COMPLEX ANALYSIS HW 11

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Let $\{D_l : k \in I\}$ be an open cover of an open set $\Omega \subseteq \mathbb{C}$ by open discs. For each index $k \in I$, let $h_k \not\equiv 0$ be a meromorphic function on D_k, and assume that for all indices $k, \ell \in I$, the function $g_{k,\ell} := h_k/h_\ell$ is holomorphic on $D_k \cap D_\ell$. Prove that for all indices $k, \ell \in I$ there exist holomorphic functions f_k on D_k without any zero and such that $f_k = g_{k,\ell} f_\ell$ on $D_k \cap D_\ell$.

Proof. First, note that, for all $k, \ell \in I$,

$$g_{k,\ell} = \frac{h_k}{h_\ell} = \frac{1}{\frac{h_\ell}{h_k}} = \frac{1}{g_{\ell,k}},$$

so $g_{\ell,k}$ is non-vanishing. Since our choice of k, ℓ was arbitrary, we see that $g_{k,\ell}$ is non-vanishing for all $k, \ell \in I$. Now, the intersection of two open discs is simply connected, so, since $g_{k,\ell} \in \mathcal{H}(D_k \cap D_\ell)$ for all $k, \ell \in I$ and $D_k \cap D_\ell$ is simply connected, there exists $\alpha_{k,\ell} \in \mathcal{H}(D_k \cap D_\ell)$ such that

$$e^{\alpha_{k,\ell}} = g_{k,\ell}.$$

Now, on $D_j \cap D_k \cap D_\ell$ for any $j, k, \ell \in I$,

$$e^{\alpha_{j,k}+\alpha_{k,\ell}} = e^{\alpha_{j,k}} e^{\alpha_{k,\ell}} = g_{j,k} g_{k,\ell} = \frac{h_j}{h_k}\frac{h_k}{h_\ell} = \frac{h_j}{h_\ell} = g_{j,\ell} = e^{\alpha_{j,\ell}}.$$

Hence, $\alpha_{j,k} + \alpha_{k,\ell} = \alpha_{j,\ell} + 2\pi i n_{j,\ell}$ for $n \in \mathbb{Z}$. Now, we need the following lemma:

Lemma 0.1. *There exists a family of functions $\{\beta_k\}_{k\in I}$ where $\beta_k \in \mathcal{H}(D_k)$ such that*

$$\beta_k - \beta_\ell = \alpha_{k,\ell} + 2\pi i m_{k,\ell}$$

on $D_k \cap D_\ell$ for all $k, \ell \in I$ and some $m_{k,\ell} \in \mathbb{Z}$.

Proof. Let $\{\gamma_k\}_{k\in I}$ be a partition of unity subordinate to $\{D_k\}_{k\in I}$. Define

$$\psi_{k,\ell}(z) = \begin{cases} \gamma_\ell(z)(\alpha_{k,\ell}(z) + 2\pi i n_{k,\ell}) & \text{if } z \in D_k \cap D_\ell \\ 0 & \text{if } z \in D_k \setminus D_k \cap D_\ell. \end{cases}$$

Then $\psi_{k,\ell} \in C^\infty(D_k)$. Define

$$\phi_k = \sum_{\ell\in I} \psi_{k,\ell};$$

since only finitely many γ_ℓ are supported on any neighborhood of a point in D_k, $\phi_k \in C^\infty(D_k)$.

Now, letting $j = k = \ell$,

$$\alpha_{j,j} + \alpha_{j,j} = \alpha_{j,j} + 2\pi i n_{j,j},$$

so $\alpha_{j,j} = 2\pi i n_{j,j}$. Also, if $\ell = j$, then

$$\alpha_{j,k} + \alpha_{k,j} = \alpha_{j,j} = 2\pi i n_{j,j}$$

or

$$\alpha_{j,k} + \alpha_{k,j} = \alpha_{k,j} + \alpha_{j,k} = \alpha_{k,k} = 2\pi i n_{k,k},$$

so note that $n_{j,j} = n_{k,k}$. Now, for $k, \ell \in I$,

$$\phi_k - \phi_\ell = \sum_j \gamma_j(\alpha_{k,j} + 2\pi i n_{k,j} - (\alpha_{\ell,j} + 2\pi i n_{\ell,j}));$$

since

$$\alpha_{k,j} - \alpha_{\ell,j} = \alpha_{k,j} + \alpha_{j,\ell} - 2\pi i n_{\ell,\ell} = \alpha_{k,\ell} + 2\pi i(n_{k,\ell} + n_{\ell,\ell}),$$

we see that

$$\phi_k - \phi_\ell = \sum_j \gamma_j(\alpha_{k,\ell} + 2\pi i(n_{k,\ell} + n_{\ell,\ell})) = \left(\sum_{j\in I} \gamma_j\right)(\alpha_{k,\ell} + 2\pi i(n_{k,\ell} + n_{\ell,\ell}) = \alpha_{k,\ell} + 2\pi i(n_{k,\ell} + n_{\ell,\ell}).$$

Hence, we see that there exists a family $\{\phi_j\}_{j\in I}$ such that

$$\phi_k - \phi_\ell = \alpha_{k,\ell} + 2\pi i m_{k,\ell}$$

on $D_k \cap D_\ell$ for $m_{k,\ell} \in \mathbb{Z}$. Since $\alpha_{k,\ell} \in \mathcal{H}(D_k \cap D_\ell)$ for all $k, \ell \in I$, this in turn implies that

$$0 = \frac{\partial}{\partial \bar{z}}(\alpha_{k,\ell} + 2\pi i m_{k,\ell}) = \frac{\partial \phi_k}{\partial \bar{z}} - \frac{\partial \phi_\ell}{\partial \bar{z}}.$$

Hence, there exists $\phi \in C^\infty(\Omega)$ such that $\phi|_{D_k} = \frac{\partial \phi_k}{\partial \bar{z}}$ for all $k \in I$.

Now, let $u \in C^\infty(\Omega)$ such that $\frac{\partial u}{\partial \bar{z}} = \phi$ on Ω and let $\beta_k = \phi_k - u$ on D_k for all $k \in I$. Then

$$\frac{\partial \beta_k}{\partial \bar{z}} = \frac{\partial \phi_k}{\partial \bar{z}} - \frac{\partial u}{\partial \bar{z}} = \frac{\partial \phi_k}{\partial \bar{z}} - \phi = 0$$

on D_k, so $\beta_k \in \mathcal{H}(D_k)$. Now, if $k, \ell \in I$, then

$$\beta_k - \beta_\ell = (\phi_k - u) - (\phi_\ell - u) = \phi_k - \phi_\ell = \alpha_{k,\ell} + 2\pi i m_{k,\ell},$$

so $\{\beta_k\}_{k\in I}$ is the desired family of functions. □

With this lemma in hand, let $\{\beta_k\}_{k\in I}$ be as in the lemma and define $f_k := e^{\beta_k}$ for all $k \in I$. Then, since $\beta_k \in \mathcal{H}(D_k)$ and exp is entire, $f_k \in \mathcal{H}(D_k)$. Furthermore, since the exponential function is non-vanishing, f_k has no zeros. Now, for all $k, \ell \in I$,

$$\frac{f_k}{f_\ell} = \frac{e^{\beta_k}}{e^{\beta_\ell}} = e^{\beta_k - \beta_\ell} = e^{\alpha_{k,\ell} + 2\pi i m_{k,\ell}} = e^{\alpha_{k,\ell}} e^{2\pi i m_{k,\ell}} = g_{k,\ell}.$$

Hence, for all $k, \ell \in I$,

$$f_k = g_{k,\ell} f_\ell$$

on $D_k \cap D_\ell$. $\square$

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Let R_1 and R_2 be rectangles in $\mathbb{C}$ whose union $R_1 \cup R_2$ is also a rectangle. Suppose that $f : (R_1 \cap R_2) \to \mathbb{C}$ is a bounded holomorphic function on their intersection. Prove that there exist bounded holomorphic functions $f_1 \in \mathcal{H}(R_1)$ and $f_2 \in \mathcal{H}(R_2)$ such that $f = f_1 - f_2$ on $R_1 \cap R_2$.

Proof. Let $\alpha_1 \in C_c^\infty(R_1)$ and $\alpha_2 \in C_c^\infty(R_2)$ such that $\alpha_1 + \alpha_2 = 1$ on $R_1 \cap R_2$. Define

$$g_1(z) = \begin{cases} (\alpha_2 f)(z) & z \in R_1 \cap R_2 \\ 0 & z \in R_1 \setminus R_1 \cap R_2 \end{cases}$$

and

$$g_2(z) = \begin{cases} -(\alpha_1 f)(z) & z \in R_1 \cap R_2 \\ 0 & z \in R_2 \setminus R_1 \cap R_2. \end{cases}$$

Then $g_1 \in C^\infty(R_1)$ and $g_2 \in C^\infty(R_2)$. Furthermore, for $z \in R_1 \cap R_2$,

$$g_1(z) - g_2(z) = (\alpha_2 f)(z) - (-\alpha_1 f)(z) = (\alpha_1 + \alpha_2)(z) f(z) = f(z).$$

Now,

$$0 = \frac{\partial f}{\partial \bar{z}} = \frac{\partial g_1}{\partial \bar{z}} - \frac{\partial g_2}{\partial \bar{z}}$$

on $R_1 \cap R_2$, so there exists $\phi \in C^\infty(R_1 \cup R_2)$ such that $\phi|_{R_i} = \frac{\partial g_i}{\partial \bar{z}}$.

Now, let $u \in C^\infty(R_1 \cup R_2)$ such that $\frac{\partial u}{\partial \bar{z}} = \phi$ and let $f_i = g_i - u$. Then

$$\frac{\partial f_i}{\partial \bar{z}} = \frac{\partial g_i}{\partial \bar{z}} - \frac{\partial u}{\partial \bar{z}} = \frac{\partial g_i}{\partial \bar{z}} - \phi = 0$$

on R_i, so $f_i \in \mathcal{H}(R_i)$. Furthermore, for $z \in R_1 \cap R_2$,

$$f_1(z) - f_2(z) = (g_1 - u)(z) - (g_2 - u)(z) = g_1(z) - g_2(z) = f(z).$$

Hence, $f = f_1 - f_2$ on $R_1 \cap R_2$. $\square$

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Consider a simply connected open set $\Omega \subseteq \mathbb{C}$ and a meromorphic function f on Ω with the set of poles of f denoted by E. Assume that the residue of f at each point of E is an integer. Prove that there exists a meromorphic function g on Ω that is holomorphic on $\Omega \setminus E$ and such that the function

$$\frac{g'}{g} - f \in \mathcal{H}(\Omega \setminus E)$$

has a primitive in $\mathcal{H}(\Omega \setminus E)$. Deduce that if all the poles of f are simple, then there exists a meromorphic function h on Ω such that on $\Omega \setminus E$

$$\frac{h'}{h} = f.$$

Proof. By Weierstrass' Theorem, there exists $g \in \mathcal{H}(\Omega \setminus E)$ such that $g(z) \neq 0$ for all $z \in \Omega \setminus E$ and

$$\mathrm{ord}_a(g) = \mathrm{res}_a(f)$$

for all $a \in E$. Then if γ is a closed loop in $\Omega \setminus E$,

$$\begin{aligned}\int_\gamma \left[\frac{g'(z)}{g(z)} - f(z)\right] dz &= \int_\gamma \frac{g'(z)}{g(z)} dz - \int_\gamma f(z) dz \\ &= \sum_{a \in E} \mathrm{ord}_a(g) n(\gamma, a) - \sum_{a \in E} \mathrm{res}_a(f) n(\gamma, a) \\ &= \sum_{a \in E} (\mathrm{ord}_a(g) - \mathrm{res}_a(f)) n(\gamma, a) \\ &= 0,\end{aligned}$$

where the first term in the second line is by the Argument Principle and the second term is by the Residue Theorem. Since our choice of closed loop γ was arbitrary, we see that $\int_\gamma \frac{g'}{g} - f = 0$ for all closed loops γ in $\Omega \setminus E$, meaning that $\frac{g'}{g} - f$ has a primitive in $\mathcal{H}(\Omega \setminus E)$.

Now, suppose all poles of f are simple. Then, with g defined as above,

$$\mathrm{res}_a\left(\frac{g'}{g}\right) = \mathrm{ord}_a(g) = \mathrm{res}_a(f)$$

for all $a \in E$. Hence, since $\frac{g'}{g}$ has only simple poles, the principal parts of $\frac{g'}{g}$ and f are identical at all $a \in E$; since for all other points in Ω, both $\frac{g'}{g}$ and f are holomorphic, this implies that

$$\frac{g'}{g} - f$$

has 0 principal part on all of Ω, so $h_1 = \frac{g'}{g} - f \in \mathcal{H}(\Omega)$. Since Ω is simply connected, h_1 has a primitive $H \in \mathcal{H}(\Omega)$. Now, define

$$h_2(z) = e^{-H(z)}$$

for all $z \in \Omega$. Then $h_2 \in \mathcal{H}(\Omega)$. Furthermore,

$$\frac{h_2'(z)}{h_2(z)} = \frac{-H'(z)e^{-H(z)}}{e^{-H(z)}} = -H'(z) = -h_1(z).$$

Hence,

$$0 = \frac{g'}{g} - f - \left(\frac{g'}{g} - f\right) = \frac{g'}{g} - f - h_1 = \frac{g'}{g} - f + \frac{h_2'}{h_2} = \left(\frac{g'}{g} + \frac{h_2'}{h_2}\right) - f. \tag{1}$$

Define $h = gh_2$. Then

$$\frac{h'}{h} = \frac{(gh_2)'}{gh_2} = \frac{g'h_2 + gh_2'}{gh_2} = \frac{g'}{g} + \frac{h_2'}{h_2},$$

so, by (1), $\frac{h'}{h} - f = 0$. Furthermore, since g is meromorphic on Ω and $h_2 \in \mathcal{H}(\Omega)$, h is meromorphic on Ω, so h is exactly the desired function. $\square$

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(285.1): Prove that for each sequence of complex numbers $(w_\ell)_{\ell=0}^{\infty}$, the product

$$\prod_{\ell=1}^{\infty}(1+|w_\ell|) := \lim_{L\to\infty}\prod_{\ell=1}^{L}(1+|w_\ell|) = P$$

converges to a limit P if but only if the series

$$\sum_{\ell=1}^{\infty}|w_\ell| := \lim_{L\to\infty}\sum_{\ell=1}^{L}|w_\ell| = S$$

converges to a limit S.

Proof. Suppose $\prod_{\ell=1}^{\infty}(1+|w_\ell|) = P$. Then, if $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that, if $L \geq N$,

$$\begin{aligned}\epsilon &> \left|\prod_{\ell=1}^{L}(1+|w_\ell|) - P\right| \\ &= \left|\left(1+\sum_{\ell=1}^{L}|w_\ell| + \sum_{\ell=1}^{L-1}\sum_{k=\ell+1}^{L}|w_\ell||w_k| + \cdots + \sum_{k=1}^{L}\prod_{\ell\neq k}|w_\ell| + \prod_{\ell=1}^{L}|w_\ell|\right) - P\right| \\ &= \left|\sum_{\ell=1}^{L}|w_\ell| - \left[P - \left(1+\sum_{\ell=1}^{L-1}\sum_{k=\ell+1}^{L}|w_\ell||w_k| + \cdots + \sum_{k=1}^{L}\prod_{\ell\neq k}|w_\ell| + \prod_{\ell=1}^{L}|w_\ell|\right)\right]\right|,\end{aligned}$$

Hence,

$$\lim_{L\to\infty}\sum_{\ell=1}^{L}|w_\ell| = P - \left(1+\sum_{\ell=1}^{L-1}\sum_{k=\ell+1}^{L}|w_\ell||w_k| + \cdots + \sum_{k=1}^{L}\prod_{\ell\neq k}|w_\ell| + \prod_{\ell=1}^{L}|w_\ell|\right)$$

converges.

On the other hand, suppose $\sum_{\ell=1}^{\infty}|w_\ell| = S$. Now, if $x \in \mathbb{R}$ such that $x \geq 0$, then

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n} \geq 1+x.$$

Hence, for each $\ell \in \{1,2,\ldots,\}$, $1+|w_\ell| \leq e^{|w_\ell|}$ since $|w_\ell| \geq 0$. Therefore, for all $N \in \mathbb{N}$,

$$\prod_{\ell=1}^{N}(1+|w_\ell|) \leq \prod_{\ell=1}^{N}e^{|w_\ell|} = \exp\left(\sum_{\ell=1}^{N}|w_\ell|\right).$$

Hence, $\prod_{\ell=1}^{N}(1+|w_\ell|) \leq e^S$ for all N. Therefore, the sequence of partial products is bounded; since it is a monotone increasing function, it must converge to some limit P. Therefore, we conclude that

$$\prod_{\ell=1}^{\infty}(1+|w_\ell|) = P.$$

□

(285.2): Prove that if the product $\prod_{\ell=1}^{\infty}(1+|w_\ell|)$ converges, then the product $\prod_{\ell=1}^{\infty}(1+w_\ell)$ also converges.

Proof.

Lemma 0.2. *Suppose* $P_N = \prod_{\ell=1}^{N}(1+|w_\ell|)$ *and* $P'_N = \prod_{\ell=1}^{N}(1+w_\ell)$. *Then*

$$|P'_N - 1| \leq P_N - 1.$$

Proof. We prove this by induction. If $N = 1$, then

$$|P'_N - 1| = |(1+w_\ell) - 1| = |w_\ell| \leq |w_\ell| + 1 = P_N.$$

Now, suppose $|P'_k - 1| \leq P_k + 1$. Then

$$\begin{aligned} |P'_{k+1} - 1| = |P'_k(1+w_{k+1}) - 1| &= |(P'_k - 1)(1+w_{k+1}) + w_{k+1}| \\ &\leq |P'_k - 1||1+w_{k+1}| + |w_{k+1}| \\ &\leq (P_k - 1)(1+|w_{k+1}|) + |w_{k+1}| \\ &= P_k(1+|w_{k+1}|) - 1 \\ &= P_{k+1} - 1. \end{aligned}$$

Hence, by induction, $|P'_N - 1| \leq P_N - 1$ for all $N \in \mathbb{N}$ □

Suppose $\prod_{\ell=1}^{\infty}(1+|w_\ell|)$ converges and let P_N and P'_N denote the partial products as in Lemma 0.2. Then, since $\prod(1+|w_\ell|)$ converges, the sequence (P_N) must be bounded. By the lemma, the sequence $(|P'_N|)$ is bounded as well; let B be a bound on the sequence (P'_N).

Now, since $\prod_{\ell=1}^{\infty}(1+|w_\ell|)$ converges, we know, by (285.1) above, that

$$\sum_{\ell=1}^{\infty}|w_\ell| = S$$

for some S. Hence, if $0 < \epsilon < 1/2$, then there exists N_0 such that $\sum_{\ell=N_0}^{\infty}|w_\ell| < \frac{\epsilon}{2B}$. Now, if $N > N_0$ and $M \geq N$, then

$$P'_M - P'_N = P'_N \prod_{\ell=N+1}^{M}(1+w_\ell) - P'_N = P'_N\left(\prod_{\ell=N+1}^{M}(1+w_\ell) - 1\right).$$

Define a new sequence (v_ℓ) by $v_\ell = 0$ for $\ell < N$ and $v_\ell = w_\ell$ for $\ell \geq N$. Then,

$$|P'_M - P'_N| = |P'_N|\left|\prod_{\ell=N+1}^{M}(1+w_\ell) - 1\right| = |P'_N|\left|\prod_{\ell=1}^{M}(1+v_\ell) - 1\right| \leq |P'_N|\left(\prod_{\ell=1}^{M}(1+|v_\ell|) - 1\right)$$

by Lemma 0.2. Hence,

$$|P'_M - P'_N| \leq |P'_N|(e^{\epsilon/2B} - 1) < 2B\frac{\epsilon}{2B} = \epsilon$$

since $e^\delta - 1 < 2\delta$ for all $\delta < 1/2$. Hence, since our choice of $\epsilon > 0$ was arbitrary, we see that the sequence of partial products (P'_N) is Cauchy, and so has a limit. Therefore, we conclude that

$$\prod_{\ell=1}^{\infty}(1 + w_\ell) = \lim_{N\to\infty}\prod_{\ell=1}^{N}(1 + w_\ell)$$

exists. □

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