ANALYSIS HW 4

CLAY SHONKWILER

1

Let $f(x) = x$ for $-\pi \leq x < \pi$. Use orthogonal functions e^{ikx}, $k = 0, \pm 1, \pm 2, \ldots$, in $L^2(-\pi, \pi)$.

(a) Find the Fourier expansion $\sum c_k e^{ikx}$ of f.

Answer: We expand with respect to the orthonormal set $\frac{e^{ikx}}{\sqrt{2\pi}}$. Then $f(x) = \sum_{-\infty}^{\infty} a_k \frac{e^{ikx}}{\sqrt{2\pi}}$ where, for $k \neq 0$,

$$\begin{aligned}
a_k &= \langle f, \tfrac{e^{ikx}}{\sqrt{2\pi}} \rangle \\
&= \tfrac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx \\
&= \tfrac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} x e^{-ikx} dx \\
&= \tfrac{1}{\sqrt{2\pi}} \left(\left[\tfrac{-xe^{-ikx}}{ik} \right]_{-\pi}^{\pi} - \tfrac{1}{ik} \int_{-\pi}^{\pi} -e^{-ikx} dx \right) \\
&= \tfrac{1}{\sqrt{2\pi}} \left(\left[\tfrac{-\pi e^{-ik\pi}}{ik} - \tfrac{\pi e^{ik\pi}}{ik} \right] + \left[\tfrac{e^{-ikx}}{k^2} \right]_{-\pi}^{\pi} \right) \\
&= \tfrac{1}{\sqrt{2\pi}} \left(\tfrac{2\pi}{ik} (-1)^{k+1} + \left[\tfrac{e^{-ik\pi}}{k^2} - \tfrac{e^{ik\pi}}{k^2} \right] \right) \\
&= (-1)^{k+1} \tfrac{\sqrt{2\pi}}{ik}
\end{aligned}$$

If $k = 0$, then $a_k = \langle f, 1 \rangle = \int_{-\pi}^{\pi} x dx = 0$. Hence,

$$f(x) = \sum_{-\infty}^{\infty} c_k e^{ikx}$$

where $c_k = \frac{a_k}{\sqrt{2\pi}}$.

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(b) Compute $||f||^2$ and $\sum_{|k| \leq N} |c_k|^2$. What can you conclude as $N \to \infty$?

Answer:

$$||f||^2 = \langle f, f, \rangle = \int_{-\pi}^{\pi} x^2 dx = \left[\frac{x^3}{3} \right]_{-\pi}^{\pi} = \frac{2\pi^3}{3}.$$

On the other hand,

$$\sum_{|k| \leq N} |c_k|^2 = \sum_{|k| \leq N} \left| \frac{(-1)^{k+1}}{ik} \right|^2 = \sum_{|k| \leq N} \frac{1}{k^2}.$$

So, as $N \to \infty$, this sum approaches $\frac{\pi^2}{3}$.

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2

(a) Compute $\min_{a,b,c} \int_{-1}^{1} |x^3 - a - bx - cx^2|^2 dx$.

Answer: Consider the subspace A of all polynomials of degree less than or equal to 2. Then there exists some value d such that d is the distance from x^3 to A. In fact, if $p(x^3)$ is the projection of x^3 onto A, then this distance $d = ||v||$, where $v = x^3 - p(x^3)$. Now, $p(x^3) = a + bx + cx^2$ for some a, b, c, so we see that the integral we are trying to compute is, in fact, just d. To compute it, first we need to find a basis $\{u_k\}$ for A. By the Gram-Schmidt process, then, we let

$$u_0 = \frac{x^2}{||1||} = \sqrt{\frac{5}{2}}x^2,$$

since $\langle x^2, x^2\rangle = \int_{-1}^{1} x^4 dx = \left[\frac{x^5}{5}\right]_{-1}^{1} = \frac{2}{5}$. Now, notice that

$$\langle x, x^2\rangle = \int_{-1}^{1} x^3 dx = \left[\frac{x^4}{4}\right]_{-1}^{1} = 0$$

so we need only normalize x.

$$\langle x, x\rangle = \int_{-1}^{1} x^2 dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{2}{3},$$

so $u_1 = \sqrt{\frac{3}{2}}x$. Now, to get u_2 first we compute an un-normalized u_2'.

$$\begin{aligned} u_2' &= 1 - \langle 1, x^2\rangle \tfrac{5}{2}x^2 - \langle 1, x\rangle \tfrac{3}{2}x \\ &= 1 - \left(\int_{-1}^{1} x^2 dx\right) \tfrac{5}{2}x^2 - \left(\int_{-1}^{1} x dx\right) \tfrac{3}{2}x \\ &= 1 - \tfrac{2}{3}\tfrac{5}{2}x^2 - 0 \cdot \tfrac{3}{2}x \\ &= 1 - \tfrac{5}{3}x^2 \end{aligned}$$

Now, to normalize, we note that

$$\begin{aligned} \langle 1 - \tfrac{5}{3}x^2, 1 - \tfrac{5}{3}x^2\rangle &= \int_{-1}^{1} (1 - \tfrac{10}{3}x^2 + \tfrac{25}{9}x^4) dx \\ &= \left[x - \tfrac{10}{9}x^3 + \tfrac{25}{45}x^5\right]_{-1}^{1} \\ &= \tfrac{8}{9} \end{aligned}$$

so $||1 - \frac{5}{3}x^2|| = \frac{2\sqrt{2}}{3}$. Hence,

$$u_2 = \frac{3}{2\sqrt{2}}\left(1 - \frac{5}{3}x^2\right) = \frac{3}{2\sqrt{2}} - \frac{5}{2\sqrt{2}}x^2.$$

Therefore, the basis for A is

$$\left\{\sqrt{\frac{5}{2}}x^2, \sqrt{\frac{3}{2}}, \frac{3}{2\sqrt{2}} - \frac{5}{2\sqrt{2}}x^2\right\}.$$

Now, we project x^3 onto A.

$$p(x^3) = \langle x^3, u_0\rangle u_0 + \langle x^3, u_1\rangle u_1 + \langle x^3, u_2\rangle u_2.$$

First, we calculate $\langle x^3, u_0\rangle$:

$$\langle x^3, u_0\rangle = \int_{-1}^{1} x^3 \sqrt{\frac{5}{2}} x^2 dx = \sqrt{\frac{5}{2}} \int_{-1}^{1} x^5 dx = \sqrt{\frac{5}{2}} \left[\frac{x^6}{6}\right]_{-1}^{1} = 0.$$

Now, we calculate $\langle x^3, u_1\rangle$:

$$\langle x^3, u_1\rangle = \int_{-1}^{1} x^3 \sqrt{\frac{3}{2}} x dx = \sqrt{\frac{3}{2}} \int_{-1}^{1} x^4 dx = \sqrt{\frac{3}{2}} \left[\frac{x^5}{5}\right]_{-1}^{1} = \frac{\sqrt{6}}{5}.$$

Finally, we calculate $\langle x^3, u_2\rangle$:

$$\langle x^3, u_2\rangle = \int_{-1}^{1} x^3 \left(\frac{3}{2\sqrt{2}} - \frac{5}{2\sqrt{2}} x^2\right) dx = \int_{-1}^{1} \left(\frac{3}{2\sqrt{2}} x^3 - \frac{5}{2\sqrt{2}} x^5\right) dx = \left[\frac{3}{8\sqrt{2}} x^4 - \frac{5}{12\sqrt{2}} x^6\right]_{-1}^{1} = 0.$$

Hence,

$$p(x^3) = \frac{\sqrt{6}}{5} \sqrt{\frac{3}{2}} = \frac{3}{5} x,$$

so our desired minimum is the square of the norm of the vector $x^3 - p(x^3)$, that is

$$\begin{aligned}\langle x^3 - \tfrac{3}{5}, x^3 - \tfrac{3}{5}\rangle &= \int_{-1}^{1} \left(x^6 - \tfrac{6}{5} x^4 + \tfrac{9}{25} x^2\right) dx \\ &= \left[\tfrac{x^7}{7} - \tfrac{6}{25} x^5 + \tfrac{9}{75} x^3\right]_{-1}^{1} \\ &= \tfrac{2}{7} - \tfrac{12}{25} + \tfrac{18}{75} \\ &= \tfrac{8}{175}\end{aligned}$$

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(b) Compute $\max \int_{-1}^{1} x^3 h(x) dx$ where $h \in L^2(-1,1)$ is subject to the restrictions

$$\int_{-1}^{1} h(x) dx = \int_{-1}^{1} x h(x) dx = \int_{-1}^{1} x^2 h(x) dx = 0; \int_{-1}^{1} |h(x)|^2 dx = 1.$$

Answer: We re-write as follows:

$$\begin{aligned}\int_{-1}^{1} x^3 h(x) dx &= \int_{-1}^{1} \left[(x^3 - a - bx - cx^2) + a + bx + cx^2\right] h(x) dx \\ &= \int_{-1}^{1} (x^3 - a - bx - cx^2) h(x) + a \int_{-1}^{1} h(x) dx + b \int_{-1}^{1} x h(x) dx + c \int_{-1}^{1} x^2 h(x) dx \\ &= \int_{-1}^{1} (x^3 - a - bx - cx^2) h(x) dx \\ &\leq \int_{-1}^{1} |(x^3 - a - bx - cx^2) h(x)| dx \\ &\leq \left[\int_{-1}^{1} |x^3 - a - bx - cx^2|^2 dx\right]^{1/2} \left[\int_{-1}^{1} |h(x)|^2 dx\right]^{1/2} \\ &= \left[\int_{-1}^{1} |x^3 - a - bx - cx^2|^2 dx\right]^{1/2}\end{aligned}$$

by Holder and the the restrictions on h. Now, from part (a) above, we know how to minimize the right side, so we merely insert that minimum into the inequality:

$$\int_{-1}^{1} x^3 h(x) dx \leq \sqrt{\frac{8}{175}}.$$

Notice, though, that, as we showed in (a), $x^3 - \frac{3}{5}x$ has norm $\sqrt{\frac{8}{175}}$ and is perpendicular to 1, x and x^2, so

$$h(x) = \sqrt{\frac{175}{8}}\left(x^3 - \frac{3}{5}\right)$$

meets the hypothesized restrictions. Furthermore,

$$\begin{aligned} \int_{-1}^1 x^3\sqrt{\tfrac{175}{8}}\left(x^3 - \tfrac{3}{5}x\right)dx &= \sqrt{\tfrac{175}{8}}\int_{-1}^1\left(x^6 - \tfrac{3}{5}x^4\right)dx \\ &= \sqrt{\tfrac{175}{8}}\left[\tfrac{x^7}{7} - \tfrac{3x^5}{25}\right]_{-1}^1 \\ &= \sqrt{\tfrac{175}{8}}\left(\tfrac{8}{175}\right) \\ &= \sqrt{\tfrac{8}{175}}. \end{aligned}$$

Since the upper bound $\sqrt{\frac{8}{175}}$ is achieved, it is, in fact, the maximum of $\int_{-1}^1 x^3h(x)dx$.

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3

Let $\{u_k\}$, $k = 1, 2, \ldots$ be an orthonormal set in H and let

$$Q = \left\{x \in H | x = \sum_1^\infty c_k u_k, \quad \text{where} \quad |c_k| \leq \frac{1}{k}\right\}.$$

(a) Show that Q (often called the *Hilbert cube*) is a compact set.

Proof. Let $\{x_j\}$ be a sequence in Q. We want to demonstrate that $\{x_j\}$ contains a convergent subsequence. Now,

$$||x_j||^2 = \langle x_j, x_j\rangle = \langle\sum_{k=1}^\infty c_{j_k}u_k, \sum_{k=1}^\infty c_{j_k}u_k\rangle = \sum_{k=1}^\infty c_{j_k}\overline{c_{j_k}}\langle u_k, u_k\rangle = \sum_{k=1}^\infty |c_{j_k}|^2.$$

Since $|c_{j_k}| \leq \frac{1}{k}$,

$$||x_j||^2 = \sum_{k=1}^\infty |c_{j_k}|^2 \leq \sum_{k=1}^\infty \frac{1}{k^2} = \frac{\pi^2}{6}.$$

This means that $\{||x_j||\}$ is a bounded sequence and so contains a convergent subsequence by Bolzano-Weierstrass. Denote this subsequence by $||y_i||$ and let its limit be L. □

(b) Let $\{\delta_k\} \geq 0$ be given and

$$R = \{x \in H | x = \sum_1^\infty c_k u_k, \quad \text{where} \quad |c_k| \leq \delta_k\}.$$

Show that the set R is compact if and only if $\sum \delta_k^2 < \infty$.

4

Let V be a closed subspace of a Hilbert space H. Prove that $(V^{\perp})^{\perp} = V$. What if V is not closed?

Proof. Let $x \in H$. Since V is a closed subspace and hence a closed, convex subset of H, we know that there exists a vector $v \in V$ that minimizes the distance between V and x. Define $y := x - v$. Then, for any $z \in V$ and any $t \in \mathbb{R}$,

$$||y||^2 \leq ||y + tz||^2 = ||y||^2 + 2t\text{Re}\langle y, z\rangle + t^2||z||^2,$$

which in turn implies that

$$0 \leq 2t\text{Re}\langle y, z\rangle + t^2||z||^2.$$

From this, we see that $\text{Re}\langle y, z\rangle = 0$ for all $z \in V$. Hence, $y \perp V$; in other words $y \in V^{\perp}$. From our definition of y, then, we see that $x = v + y$, the sum of a vector from V and one from $V^{\perp}$.

Now, suppose $x = v + y = v' + y'$, where $v, v' \in V$, $y, y' \in V^{\perp}$. Then $y - y' = v' - v$ would belong to both V and $V^{\perp}$; it would be orthogonal to itself. We know that the only verctor orthogonal to itself is the zero vector, so,

$$0 = y - y' = v' - v,$$

meaning $y = y'$, $v = v'$. Hence, the decomposition we demonstrated above is unique.

Since any vector in H can be uniquely decomposed as the sum of a vector in V and one in $V^{\perp}$, we see that V and $V^{\perp}$ are complementary subspaces. In turn, this means $V^{\perp}$ and $(V^{\perp})^{\perp}$ must be complementary, but we already know that the complement of $V^{\perp}$ is V, so it must be the case that $(V^{\perp})^{\perp} = V$. □

5

Let $L : H \to \mathbb{C}$ be a continuous linear map (here $\mathbb{C}$ is the complex numbers) and let K be the kernel of L. Prove that either $K = H$ or $K^{\perp}$ has dimension 1.

Proof. Suppose $K \neq H$. Let $x, x' \in K^{\perp}$. Then, by the linearity of L, we know that for some constant c,

$$Lx' = cLx = L(cx).$$

This, in turn, means that

$$0 = Lx' - L(cx) = L(x' - cx),$$

so $x' - cx \in K$. However, since $x, x' \in K^b ot$, we know that $x' - cx \in K^{\perp}$. The only vector orthogonal to itself is the zero vector, so

$$x' - cx = 0,$$

which means that $x' = cx$. Since our choice of x, x' was arbitrary, we see that any two elements of $K^\perp$ are multiples of eachother, so $K^\perp$ has dimension 1. □

6

Let $f \in C(\mathbb{R})$ have period 1 and α be an irrational number. Prove that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(n\alpha) = \int_0^1 f(x)dx.$$

Proof. Let $[x]$ denote the greatest integer function. Then, since f has period 1, for any $x \in \mathbb{R}$, $f(x) = f(x - [x])$. Now, let $[a, b] \subseteq$ □

DRL 3E3A, University of Pennsylvania
E-mail address: `shonkwil@math.upenn.edu`