))dy\right)\right)dy|| \\
&= |\lambda|^2||\textstyle\int_0^1 K(x,y)\left(\int_0^1 K(x,y)(u_{j-1}(y) - u_{j-2}(y))dy\right)dy|| \\
&\vdots \\
&= |\lambda|^j||\textstyle\int_0^1 K(x,y)\left(\int_0^1 K(x,y)\left(\ldots\left(\int_0^1 K(x,y)(u_1(y) - u_0(y))dy\right)\ldots\right)dy\right)dy|| \\
&= |\lambda|^j||\textstyle\int_0^1 K(x,y)\left(\int_0^1 K(x,y)\left(\ldots\left(\int_0^1 K(x,y)f(y)dy\right)\ldots\right)dy\right)dy|| \\
&\leq |\lambda|^j M||\textstyle\int_0^1 K(x,y)\left(\int_0^1 K(x,y)\left(\ldots\left(\int_0^1 K(x,y)dy\right)\ldots\right)dy\right)dy|| \\
&= |\lambda|^j M||\textstyle\int_0^1 K(x,y)dy|| \\
&\leq |\lambda|^j MM'.
\end{aligned}$$

Hence, for $m > n > N$,

$$\begin{aligned}
||u_m - u_m|| \quad &= ||(u_m - u_{m-1}) + (u_{m-1} - u_{m-2}) + \ldots + (u_{n+1} - u_n)|| \\
&\leq ||u_m - u_{m-1}|| + \ldots + ||u_{n+1} - n_n|| \\
&\leq |\lambda|^{m-1}MM' + \ldots + |\lambda|^n MM' \\
&= MM'(|\lambda|^{m-1} + \ldots + |\lambda|^n) \\
&< MM'\tfrac{\epsilon}{MM'} \\
&= \epsilon.
\end{aligned}$$

Therefore, the sequence $\{u_j\}$ is Cauchy, so it converges to some continuous u. Furthermore,

$$\begin{aligned}
u(x) \quad &= \lim_{n\to\infty} u_n(x) \\
&= \lim_{n\to\infty}\left(f(x) + \lambda \textstyle\int_0^1 K(x,y)u_n(y)dy\right) \\
&= f(x) + \lambda \lim_{n\to\infty} \textstyle\int_0^1 K(x,y)u_n(y)dy \\
&= f(x) + \lambda \textstyle\int_0^1 K(x,y)u(y)dy.
\end{aligned}$$

To see that u is a unique solution of this equation, suppose there existed u' that was also a solution. Then

$$\begin{aligned}
||u - u'|| &= || \left(f(x) + \lambda \int_0^1 K(x,y)u(y)dy \right) - \left(f(x) + \lambda \int_0^1 K(x,y)u'(y)dy \right) || \\
&= |\lambda| || \int_0^1 K(x,y) \left(u(y) - u'(y) \right) dy || \\
&= |\lambda| || \int_0^1 K(x,y) \left(\left(f(x) + \lambda \int_0^1 K(x,y)u(y)dy \right) - \left(f(x) + \lambda \int_0^1 K(x,y)u'(y)dy \right) \right) || \\
&= |\lambda|^2 || \int_0^1 K(x,y) \left(\int_0^1 K(x,y) \left(u(y) - u'(y) \right) dy \right) dy || \\
&= |\lambda|^2 \int_0^1 K(x,y) ||u - u'|| dy \\
&= |\lambda|^2 ||u - u'|| \int_0^1 K(x,y) \\
&\leq |\lambda|^2 M' ||u - u'||.
\end{aligned}$$

Since $|\lambda|^2 M' < 1$, this implies that $||u-u'|| = 0$. In other words, the solution u is unique. □

5

Let $K(x, y : z)$ be continuous for all $x, y \in [a, b] \subset \mathbb{R}$ and, for all real z_1, z_2 satisfy the *Lipschitz condition*

$$|K(x, y : z_2) - K(x, y : z_1)| \leq M|z_2 - z_1|,$$

where the constant M is independent of x, y, z_1, z_2. Generalize (1) by showing that the nonlinear integral equation

$$u(x) = f(x) + \lambda \int_a^b K(x, y : u(y))dy$$

has a unique solution for all $|\lambda| < \frac{1}{M(b-a)}$.

Proof. Let $|\lambda| < \frac{1}{M(b-a)}$ and define

$$g(u(x)) = f(x) + \lambda \int_a^b K(x, y; u(y))dy.$$

Then, for all $u, u' \in C([a, b])$,

$$\begin{aligned}
||g(u) - g(u') &= ||f(x) + \lambda \int_a^b K(x,y;u(y))dy - \left(f(x) + \lambda \int_a^b K(x,y;u'(y))dy \right) || \\
&= |\lambda| || \int_a^b \left(K(x,y;u(y)) - K(x,y;u'(y)) \right) dy || \\
&\leq |\lambda| M(b-a) ||u - u'||
\end{aligned}$$

where $|\lambda|M(b-a) < 1$. Therefore, g is a contraction, meaning it has exactly one fixed point v. To say that v is a fixed point is to say that

$$v(x) = f(x) + \lambda \int_a^b K(x, y; v(y))dy.$$

Thus, v is the unique solution of the nonlinear equation. □

DRL 3E3A, University of Pennsylvania
E-mail address: shonkwil@math.upenn.edu