COMPLEX ANALYSIS HW 10

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For each compact $K \subset \mathbb{R} \subset \mathbb{C}$, prove that $\mathcal{O}(K)$ is dense in $C_0(K)$ and that $\mathcal{O}(K) = C_c(K)$ if and only if K is a finite set.

Proof. Note, first of all, that if $h \in \mathcal{O}(K)$, then h is certainly continuous on K, so $h \in C_0(K)$. Now, let $f \in C_0(K)$. Then $f(z) = f_1(z) + if_2(z)$ for $z \in K$ and f_1, f_2 real-valued, continuous functions on K. Since K is compact, we know, by the Stone-Weierstrass Theorem, that there exist sequences of polynomials $\{p_n\}, \{q_n\} \in \mathcal{P}$ (where $\mathcal{P}$ denotes polynomials on $\mathbb{R}$) such that $\{p_n\}$ uniformly approximates f_1 and $\{q_n\}$ uniformly approximates f_2. Therefore, for $\epsilon > 0$, there exist $N_1, N_2 \in \mathbb{N}$ such that $m \geq N_1$ implies that

$$|p_m(z) - f_1(z)| < \epsilon/2$$

for all $z \in K$ and $n \geq N_2$ implies that

$$|q_n(z) - f_2(z)| < \epsilon/2$$

for all $z \in K$. Let $N = \max\{N_1, N_2\}$. Then, for $n \geq N$,

$$\begin{aligned} |p_n(z) + iq_n(z) - f(z)| &= |(p_n(z) - f_1(z)) + i(q_n(z) - f_2(z))| \\ &\leq |p_n(z) - f_1(z)| + |q_n(z) - f_2(z)| \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon \end{aligned}$$

for all $z \in K$. Therefore, $g_k := p_k(z) + iq_k(z)$ uniformly approximates f on K. Now, we can extend p_k and q_k to polynomials on $\mathbb{C}$ and, considered as such, $p_k, q_k \in \mathcal{O}(K)$. Hence, $g_k = p_k + iq_k \in \mathcal{O}(K)$. Since our choice of f was arbitrary, we see that, for any $f \in C_0(K)$, there exists a sequence $\{g_k\} \subset \mathcal{O}(K)$ that uniformly approximates f on K, so $\mathcal{O}(K)$ is dense in $C_0(K)$.

Now, suppose K is a finite set. If $f(z) = f_1(z) + if_2(z)$ is continuous on K, then there exist polynomials $p, q \in \mathcal{P}$ such that $p(z) = f_1(z)$ and $q(z) = f_2(z)$ for every $z \in K$ (since there are only finitely many such z). Therefore, extending p and q to be defined on complex numbers and defining $g(z) = p(z) + iq(z)$, $g \in \mathcal{O}(K)$ and $g|_K \equiv f$, so $C_0(K) \subset \mathcal{O}(K)$. Since we already know $\mathcal{O}(K) \subset C_0(K)$, we conclude that $\mathcal{O}(K) = C_0(K)$.

On the other hand, suppose K is not finite. Then, since K is compact, there exists an accumulation point $a \in K$. Let $b \in K$ such that $b \neq a$. Then

$d = |b - a| \neq 0$; let $X = \overline{D_{d/4}(a)}$ and let $U = D_{d/2}(a)$. Then there exists a function $f \in C^\infty(\mathbb{C})$ such that $f|_X \equiv 1$, $f|_{\mathbb{C} \setminus U} \equiv 0$. Since f is smooth on all of $\mathbb{C}$, certainly $f|_K \in C_0(K)$. Note, that $f(b) = 0$, since $b \in \mathbb{C} \setminus U$. Now, suppose there exists $g \in \mathcal{O}(K)$ such that $g|_K = f|_K$. Then $g - 1$ vanishes on a neighborhood of a and so, by the identity theorem, $g \equiv 0$. However, this is impossible, since $(g - 1)(b) = g(b) - 1 = f(b) - 1 = -1$. From this contradiction, then, we conclude that there is no such g and so we conclude that if K is infinite, then $\mathcal{O}(K) \neq C_0(K)$. By the contrapositive, then, we see that if $\mathcal{O}(K) = C_0(K)$, then K is not infinite.

Having proved implications in both directions, we conclude that $\mathcal{O}(K) = C_0(K)$ if and only if K is finite. □

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Consider finitely many pairwise disjoint *closed* discs $D_1, \ldots, D_N$ in $\mathbb{C}$. Prove that for every real $\epsilon > 0$ and for each index $k \in \{1, \ldots, N\}$ there exists a complex polynomial $P \in \mathbb{C}[Z]$ such that $|P - 1| < \epsilon$ on D_k and $|P| < \epsilon$ on D_ℓ for every $\ell \neq k$.

Proof. Since the D_i are pairwise disjoint and compact,

$$\inf_{z \in D_i, w \in D_j} \{|z - w|\} > 0$$

for all $i, j \in \{1, \ldots, N\}$. Let $\delta > 0$ be such that

$$\delta < \inf_{z \in D_i, w \in D_j} \{|z - w|\}$$

for all i and j (such a δ exists because there are only finitely many i and j). If

$$D_i = \overline{D_{r_i}(a_i)}$$

for all $i \in \{1, \ldots, N\}$ and we define

$$D_i' := D_{r_i + \delta}(a_i)$$

to be the open disc of radius $r_i + \delta$ centered at a_i, then $D_i' \cap D_j' = \emptyset$ for all $i, j \in \{1, \ldots, N\}$. Let $U = \bigcup_{1 \leq i \leq N} D_i'$. Then U is open in $\mathbb{C}$ since it is the union of open sets. Note that, since there are only finitely many D_i, U is bounded, which implies $\mathbb{C} \setminus U$ is unbounded. Furthermore, $\mathbb{C} \setminus U$ consists of a single connected component,[1] which is, of necessity, unbounded and, therefore, not compact. Hence, $\mathbb{C} \setminus U$ has no compact connected components and so, by the corollary of Runge's Theorem (or, in fact, simply from the

[1] To see this, suppose $a, b \in \mathbb{C} \setminus U$. Then we construct a path from a to b in $\mathbb{C} \setminus U$ in the following way: starting at a, proceed along the straight line path L from a to b. Whenever this path intersects the boundary of a D_i', traverse that boundary counter-clockwise until it re-intersects L; at this point, continue traversing L. Since the D_i' are open and disjoint, $\partial D_i' \subset \mathbb{C} \setminus U$, so this path is contained in $\mathbb{C} \setminus U$. Furthermore, since this path is piecewise smooth, it is certainly continuous. Hence, $\mathbb{C} \setminus U$ is pathwise connected and, therefore, connected.

proof of Runge's Theorem itself), the set of restrictions to U of polynomials is dense in $\mathcal{H}(U)$.

Now, for each $i \in \{1, \ldots, N\}$, define f_i such that $f_i|_{D'_i} \equiv 1$ and $f_u|_{D'_j} \equiv 0$ for $j \neq i$. Then, since the D'_k are pairwise disjoint, $f_i \in \mathcal{H}(U)$ for all i and so, since the restrictions to U of polynomials are dense in $\mathcal{H}(U)$, for any $k \in \{1, \ldots, N\}$ and $\epsilon > 0$, there exists a complex polynomial $P_{k,\epsilon} \in \mathbb{C}[z]$ such that

$$\|P_{k,\epsilon} - f_k\|_U < \epsilon.$$

In particular, this implies that $\epsilon > |P_{k,\epsilon} - f_k| = |P_{k,\epsilon} - 1|$ on $D'_k \supset D_k$ and $\epsilon > |P_{k,\epsilon} - f_k| = |P_{k,\epsilon} - 0| = |P_{k,\epsilon}|$ on $D'_\ell \supset D_\ell$ for all $\ell \neq k$. □

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For each open set $\Omega \subseteq \mathbb{C}$ and all $f_1, \ldots, f_N \in \mathcal{H}(\Omega)$, prove that

$$U := \{z \in \Omega : |f_1(z)| < 1, \ldots, |f_N(z)| < 1\}$$

is Runge in Ω [in other words, (Ω, U) is a Runge pair].

Proof. Suppose, for the sake of deriving a contradiction, that there is some connected component C of $\Omega \setminus U$ that is compact. Then C is closed, so $\partial C = C \setminus C^\circ \subset C \subset \Omega$. On the other hand,

$$\partial C \subset \partial U \cup \partial \Omega.$$

Now, since Ω is open, $\partial\Omega \cap \Omega = \emptyset$, so the fact that $\partial C \subset \Omega$ implies that $\partial C \subset \partial U$. Furthermore,

$$\partial U = \{z \in \Omega : |f_1(z)| \leq 1, \ldots, |f_N(z)| \leq 1, |f_k(z)| = 1 \text{ for some } k \in \{1, \ldots, N\}\}.$$

Now, for each $k \in \{1, \ldots, N\}$, $f_k \in \mathcal{H}(C^\circ)$. Since $\partial C^\circ = \partial C \subset \partial U$, we know, by the maximum principle, that

$$|f_k(z)| < \sup_{\zeta \in \partial C} |f_k(\zeta)| \leq \sup_{\zeta \in \partial U} |f_k(\zeta)| \leq 1$$

for all $z \in C^\circ$ if f_k is nonconstant (if f_k is constant, then either $|f_k(z)| < 1$ for all $z \in \mathbb{C}$, or U is empty). Since our choice of k was arbitrary, we see that $|f_k(z)| < 1$ for all $z \in C^\circ$ and all $k \in \{1, \ldots, N\}$, which implies that $C^\circ \subset U$. However, this is impossible, since $C \supset C^\circ$ is a connected component of $\Omega \setminus U$.

From this contradiction, then, we see that no connected component of $\Omega \setminus U$ is compact. Therefore, by Runge's Theorem, (Ω, U) is a Runge pair. □

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Let $\overline{D(0,1)} := \{z \in \mathbb{C} : |z| \leq 1\}$ be the closed unit disc, and let $D(c, r) := \{z \in \mathbb{C} : |z - c| < r\}$. Consider a sequence of discs $D(c_n, r_n)$ with $|c_n| < 1$, $|r_n| > 0$ and such that $\overline{D(c_n, r_n)} \subset D(0, 1)$ and $\overline{D(c_m, r_m)} \cap \overline{D(c_n, r_n)} = \emptyset$ for all $m \in \mathbb{N}^*$ and $n \in \mathbb{N}^*$ with $m \neq n$. Also, assume that $\sum_{n=1}^\infty r_n < \infty$ and

$\sum_{n=1}^{\infty} r_n^2 < 1$. Moreover, suppose that $\bigcup_{n\in\mathbb{N}^*} D(c_n, r_n)$ is dense in $D(0,1)$. Let

$$K := \overline{D(0,1)} \setminus \bigcup_{n\in\mathbb{N}^*} D(c_n, r_n)$$

and for each $\phi \in C(K)$, the Banach space of continuous functions on K, define

$$F(\phi) := \int_{\partial D(0,1)} \phi(z)dz - \sum_{n=1}^{\infty} \int_{\partial D(c_n,r_n)} \phi(z)dz.$$

(274.1): Prove that in the definition of F the series converges, and that F defines a continuous linear functional on $C(K)$.

Proof. First, note that, for all $n \in \mathbb{N}^*$, $\partial D(c_n, r_n) \subset K$ since $\overline{D(c_n, r_n)} \cap \overline{D(c_m, r_m)} = \emptyset$ for all $m \neq n$. Now, since K is a compact set, if $\phi \in C(K)$, then there exists $M \in \mathbb{N}$ such that $|\phi(z)| < M$ for all $z \in K$. Specifically, for $z \in \partial D(c_n, r_n)$, $|\phi(z)| < M$. Therefore,

$$\begin{aligned} \left|\sum_{n=1}^{\infty} \int_{\partial D(c_n,r_n)} \phi(z)dz\right| &\leq \sum_{n=1}^{\infty} \left|\int_{\partial D(c_n,r_n)} \phi(z)dz\right| \\ &\leq \sum_{n=1}^{\infty} \int_{\partial D(c_n,r_n)} |\phi(z)||dz| \\ &\leq \sum_{n=1}^{\infty} M \int_{\partial D(c_n,r_n)} |dz| \\ &= \sum_{n=1}^{\infty} M(2\pi r_n) \\ &= 2\pi M \sum_{n=1}^{\infty} r_n \\ &< \infty \end{aligned}$$

since $\sum_{n=1}^{\infty} r_n < \infty$. Thus, the series in the definition of F converges.

To see that F is linear, suppose $\phi, \psi \in C(K)$ and $\alpha, \beta \in \mathbb{C}$. Then

$$\begin{aligned} F(\alpha\phi + \beta\psi) &= \int_{\partial D(0,1)} (\alpha\phi + \beta\psi)(z)dz - \sum_{n=1}^{\infty} \int_{\partial D(c_n,r_n)} (\alpha\phi + \beta\psi)(z)dz \\ &= \int_{\partial D(0,1)} [\alpha\phi(z) + \beta\psi(z)]dz - \sum_{n=1}^{\infty} \int_{\partial D(c_n,r_n)} [\alpha\phi(z) + \beta\psi(z)]dz \\ &= \alpha \int_{\partial D(0,1)} \phi(z)dz + \beta \int_{\partial D(0,1)} \psi(z)dz \\ &\quad - \sum_{n=1}^{\infty} \left[\alpha \int_{\partial D(c_n,r_n)} \phi(z)dz + \beta \int_{\partial D(c_n,r_n)} \psi(z)dz\right] \end{aligned}$$

Hence,

$$\begin{aligned}
F(\alpha\phi+\beta\psi) &= \alpha\int_{\partial D(0,1)}\phi(z)dz+\beta\int_{\partial D(0,1)}\psi(z)dz \\
&\quad -\sum_{n=1}^{\infty}\alpha\int_{\partial D(c_n,r_n)}\phi(z)dz-\sum_{n=1}^{\infty}\beta\int_{\partial D(c_n,r_n)}\psi(z)dz \\
&= \alpha\left(\int_{\partial D(0,1)}\phi(z)dz-\sum_{n=1}^{\infty}\int_{\partial D(c_n,r_n)}\phi(z)dz\right) \\
&\quad +\beta\left(\int_{\partial D(0,1)}\psi(z)dz-\sum_{n=1}^{\infty}\int_{\partial D(c_n,r_n)}\psi(z)dz\right) \\
&= \alpha F(\phi)+\beta F(\psi),
\end{aligned}$$

so we see that F is a linear functional.

Finally, suppose $\|\phi\|\leq 1$ (that is, $\sup_{z\in K}|\phi(z)|<1$) and let $M=\sum_{n=1}^{\infty} r_n$. Then

$$\begin{aligned}
|F(\phi)| &= \left|\int_{\partial D(0,1)}\phi(z)dz-\sum_{n=1}^{\infty}\int_{\partial D(c_n,r_n)}\phi(z)dz\right| \\
&\leq \left|\int_{\partial D(0,1)}\phi(z)dz\right|+\left|\sum_{n=1}^{\infty}\int_{\partial D(c_n,r_n)}\phi(z)dz\right| \\
&\leq \int_{\partial D(0,1)}|\phi(z)||dz|+\sum_{n=1}^{\infty}\left|\int_{\partial D(c_n,r_n)}\phi(z)dz\right| \\
&\leq \int_{\partial D(0,1)}|\phi(z)||dz|+\sum_{n=1}^{\infty}\int_{\partial D(c_n,r_n)}|\phi(z)||dz| \\
&\leq \int_{\partial D(0,1)}|dz|+\sum_{n=1}^{\infty}\int_{\partial D(c_n,r_n)}|dz| \\
&= 2\pi+\sum_{n=1}^{\infty}2\pi r_n \\
&= 2\pi(1+M).
\end{aligned}$$

Therefore, by the result proved in Exercise 251 on last week's homework, F is a continuous linear functional. □

(274.2): Prove that if ϕ is the restriction to K of a rational function with poles outside K, then $F(\phi)=0$.

Proof. Let P_ϕ denote the set of poles of ϕ. Then, for each $z\in P_\phi$, either $z\in D(c_n,r_n)$ for some $n\in\mathbb{N}^*$ or $z\notin D(0,1)$. Let U be a neighborhood of $\overline{D(0,1)}$ such that $U\cap P_\phi=\overline{D(0,1)}\cap P_\phi$ and for each $n\in\mathbb{N}^*$ let U_n be a neighborhood of $\overline{D(c_n,r_n)}$ such that

$U_n \cap P_\phi = D(c_n, r_n) \cap P_\phi$. Since ϕ is a rational function, $\phi \in \mathcal{H}(U \backslash P_\phi)$ and so, by the residue theorem,

$$\int_{\partial D(0,1)} \phi(z)dz = 2\pi i \sum_{a \in P_\phi \cap D(0,1)} \text{res}_\phi(a).$$

Similarly, $\phi \in \mathcal{H}(U_n \backslash P_\phi)$ for each n, so

$$\int_{\partial D(c_n,r_n)} \phi(z)dz = 2\pi i \sum_{a \in P_\phi \cap D(c_n,r_n)} \text{res}_\phi(a)$$

for all $n \in \mathbb{N}^*$. Now, since each $a \in P_\phi \cap D(0,1)$ is contained in exactly one $D(c_n, r_n)$ (since $D(c_n, r_n) \cap D(c_m, r_m) = \emptyset$ for $n \neq m$), we see that

$$\sum_{n=1}^{\infty} \int_{\partial D(c_n,r_n)} \phi(z)dz = \sum_{n=1}^{\infty} 2\pi i \sum_{a \in P_\phi \cap D(c_n,r_n)} \text{res}_\phi(a) = 2\pi i \sum_{a \in P_\phi \cap D(0,1)} \text{res}_\phi(a) = \int_{\partial D(0,1)} \phi(z)dz.$$

Therefore,

$$F(\phi) = \int_{\partial D(0,1)} \phi(z)dz - \sum_{n=1}^{\infty} \int_{\partial D(c_n,r_n)} \phi(z)dz = 0.$$

□

(274.3): Prove that if $\phi(z) = \bar{z}$ then $F(\phi) \neq 0$.

Proof. Let $\omega = \phi(z)dz = \bar{z}dz$. Then, by Green's Theorem (i.e. the result proved in Exercise 232 two weeks ago),

$$\int_{\partial\Omega} \phi(z)dz = \int_{\partial\Omega} \omega = \int\int_\Omega d\omega = \int\int_\Omega -\frac{\partial}{\partial \bar{z}}(\bar{z})dz \wedge d\bar{z} = \int\int_\Omega -dz \wedge d\bar{z}.$$

Therefore,

$$\int_{\partial D(0,1)} \phi(z)dz = \int\int_{D(0,1)} -dz \wedge d\bar{z} = \int\int_{D(0,1)} -(-2idx \wedge dy) = 2i\pi(1)^2 = 2\pi i.$$

On the other hand, for each $n \in \{1, \ldots, N\}$,

$$\int_{\partial D(c_n,r_n)} \phi(z)dz = \int\int_{D(c_n,r_n)} -dz \wedge d\bar{z} = \int\int_{D(c_n,r_n)} -(-2idx \wedge dy) = 2i\pi r_n^2.$$

Hence,

$$F(\phi) = \int_{\partial D(0,1)} \phi(z)dz - \sum_{n=1}^{N} \int_{\partial D(c_n,r_n)} \phi(z)dz = 2\pi i - \sum_{n=1}^{\infty} 2i\pi r_n^2 = 2\pi i \left(1 - \sum_{n=1}^{\infty} r_n^2\right) \neq 0$$

since $\sum_{n=1}^{\infty} r_n^2 < 1$. Therefore, we conclude that $F(\phi) \neq 0$. □

(274.4): Prove that rational functions with poles outside K are not dense in $C(K)$.

Proof. We saw in (274.2) that if ψ is a rational function with poles outside K, $F(\psi) = 0$. On the other hand, we saw in (274.3) that if $\phi(z) = \bar{z}$, then $F(\phi) \neq 0$. Thus F is a continuous linear functional that is zero on the rational functions with poles outside K but F is not zero on all of $C(K)$, so, by the Hahn-Banach Theorem, the rational functions with poles outside K are not dense in $C(K)$. □

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With the notation as in the text, prove that for each open subset $\Omega \subseteq \mathbb{C}$ and for all compact subsets K and L of Ω such that

$$K \subset L^\circ \subset \Omega,$$

the following inclusion holds:

$$\widehat{K}_\Omega \subset \left(\widehat{L}_\Omega\right)^\circ.$$

Proof. Suppose C is a connected component of $\Omega \setminus K$ such that $C \Subset \Omega$. Since $\Omega \setminus K \supset \Omega \setminus L$, either $C \subset L \subset \widehat{L}_\Omega$ or C contains a connected component C' of $\Omega \setminus L$. If the latter, then $\overline{C'} \subset \overline{C} \subset \Omega$ and $\overline{C'}$ is bounded (since $\overline{C}$ is); that is, $C' \Subset \Omega$. Since C is connected, either $C' = C$ or $C \setminus C' \subset L$; either way, $C = C' \cup (L \cap C)$, so $C \subset \widehat{L}_\Omega$. Hence, we see that if $C \Subset \Omega$ is a connected component of $\Omega \setminus K$, then $C \subset \widehat{L}_\Omega$. Since $K \subset L^\circ \subset L \subset \widehat{L}_\Omega$, this implies that $\widehat{K}_\Omega \subset \widehat{L}_\Omega$.

Now, suppose $z \in \widehat{K}_\Omega \subset \widehat{L}_\Omega$. If $z \in K$, then, since $K \subset L^\circ \subset \left(\widehat{L}_\Omega\right)^\circ$, $z \in \left(\widehat{L}_\Omega\right)^\circ$. Otherwise, $z \in C$ for some connected component C of $\Omega \setminus K$ such that $C \Subset \Omega$. Since $\Omega \setminus K$ is open, there exists an open disc D containing z such that $D \subset \Omega \setminus K$. Since C and D are connected and $C \cap D \supset \{z\} \neq \emptyset$, $C \cup D$ is connected. Therefore, since C is the connected component of $\Omega \setminus K$ containing z,

$$D \subset C \subset \widehat{K}_\Omega \subset \widehat{L}_\Omega.$$

Hence, $z \in D \subset \widehat{L}_\Omega$, so $z \in \left(\widehat{L}_\Omega\right)^\circ$. Since our choice of $z \in \widehat{K}_\Omega$ was arbitrary, we conclude that $\widehat{K}_\Omega \subset \left(\widehat{L}_\Omega\right)^\circ$. □

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