ALGEBRA HW 12

CLAY SHONKWILER

10

For each $p \neq 0$, determine the radius and center of the circle with equation $pz\bar{z} + gz + \overline{gz} = q$.

Answer: First, suppose $g = a + bi$ and $z = x + iy$. Then

$$\begin{aligned}
q &= pz\bar{z} + gz + \overline{gz} \\
&= p(x+iy)(x-iy) + (a+bi)(x+iy) + (a-bi)(x-iy) \\
&= p(x^2+y^2) + (ax-by) + (ay+bx)i + (ax-by) + (-ay-bx)i \\
&= p(x^2+y^2) + 2(ax-by) \\
&= px^2 + 2ax + py^2 - 2by \\
&= p\left(x + \frac{a}{p}\right)^2 + p\left(y - \frac{b}{p}\right)^2 - \frac{a^2}{p^2} - \frac{b^2}{p^2}.
\end{aligned}$$

Hence,

$$p\left(x + \frac{a}{p}\right)^2 + p\left(y - \frac{b}{p}\right)^2 = q + \frac{a^2}{p^2} + \frac{b^2}{p^2},$$

or

$$\left(x + \frac{a}{p}\right)^2 + \left(y - \frac{b}{p}\right)^2 = \frac{q + \frac{a^2}{p^2} + \frac{b^2}{p^2}}{p} = \left(\frac{\sqrt{qp^2 + a^2 + b^2}}{p^{3/2}}\right)^2,$$

so this equation determines a circle centered at $\frac{-a}{p} + i\frac{b}{p}$ with a radius of $\frac{\sqrt{qp^2+a^2+b^2}}{p^{3/2}}$.

♣

14

Prove that each fractional linear transformation is the composition of at most two translations, a rotation, a dilation, and an inversion.

Proof. Let $F(z) := \frac{az+b}{cz+d}$ be a fractional linear transformation. Define the following transformations:

$$\begin{aligned}
T_1(z) &:= z + \frac{d}{c}\\
T_2(z) &:= z + \frac{a}{cw}\\
R(z) &:= (\cos\theta + i\sin\theta)z\\
D(z) &:= rz\\
I(z) &:= \frac{1}{z},
\end{aligned}$$

where

$$w = \frac{bc-ad}{c^2} = r(\cos\theta + i\sin\theta).$$

Then T_1 and T_2 are translations, R a rotation, D a dilation and I an inversion. Now,

$$\begin{aligned}
(R\circ D\circ T_2\circ I\circ T_1)(z) &= (R\circ D\circ T_2\circ I)\left(z+\frac{d}{c}\right)\\
&= (R\circ D\circ T_2)\left(\frac{1}{z+\frac{d}{c}}\right)\\
&= (R\circ D)\left(\frac{1}{z+\frac{d}{c}}+\frac{a}{cw}\right)\\
&= (R\circ D)\left(\frac{cw}{cwz+dw}+\frac{a(z+\frac{d}{c}}{cwz+dw}\right)\\
&= (R\circ D)\left(\frac{az+cw+\frac{ad}{c}}{cwz+dw}\right)\\
&= R\left(\frac{r\left(az+cw+\frac{ad}{c}\right)}{cwz+dw}\right)\\
&= \frac{(\cos\theta+i\sin\theta)r\left(az+cw+\frac{ad}{c}\right)}{cwz+dw}\\
&= \frac{w\left(az+cw+\frac{ad}{c}\right)}{cwz+dw}\\
&= \frac{az+cw+\frac{ad}{c}}{cz+d}
\end{aligned}$$

$$\begin{aligned} &= \frac{az + c\frac{bc-ad}{c^2} + \frac{ad}{c}}{cz+d} \\ &= \frac{az + \frac{bc-ad}{c} + \frac{ad}{c}}{cz+d} \\ &= \frac{az + \frac{bc}{c}}{cz+d} \\ &= \frac{az+b}{cz+d}. \end{aligned}$$

Since our choice of fractional linear transformation F was arbitrary, we see that any fractional linear transformation can be written as the composition of at most two translations, a rotation, a dilation and an inversion. □

16

Prove that the composition $F \circ G$ of two fractional linear transformations F and G is also a fractional linear transformation, such that if

$$F(z) := \frac{az+b}{cz+d}, \quad G(z) := \frac{rz+p}{sz+q}$$

then

$$(F \circ G)(z) := \frac{mz+h}{nz+k}$$

with

$$\begin{pmatrix} m & h \\ n & k \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} r & p \\ s & q \end{pmatrix} = \begin{pmatrix} ar+bs & ap+bq \\ cr+ds & cp+dq \end{pmatrix}$$

Proof. With F and G as defined above,

$$\begin{aligned} (F \circ G)(z) &= F\left(\frac{rz+p}{sz+q}\right) \\ &= \frac{a\left(\frac{rz+p}{sz+q}\right)+b}{c\left(\frac{rz+p}{sz+q}\right)+d} \\ &= \frac{\frac{arz+ap}{sz+q} + \frac{bsz+bq}{sz+q}}{\frac{crz+cp}{sz+q} + \frac{dsz+dq}{sz+q}} \\ &= \frac{\frac{(ar+bs)z+ap+bq}{sz+q}}{\frac{(cr+ds)z+cp+dq}{sz+q}} \\ &= \frac{(ar+bs)z+ap+bq}{(cr+ds)z+cp+dq}, \end{aligned}$$

which corresponds to the matrix

$$\begin{pmatrix} ar+bs & ap+bq \\ cr+ds & cp+dq \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} r & p \\ s & q \end{pmatrix}.$$

□

17

Prove that each fractional linear transformation has an inverse function that is also a fractional linear transformation: $\mathcal{M}$ forms a group with the operation composition of functions.

Proof. Let $F(z) = \frac{az+b}{cz+d}$ be a fractional linear transformation, corresponding to the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix};$$

Let G be the fractional linear transformation corresponding to this matrix. Then, by our work in problem 16 above, $(F \circ G)(z) = \frac{mz+h}{nz+k}$ where

$$\begin{pmatrix} m & h \\ n & k \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence,

$$(F \circ G)(z) = \frac{1 \cdot z + 0}{0z + 1} = z.$$

Similarly, $(G \circ F)(z) = z$, so $G = F^{-1}$. □

24

For each $c \in D(0,1)$ define a fractional linear transformation L_c by

$$L_c(z) := \frac{z-c}{1-\bar{c}z}.$$

Prove that such a fractional linear transformation maps the unit disc onto the unit disc, and the unit circle onto the unit circle $S^1 := \{z \in \mathbb{C} : |z| = 1\}$.

Proof. Fix $c \in D(0,1)$. Note that L_c is defined for all $z \in \mathbb{C}$ such that $1 - \bar{c}z \neq 0$, that is, for all $z \neq \frac{-1}{-\bar{c}} = \frac{1}{\bar{c}}$. Since $|\bar{c}| = |c| < 1$,

$$\left|\frac{1}{\bar{c}}\right| > 1,$$

so L_c is defined on the entire unit disc. Furthermore, by our work in problem 17 above, L_c has an inverse, L_c^{-1} given by

$$L_c^{-1}(z) = \frac{z+c}{1+\bar{c}z}.$$

Now, L_c^{-1} is defined for all $z \in \mathbb{C}$ such that $1 + \bar{c}z \neq 0$, that is, for all $z \neq \frac{-1}{\bar{c}}$. Again, since $|\bar{c}| < 1$,

$$\left|\frac{-1}{\bar{c}}\right| > 1,$$

so L_c^{-1} is defined on the entire unit disc. Thus, L_c is bijective on $\mathbb{C}\setminus\left\{\frac{1}{\bar{c}}, \frac{-1}{\bar{c}}\right\}$, so if we can show that L_c maps the unit disc to itself, this will suffice to show that L_c maps the unit disc onto itself.

Similarly, if we can show that L_c maps S^1 into itself, this suffices to show that L_c maps S^1 onto itself. To that end, suppose $|z| = 1$. Then $1 = |z|^2 = z\bar{z}$, so $\bar{z} = z^{-1}$. Hence,

$$\begin{aligned} |L_c(z)| &= \left|\frac{z-c}{1-\bar{c}z}\right| \\ &= \frac{|z-c|}{|1-\bar{c}z|} \\ &= \frac{|z-c|}{|z(z^{-1}-\bar{c})|} \\ &= \frac{|z-c|}{|z||\bar{z}-\bar{c}|} \\ &= \frac{|z-c|}{|\overline{z-c}|} \\ &= 1, \end{aligned}$$

so we conclude that L_c maps S^1 onto itself. □

28

Prove that $(z_1, z_2, z_3, z_4) = (F(z_1), F(z_2), F(z_3), F(z_4))$ for all distinct $z_1, z_2, z_3, z_4 \in \mathbb{C}$ and for each fractional linear transformation F.

Proof. Let G be the fractional linear transformation that maps z_2, z_3, z_4 onto $1, 0, \infty$, respectively, and let H be the fractional linear transformation that maps $F(z_2), F(z_3), F(z_4)$ onto $1, 0, \infty$, respectively. Then

$$G(z_1) = (z_1, z_2, z_3, z_4)$$

and

$$H(F(z_1)) = (F(z_1), F(z_2), F(z_3), F(z_4)).$$

Furthermore, since H and F are both fractional linear transosformations and we showed, in problem 16 above, that the composition of fractional linear transformations is a fractional linear transformation, $H \circ F$ is a fractional linear transformation mapping z_2, z_3, z_4 to $1, 0, \infty$, respectively. However, this is precisely how G was defined, so we see that $H \circ F = G$, and so

$$(z_1, z_2, z_3, z_4) = G(z_1) = (H \circ F)(z_1) = (F(z_1), F(z_2), F(z_3), F(z_4)).$$

□

41

Prove that for all open sets $D_\gamma \subseteq \mathbb{R}$ and $D_f \subseteq \mathbb{C}$ and all functions $\gamma : D_\gamma \to \mathbb{C}$ with real derivative at $t \in D_\gamma$ and $f : D_f \to \mathbb{C}$ with a complex derivative at $z := \gamma(t) \in D_f$, the composite function $f \circ \gamma$ has a real derivative at t given by

$$(f \circ \gamma)'(t) = \left\{f'[\gamma(t)]\right\} \cdot \gamma'(t).$$

Proof. By definition,

$$(f \circ \gamma)'(t) = \lim_{s \to t} \frac{(f \circ \gamma)(s) - (f \circ \gamma)(t)}{s - t}.$$

Now, if $\gamma(s) - \gamma(t) \neq 0$ as $s \to t$, then

$$\begin{aligned} (f \circ \gamma)'(t) &= \lim_{s \to t} \frac{(f \circ \gamma)(s) - (f \circ \gamma)(t)}{s - t} \\ &= \lim_{s \to t} \frac{(f \circ \gamma)(s) - (f \circ \gamma)(t)}{\gamma(s) - \gamma(t)} \cdot \frac{\gamma(s) - \gamma(t)}{s - t} \\ &= f'(\gamma(t)) \cdot \gamma'(t) \end{aligned}$$

since f has a complex derivative at $\gamma(t)$.

On the other hand, if $\gamma(s) - \gamma(t) = 0$ as $s \to t$, then

$$\begin{aligned} (f \circ \gamma)'(t) &= \lim_{s \to t} \frac{(f \circ \gamma)(s) - (f \circ \gamma)(t)}{s - t} \\ &= \lim_{s \to t} \frac{f(\gamma(s)) - f(\gamma(t))}{s - t} \\ &= \lim_{s \to t} \frac{0}{s - t} \\ &= 0. \end{aligned}$$

Also,

$$\gamma'(t) = \lim_{s \to t} \frac{\gamma(s) - \gamma(t)}{s - t} = 0,$$

so we see that $(f \circ \gamma)'(t) = \gamma'(t) = 0$. Hence, since $f'(\gamma(t))$ exists, we see that

$$(f \circ \gamma)'(t) = 0 = f'(\gamma(t))\gamma'(t).$$

□

45

Prove that if $f : \mathbb{C} \to \mathbb{C}$ satisfies the Cauchy-Riemann equations at z, then the function $g : \mathbb{C} \to \mathbb{C}$ defined by $g(w) := \overline{f(\bar{w})}$ satisfies the Cauchy-Riemann equations at $w := \bar{z}$.

Proof. Suppose $z = x_0 + iy_0$ and that $f(z) = f(x_0, y_0) = u_1(x_0, y_0) + iv_1(x_0, y_0)$ and $g(w) = g(x_0, -y_0) = u_2(x_0, -y_0) + iv_2(x_0, -y_0)$. Then

$$u_2(x_0, -y_0) + iv_2(x_0, -y_0) = g(w) = \overline{f(\bar{w})} = \overline{f(z)} = u_1(x_0, y_0) - iv_1(x_0, y_0).$$

Then $u_2(x_0,-y_0) = u_1(x_0,y_0)$, so $\frac{\partial u_2}{\partial x}(x_0,-y_0) = \frac{\partial u_1}{\partial x}(x_0,y_0)$ and $\frac{\partial u_2}{\partial y} = \frac{\partial u_1}{\partial(-y)}(x_0,y_0) = -\frac{\partial u_1}{\partial y}(x_{,0},y_0)$. Similarly, $v_2(x_0,-y_0) = -v_1(x_0,y_0)$, so $\frac{\partial v_2}{\partial x}(x_0,-y_0) = -\frac{\partial v_1}{\partial x}(x_0,y_0)$ and $\frac{\partial v_2}{\partial y}(x_0,y_0) = -\frac{\partial v_1}{\partial(-y)}(x_0,y_0) = \frac{\partial v_1}{\partial y}(x_0,y_0)$. Hence,

$$\frac{\partial g}{\partial x}(w) = \frac{\partial u_2}{\partial x}(x_0,-y_0) + i\frac{\partial v_2}{\partial x}(x_0,-y_0) = \frac{\partial u_1}{\partial x}(x_0,y_0) - i\frac{\partial v_1}{\partial x}(x_0,y_0)$$

and

$$\begin{aligned} -i\frac{\partial g}{\partial y}(w) &= -i\left(\frac{\partial u_2}{\partial y}(x_0,-y_0) + i\frac{\partial v_2}{\partial y}(x_0,-y_0)\right) \\ &= -i\left(-\frac{\partial u_1}{\partial y}(x_0,y_0) + i\frac{\partial v_1}{\partial y}(x_0,y_0)\right) \\ &= \frac{\partial v_1}{\partial y}(x_0,y_0) + i\frac{\partial u_1}{\partial y}(x_0,y_0). \end{aligned}$$

Now, since f satisfies the Cauchy-Riemann equations,

$$\frac{\partial u_1}{\partial x}(x_0,y_0) = \frac{\partial v_1}{\partial y}(x_0,y_0)$$

and

$$\frac{\partial u_1}{\partial y}(x_0,y_0) = -\frac{\partial v_1}{\partial x}(x_0,y_0);$$

plugging this into the above equations, we see that

$$\frac{\partial g}{\partial x}(w) = \frac{\partial u_1}{\partial x}(x_0,y_0) - i\frac{\partial v_1}{\partial x}(x_0,y_0) = \frac{\partial v_1}{\partial y}(x_0,y_0) + i\frac{\partial u_1}{\partial y}(x_0,y_0) = -i\frac{\partial g}{\partial y}(w),$$

so g satisfies the Cauchy-Riemann equations at w. □

67

Prove that for each power series $\sum_{n=0}^{\infty} c_n w^n$ the following formula, attributed to Jacques Hadamard, gives the radius of convergence R:

$$R = \frac{1}{\limsup(\sqrt[n]{|c_n|})}$$

Proof. Construct the series

$$\sum_{n=0}^{\infty} \sup\{|c_n|, |c_{n+1}|, \ldots\} w^n.$$

Then

$$\sum_{n=0}^{\infty} |c_n w^n| \leq \sum_{n=0}^{\infty} \sup\{|c_n|, |c_{n+1}|, \ldots\} |w^n|$$

so the radius of convergence of the original series must be at least as big as the radius of convergence of the new series. Now, by the root test, if $r \in \mathbb{R}$ such that $r > 0$, then the new series converges if

$$\begin{aligned} 1 &> \lim_{n\to\infty} (|\sup\{|c_n|, |c_{n+1}|, \ldots\} r^n|^{1/n}) \\ &= \lim_{n\to\infty} (|\sup\{|c_n|, |c_{n+1}|, \ldots\}|^{1/n} |r^n|^{1/n}) \\ &= \lim_{n\to\infty} (\sqrt[n]{\sup\{|c_n|, |c_{n+1}|, \ldots\}} r) \\ &= \limsup_{n\to\infty} \sqrt[n]{|c_n|} r, \end{aligned}$$

which is to say if

$$r < \frac{1}{\limsup_{n\to\infty} \sqrt{|c_n|}}.$$

Hence, the radius of convergence of the new series $R \geq \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|c_n|}}$.

On the other hand, if $r > \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|c_n|}}$, then

$$\begin{aligned} \sum_{n=0}^{\infty} |c_n| r^n &\geq \sum_{n=0}^{\infty} |c_n| \left(\frac{1}{\limsup_{m\to\infty} \sqrt[m]{|c_m|}} \right)^n \\ &= \sum_{n=0}^{\infty} \frac{|c_n|}{\limsup_{m\to\infty} \sqrt[m]{|c_m|}^n} \\ &\geq \sum_{n=0}^{\infty} \frac{|c_n|}{\sqrt[n]{|c_n|}^n} \\ &= \sum_{n=0}^{\infty} \frac{|c_n|}{|c_n|} \\ &= \sum_{n=0}^{\infty} 1 \end{aligned}$$

which diverges. Hence, we conclude that, indeed,

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|c_n|}}$$

□

70

(70.1): Give an example of a power series with a radius of convergence equal to 1 that *converges* everywhere on the unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$.

Example: Consider the series $\sum_{n=0}^{\infty} \frac{1}{n^2} z^n$. Then, by the ratio test,

$$R = \lim_{n\to\infty} \frac{|\frac{1}{n^2}|}{|\frac{1}{(n+1)^2}|} = \lim_{n\to\infty} \frac{(n+1)^2}{n^2} = 1.$$

Furthermore, if $|z| = 1$, then

$$\sum_{n=0}^{\infty} \left| \frac{1}{n^2} z^n \right| = \sum_{n=0}^{\infty} \frac{1}{n^2},$$

which converges, so we see that $\sum_{n=0}^{\infty}$ converges on all of S^1.

♣

(70.2): Give an example of a power series with the radius of convergence equal to 1 that *diverges* everywhere on the unit circle S^1.

Example: Consider the series $\sum_{n=0}^{\infty} z^n$. Then, by the Hadamard formula proved in problem 67 above,

$$R = \frac{1}{\limsup \sqrt[n]{|1|}} = 1.$$

Now, suppose $|z| = 1$. Then $z = e^{i\theta}$ for $\theta \in \mathbb{R}$. Furthermore, for each n, $z^n = (e^{i\theta})^n = e^{in\theta}$, so Re $z = \cos n\theta$. Hence, the real part of the series is given by

$$\sum_{n=0}^{\infty} \text{Re } z^n = \sum_{n=0}^{\infty} \cos n\theta.$$

Since the terms of this sum do not converge to zero, the sum diverges, and so the series diverges at z. Since our choice of $z \in S^1$ was arbitrary, we see that the series diverges everywhere on S^1.

♣

(70.3): Give an example of a power series with the radius of convergence equal to 1 that converges at some points but diverges at other points on the unit circle S^1.

Example: Consider the series $\sum_{n=0}^{\infty} \frac{1}{n} z^n$. Then, by the rtio test,

$$R = \lim_{n\to\infty} \frac{|\frac{1}{n}|}{|\frac{1}{n+1}|} = \lim_{n\to\infty} \frac{n+1}{n} = 1.$$

Now, if $z = 1$,

$$\sum_{n=0}^{\infty} \frac{1}{n} z^n = \sum_{n=0}^{\infty} \frac{1}{n},$$

which diverges, whereas if $z = -1$, then

$$\sum_{n=0}^{\infty} \frac{1}{n} z^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n},$$

which is an alternating series and hence converges. Therefore, we see that this series converges at some points of S^1 and diverges at others.

♣

(70.4): Determine the cardinality of the set S of all power series with a radius of convergence equal to 1.

Answer: Let $c \in \mathbb{C}$ such that $|c| < 1$. Form the series $\sum_{n=0}^{\infty} cz^n$. Then this series is simply equal to $c\sum_{n=0}^{\infty} z^n$; since $\sum_{n=0}^{\infty} z^n$ has a radius of convergence equal to 1, so does our series. Hence, we see that the cardinality of S is at least the cardinality of the open unit disc, which has the same cardinality as $\mathbb{C}$, namely $2^{\aleph_0}$.

On the other hand, since a series $\sum_{n=0}^{\infty} c_n z^n$ is determined simply by the sequence $(c_0, c_1, \ldots)$, the cardinality of S can be no greater than the cardinality of the set of all complex sequences. Since $\aleph_0 \cdot 2^{\aleph_0} = 2^{\aleph_0}$, we conclude that

$$\text{card}(S) = 2^{\aleph_0}.$$

♣

(70.5): Deduce that the following statement is false: "Given any subset E of the unit circle S^1, there exists an element of S converging at every point of E and diverging at every point of its complement $S^1 \backslash E$."

Proof. Consider the set of all subsets of S^1; that is, the power set of S^1, $\mathcal{P}(S^1)$. Since card($S^1 = 2^{\aleph_0}$ and the power set of any set has cardinality strictly larger than the original set,

$$\text{card}(\mathcal{P}(S^1)) > \text{card}(S^1) = 2^{\aleph_0} = \text{card}(S).$$

Therefore, since there are strictly more subsets of S^1 than there are power series with radius of convergence equal to 1, we conclude that there cannot be an element of S converging at every point of E and diverging at every point of $S^1 \backslash E$ for every subset $E \subset S^1$. □

DRL 3E3A, University of Pennsylvania
E-mail address: shonkwil@math.upenn.edu