COMPLEX ANALYSIS HW 12

CLAY SHONKWILER

296

Prove that an entire function is proper if and only if it is a non-constant polynomial.

Proof. ($\Rightarrow$) Suppose $f \in \mathcal{H}(\mathbb{C})$ is not a polynomial. Then f has the power-series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

where infinitely many of the a_k are non-zero. Define $F : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ by

$$z \mapsto f\left(\frac{1}{z}\right).$$

Then, since f and $z \mapsto \frac{1}{z}$ are both holomorphic on the punctured plane, $F \in \mathcal{H}(\mathbb{C} \setminus \{0\})$. Also,

$$F(z) = f\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{a_n}{z^n},$$

which, since it converges uniformly on the complement of any open neighborhood of the origin, is a Laurent series expansion of F. By the uniqueness of the Laurent series and given that infinitely many of the a_k are non-zero, we see that F has an essential singularity at the origin, which is to say that f has an essential singularity at ∞. Then, by Casorati-Weierstrass, if $r > 0$ and $D_r^* = \{z \in \mathbb{C} : 0 < |z| < r\}$, $F(D_r^*)$ is dense in $\mathbb{C}$ for all $r > 0$; which is to say that $f(\mathbb{C} \setminus D_{1/r})$ is dense in $\mathbb{C}$ for all $r > 0$. Hence, if D is any closed disc of finite radius, $f^{-1}(D) \cap (\mathbb{C} \setminus D_R) \neq \emptyset$ for all $R > 0$. Therefore, $f^{-1}(D)$ is unbounded and, thus, not compact, so we see that f is not proper. Since our choice of f was arbitrary, we see that all entire non-polynomials are non-proper. By contrapositive, then, we conclude that if f is an entire, proper function, then f is a polynomial.

($\Leftarrow$) On the other hand, suppose $f \in \mathcal{H}(\mathbb{C})$ is a polynomial and that K is compact. Note that, since f is continuous, $f^{-1}(K)$ is closed, so to demonstrate that $f^{-1}(K)$ is compact, it suffices to show that $f^{-1}(K)$ is bounded. Since f is polynomial, $f(z) = a_n z^n + \ldots + a_1 z + a_0$ for $a_0, \ldots, a_n \in$

$\mathbb{C}$. Define $F : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ by $F(z) = f\left(\frac{1}{z}\right)$. Then certainly $F \in \mathcal{H}(\mathbb{C} \setminus \{0\})$, since both f and $z \mapsto \frac{1}{z}$ are holomorphic on the punctured plane. Now,

$$F(z) = f\left(\frac{1}{z}\right) = \frac{a_n}{z^n} + \ldots + \frac{a_1}{z} + a_0.$$

This is a Laurent expansion of F at the origin; since there are only finitely many terms, we see that F is meromorphic at the origin and, hence, $|F(z)| \to \infty$ and $|z| \to 0$. Therefore, $|f(z)| \to \infty$ as $|z| \to \infty$. Hence, if $M = \max_{a \in K}\{|a|\}$, there exists $R \in \mathbb{R}$ such that, for all z in the complement of $D_R(0)$, $|f(z)| > M$. Hence,

$$f^{-1}(K) \subset D_R(0),$$

so $f^{-1}(K)$ is bounded and, therefore, compact. Since our choice of compact K was arbitrary, we see that $f^{-1}(K)$ is compact for all compact K, so f is proper. Since our choice of polynomial f was arbitrary, we conclude that all polynomials are proper. ☐

299

Consider a bounded connected open set $\Omega \subset \mathbb{C}$ and a holomorphic map $f : \Omega \to \Omega$. Assume that there exists a point $c \in \Omega$ where $f(c) = c$ and $f'(c) = 1$.

(299.1): Prove that f is the identity.

Proof. Denote $f \circ f$ by f^2, $f \circ f \circ f$ by f^3, etc. Now, certainly, if I is the identity map, $I'(z) = 1$ for all $z \in \Omega$. Now, we want to show $(f^n)'(c) = 1$ for all $n \in \mathbb{N}$. We do so by induction. Certainly $(f^1)'(c) = f'(c) = 1$, by hypothesis. Now, suppose $(f^k)'(c) = 1$. Then

$$(f^{k+1})'(c) = (f \circ f^k)'(c) = (f' \circ f^k)(c)(f^k)'(c) = f'(f^k(c)) = f'(c) = 1.$$

Thus, we see that $(f^n)'(c) = 1$ for all $n \in \mathbb{N}$. Also by induction, we show that $(f^n)''(c) = nf''(c)$: clearly, $(f^1)''(c) = f''(c)$. Now, if $(f^k)''(c) = kf''(c)$, then

$$(f^{k+1})'(c) = (f \circ f^k)'(c) = (f' \circ f^k)(c)(f^k)'(c),$$

so

$$(f^{k+1})''(c) = f''(f^k(c))(f^k)'(c) + (f^k)''(c) = f''(c) + kf''(c) = (k+1)f''(c).$$

Hence, by induction, $(f^n)''(c) = nf''(c)$ for all $n \in \mathbb{N}$. More generally,

$$(f^n)^{(k)}(c) = nf^{(k)}(c)h_{n,k}(c) + p_{n,k}(c),$$

where $h_{n,k}$ is a linear combination of powers of $f'(c)$ and $p_{n,k}$ is a polynomial in $f^{(j)}$ for $1 \leq j < k$. Since $f'(c) = 1$, we can ignore these terms, and so $(f^n)^{(k)}(c) = nf^{(k)}(c) + p_{n,k}(c)$ where $p_{n,k}(c)$ is a polynomial in $f^{(j)}(c)$ for $2 \leq j < k$.

Let $D_r(c)$ be an open disc centered at c and contained in Ω. Then, for $z \in D_r(c)$,

$$f^n(z) - z = \sum_{k=0}^{\infty} \frac{(f^n)^{(k)}(c)}{k!}(z-c)^k = c + (z-c) + \sum_{k=2}^{\infty} \frac{(f^n)^{(k)}(c)}{k!}(z-c)^k.$$

since $f^n(c) = c$ and $(f^n)'(c) = 1$. Hence, if we define $g_n(z) = f^n(z) - z$, then, for $z \in D_r(c)$,

$$g_n(z) = f^n(z)-z = f^n(z)-c+(z-c) = \sum_{k=2}^{\infty} \frac{(f^n)^{(k)}(c)}{k!}(z-c)^k = \frac{n}{2}f''(c)(z-c)^2+\sum_{k=3}^{\infty} \frac{(f^n)^{(k)}(c)}{k!}(z-c)^k. \tag{1}$$

Note that for $z \in D_r(c)$,

$$|g_n(z)| = |f^n(z) - z| \le |f^n(z)| + |z| \le 2M,$$

Now, fix $w \in D_r(c)$ with $w \neq c$ and let $|w - c| < \rho < r$. Let

$$\kappa = \min_{|\zeta - c| = \rho} \{|\zeta - w|\}.$$

Then, as we saw in Exercise 94 (HW 2), the remainder term

$$R_{g_n,3,c}(w) = \frac{1}{2\pi i}\int_{|\zeta-c|=\rho} \frac{g_n(\zeta)(w-c)^4}{(\zeta-w)(\zeta-c)^4}d\zeta.$$

Hence,

$$\begin{aligned}
|R_{g_n,3,c}(w)| &= \left|\frac{1}{2\pi i}\int_{|\zeta-c|=\rho} \frac{g_n(\zeta)(w-c)^4}{(\zeta-w)(\zeta-c)^4}d\zeta\right| \\
&\le \frac{1}{2\pi i}\int_{|\zeta-c|=\rho} \left|\frac{g_n(\zeta)(w-c)^4}{(\zeta-w)(\zeta-c)^4}\right| |d\zeta| \\
&\le \frac{1}{2\pi i}\int_{|\zeta-c|=\rho} \frac{|g_n(\zeta)|}{|\zeta-w|}|d\zeta| \\
&\le \frac{1}{2\pi i}\int_{|\zeta-c|=\rho} \frac{2M}{\kappa}|d\zeta| \\
&= \frac{2\rho M}{\kappa},
\end{aligned}$$

so $g_n(w)$ is uniformly bounded for all $n \in \mathbb{N}$. On the other hand, $\frac{n}{2}f''(c)(w-c)^2$ gets arbitrarily large as $n \to \infty$ if $f''(c) \neq 0$, so we see that, in light of (1), $g_n(w)$ gets arbitrarily large. However, this contradicts the fact that g_n is bounded, so we see that $f''(c) = 0$. In turn, this implies that $g_n''(c) = (f^n)''(c) = nf''(c) = 0$.

Now, as we saw above, $(f^n)^{(k)} = nf^{(k)}(c) + p_{n,k}(c)$ where $p_{n,k}$ is a polynomial in the $f^{(j)}(c)$ for $2 \le j < k$. Hence, as an inductive hypothesis, suppose $f^{(j)}(c) = 0$ for all $2 \le j < k$. Then $p_{n,k}(c) = 0$

and hence $(f^n)^{(k)}(c) = nf^{(k)}(c)$. Thus,

$$g_n(z) = \frac{n}{k!}f^{(k)}(c)(z-c)^k + \sum_{j=k+1}^{\infty} \frac{(f^n)^{(j)}}{j!}(z-c)^j.$$

As before, the remainder term is uniformly bounded for all n, but, for any $w \in D_r(c)$, $\frac{n}{k!}f^{(k)}(c)(z-c)^k$ gets arbitrarily large as $n \to \infty$ unless $f^{(k)}(c) = 0$. Therefore, since we showed the base case $k = 2$ above, we conclude that $f^{(k)}(c) = 0$ for all $k \geq 2$. Therefore, the Taylor series for f reduces to

$$f(z) = f(c) + f'(c)(z-c) = c + (z-c) = z$$

for all $z \in D_r(c)$. As a result we see, by the identity theorem, that $f \equiv id$ on all of Ω. □

(299.2): Determine whether the foregoing result holds without the hypothesis that Ω be bounded.

Answer: No. Let $\Omega = \mathbb{C}$ and consider the function $f(z) = \frac{z^2}{2} + \frac{1}{2}$. Then $f \in \mathcal{H}(\mathbb{C})$. Furthermore,

$$f(1) = \frac{1^2}{2} + \frac{1}{2} = \frac{1}{2} + \frac{1}{2} = 1$$

and $f'(z) = \frac{2z}{2} = z$, so

$$f'(1) = 1.$$

However, since $f(0) = \frac{0^2}{2} + \frac{1}{2} = \frac{1}{2} \neq 0$, it is clear that f is not the identity.

♣

300

Consider a bounded connected open set $\Omega \subset \mathbb{C}$ with any point $c \in \Omega$. Let G be the group of analytic automorphisms of Ω that leave c fixed. Prove that G is compact: every sequence of elements of G has a subsequence converging uniformly on compact subsets of Ω to an element of G.

Proof. Let $\langle f_n \rangle$ be a sequence of elements in G. Let M be such that $|z| < M$ for all $z \in \Omega$. Then, for all n, $|f_n(z)| < M$ for all $z \in \Omega$. Therefore, by Montel's Theorem, there exists a convergent subsequence $\langle f_{n_k} \rangle$ that converges to a function f, uniformly on any compact subset of Ω. By Weierstrass' Theorem, $f \in \mathcal{H}(\Omega)$. Furthermore, $f_{n_k}^{-1}$ is an analytic automorphism of Ω for all n_k, and so $|f_{n_k}^{-1}(z)| < M$. Again using Montel's Theorem and Weierstrass' Theorem, there is a subsequence $\langle f_{n_{k_\ell}}^{-1} \rangle$ converging to $g \in \mathcal{H}(\Omega)$. Note that $\langle f_{n_{k_\ell}} \rangle$ converges to f. Now,

$$f(c) = \lim_{n_{k_\ell} \to \infty} f_{n_{k_\ell}}(c) = \lim_{n_{k_\ell} \to \infty} c = c$$

and

$$g(c) = \lim_{n_{k_\ell}\to\infty} f_{n_{k_\ell}}^{-1}(c) = \lim_{n_{k_\ell}\to\infty} c = c.$$

Now, let $z \in \Omega$ and let $\epsilon > 0$ such that $D_\epsilon(z) \subset \Omega$. Let $\delta = \min\{\epsilon, d(f(z), b\Omega)\}$. Then there exists $N_1 \in \mathbb{N}$ such that, if $n_{k_\ell} > N_1$,

$$|f_{n_{k_\ell}}(z) - f(z)| < \delta.$$

Let $D = \overline{D_\delta(f(z))}$. Then, since D is compact, there exists $N_2 \in \mathbb{N}$ such that, if $n_{k_\ell} > N_2$,

$$|f_{n_{k_\ell}}^{-1}(w) - g(w)| < \epsilon$$

for all $w \in D$. Let $N = \max\{N_1, N_2\}$. Then, for $n_{k_\ell} > N$,

$$|f_{n_{k_\ell}}(z) - f(z)| < \epsilon,$$

which is to say that $f_{n_{k_\ell}}(z)$ and so

$$\epsilon > |f_{n_{k_\ell}}^{-1}(f_{n_{k_\ell}}(z)) - g(f(z))| = |z - (g \circ f)(z)|.$$

Since our choice of $\epsilon > 0$ was arbitrary, we see that, in fact, $(g \circ f)(z) = z$. Since our choice of $z \in \Omega$ was arbitrary, we see that $g \circ f \equiv id_\Omega$. A completely parallel argument demonstrates that $f \circ g \equiv id_\Omega$, so we see that $g = f^{-1}$. Hence, $f, f^{-1} \in \mathcal{H}(\Omega)$ and, therefore, f is an analytic automorphism of Ω; since $f(c) = c$, this means $f \in G$. Since our choice of sequence $\langle f_n \rangle$ was arbitrary, we see that every sequence in G has a subsequence converging uniformly on compact subsets of Ω to an element of G, so G is compact. □

301

(301.1): Consider a function $f \in H^\infty[D(c,R)]$; thus $f \in \mathcal{H}[D(c,R)]$ with $|f| \le M$. Also let r be a real with $0 < r < R$. Prove that if $f^{(n)}(c) = 0$ for every $n \in \{0, \dots, m\}$, then

$$|f(z)| \le M \cdot \left(\frac{r}{R}\right)^m$$

for every $z \in D(c, r)$.

Proof. We may as well assume $c = 0$ since, if not, we can just translate results in $D(0,R)$ appropriately. Now, define $g(z) = \frac{f(z)}{z^m}$ for $z \neq 0$ and $g(0) = f^{(m)}(0)$. Then $g \in \mathcal{H}(D(0,R))$. If $a \in bD(0,R)$, then

$$\limsup_{z\to a} |g(z)| = \limsup_{z\to a} \frac{|f(z)|}{|z|^m} = \limsup_{z\to a} \frac{|f(z)|}{R^m} = \frac{M}{R^m}.$$

By the maximum principle, $|g(z)| < \frac{M}{R^m}$ unless g is constant. Since $f(0) = 0$ this implies that, for $z \in D(0,R)$,

$$|f(z)| = |z||g(z)| \le |z|^m M \frac{1}{R^m}.$$

Hence, if $z \in D(0,r)$,

$$|f(z)| \leq M \cdot \left(\frac{|z|}{R}\right)^m \leq M \cdot \left(\frac{r}{R}\right)^m .$$

$\square$

(301.2): For each connected open set $\Omega \subseteq \mathbb{C}$, for each compact subset $K \subset \Omega$, and for each real $\epsilon > 0$, prove that there exists a linear subspace $V \subset H^\infty(\Omega)$ with finite codimension such that for every $z \in K$ and every $f \in V$

$$|f(z)| \leq \epsilon \cdot \|f\|_{\Omega,\infty}.$$

Proof. Let $\delta > 0$ such that $D_i = \overline{D_\delta(c_i)}$ is a finite cover of K for $c_i \in K$, $i = 1, \ldots, n$. Then, since each D_i is compact and contained in Ω, there exists $\gamma > 0$ such that $D_{\delta+\gamma}(c_i) \subset \Omega$ for all $i = 1, \ldots, n$. Now, for any $f \in \mathcal{H}(\Omega)$, if $f(c_i) = f'(c_i) = \ldots = c^{(m)}(c_i) = 0$ for all $i = 1, \ldots, n$, then, by our work above,

$$|f(z)| \leq \|f\|_{\Omega,\infty} \left(\frac{\delta}{\delta+\gamma}\right)^m$$

for all $z \in D_i$ and for each i. Since the D_i cover K, we see that

$$|f(z)| \leq \|f\|_{\Omega,\infty} \left(\frac{\delta}{\delta+\gamma}\right)^m$$

for all $z \in K$. Now, since $\delta < \delta + \gamma$, there exists $M \in \mathbb{N}$, such that, if $m > M$,

$$\left(\frac{\delta}{\delta+\gamma}\right)^m < \epsilon.$$

Let V be the subspace of $H^\infty(\Omega)$ consisting of all $f \in H^\infty(\Omega)$ such that f vanishes to order greater than M at all the c_i. Then, for all $f \in V$ and all $z \in K$,

$$|f(z)| \leq \epsilon \cdot \|f\|_{\Omega,\infty}.$$

Now, if f is an arbitrary element of $H^\infty(\Omega)$, then f need not be in V. However, for each such f, f has a power series expansion centered at c_i for each i:

$$f(z) = \sum_{j=0}^{\infty} a_j (z - c_i)^j$$

for all $z \in D_i$. Then define

$$g_{f,i} := \sum_{j=0}^{M} a_j (z - c_i)^j.$$

Now, for $k \leq M$, $g_{f,i}^{(k)}(c_i) = f^{(k)}(c_i)$ for all $i = 1, \dots, n$, so, if we define $h_{f,i} := f - g_{f,i}$, then

$$h_{f,i}^{(k)}(c_i) = 0$$

for all $k \leq M$. Hence, $h_{f,i} \in V$ for all $f \in V$ and all $i = 1, \dots, n$. Now,

$$f(z) = \frac{1}{n}\sum_{i=1}^{n}[h_{f,i}(z) + g_{f,i}(z)] = \frac{1}{n}\left(\sum_{i=1}^{n} h_{f,i} + \sum_{i=1}^{n} g_{f,i}(z)\right),$$

so we see that f is a linear combination of elements of V and polynomials of degree $\leq M$. Since polynomials of degree $\leq M$ form a finite subspace of $H^\infty(\Omega)$, we see that V has finite codimension. □

303

Consider a convex open set $\Omega \subset \mathbb{C}$ and a biholomorphic map $f : D(0,1) \to \Omega$. For each real r with $0 < r < 1$ let $\Omega_r := f[D(0,r)]$. Prove that Ω_r is convex.

Proof. Assume $f(0) = 0$. Let $w, w' \in \Omega_r$. Then $w = f(z)$ and $w' = f(z')$ for $z, z' \in D(0,r)$. Now, suppose $|z'| \leq |z|$; then $z' = \frac{z'}{z}z$ (if $|z| \leq |z'|$, then we have $z = \frac{z}{z'}z'$, and a parallel argument follows). Hence we can parametrize the line segment from w' to w by

$$(1-t)f(z') + tf(z) = (1-t)f\left(\frac{z'}{z}z\right) + tf(z).$$

Therefore, if we can show that $f^{-1}\left((1-t)f\left(\frac{p}{q}z\right) + tf(z)\right) \subset D(0,r)$ for all $p, q \in D(0,r)$ such that $|p| \leq |q| < 1$, this will suffice to prove the desired result. Note that this mapping is well-defined, since Ω is convex.

□

6

Show that if $h \in \mathrm{Aut}(D_1)$ with $h(w) = w$ and $h'(w) \in (0, \infty)$, for some $w \in D_1$, then $h(z) \equiv z$. Prove this directly, without using the results of Exercise 299. Show that given two sets of distinct points

$$\{w_1, w_2, w_3\}, \{z_1, z_2, z_3\}$$

on bD_1 there is a unique $h \in \mathrm{Aut}(D_1)$ with

$$h(w_j) = z_j \text{ for } j = 1, 2, 3.$$

Explain why this is false for points in the interior of D_1. Can you describe in a simple manner those pairs of triples in D_1 for which such a map exists?

Proof. Suppose $h \in \mathrm{Aut}(D_1)$. Then

$$h(z) = e^{i\theta}\frac{z-a}{1-\bar{a}z}$$

for some $a \in D_1$. Now,

$$w = h(w) = e^{i\theta}\frac{w-a}{1-\bar{a}w}$$

and

$$h'(w) = e^{i\theta}\frac{(1-\bar{a}w)+\bar{a}(w-a)}{(1-\bar{a}w)^2} = e^{i\theta}\frac{1-|a|^2}{(1-\bar{a}w)^2} \in (0,\infty).$$

Let

$$F(w) = \frac{w-w_2}{w-w_3}\frac{w_1-w_3}{w_1-w_2}.$$

Then F is a fractional linear transformation, and $F(w_1) = 1$, $F(w_2) = 0$, $F(w_3) = \infty$. Now, define

$$G(z) = \frac{z-z_2}{z-z_3}\frac{z_1-z_3}{z_1-z_2}.$$

Then G is a fractional linear transformation mapping (z_1, z_2, z_3) to $(1, 0, \infty)$. Then G^{-1} is also a fractional linear transformation, as is $G^{-1} \circ F$, and

$$h = G^{-1} \circ F : (w_1, w_2, w_3) \mapsto (z_1, z_2, z_3).$$

Suppose g is another fractional linear transformation mapping (w_1, w_2, w_3) to (z_1, z_2, z_3). Then as we showed in Exercise 28 (HW 1), the cross ratios are equal:

$$(g(w), g(w_1), g(w_2), g(w_3)) = (w, w_1, w_2, w_3) = (h(w), h(w_1), h(w_2), h(w_3))$$

for all w. Since $g(w_i) = h(w_i)$ for $i = 1, 2, 3$, this implies that $g(w) = h(w)$, so h is the unique fractional linear transformation mapping (w_1, w_2, w_3) to (z_1, z_2, z_3).

The result fails to hold for points in the interior of D_1 because elements of $\mathrm{Aut}(D_1)$ map circles to circles (if we include lines as degenerate circles). To see why, recall from Exercise 14 (HW 1) that every fractional linear transformation (of which all elements of $\mathrm{Aut}(D_1)$ are) is a composition of at most two translations, a rotation, a dilation and an inversion. Since each of these types of maps take circles to circles, it must be the case that fractional linear transformations do as well. Since not all circles generated by triples in D_1 are contained in D_1, it's clear that, if triple generates a circle not contained in D_1, then no triple generating a circle that is contained in D_1 can be mapped to it. Hence, the pairs of triples in D_1 for which such a map exists will be those pairs such that the circles (again, including degenerate circles) generated by both pairs are either both contained in D_1 or both not contained in D_1. □

7

If $0 < \alpha, \beta < 1$, then the holomorphic function

$$f(z) = \int_0^z \zeta^{\alpha-1}(1-\zeta)^{\beta-1} d\zeta$$

maps the upper half plane onto a triangle with angles $\pi\alpha, \pi\beta, \pi(1-\alpha-\beta)$. What are the vertices of the triangle?

Answer: As we saw in class, the vertices of the triangle are $f(0), f(1)$ and $f(\infty)$. Furthermore, if we label the sides opposite the angles $\pi\alpha, \pi\beta, \pi(1-\alpha-\beta)$ by a, b, c, respectively, then the length of c (the side from $f(0)$ to $f(1)$) is given by

$$\frac{\sin \pi(1-\alpha-\beta)}{\pi}\Gamma(\alpha)\Gamma(\beta)\Gamma(1-\alpha-\beta).$$

Now, if $0 < z_0 < 1$,

$$f(z_0) = \int_0^{z_0} \zeta^{\alpha-1}(1-\zeta)^{\beta-1} d\zeta$$

has derivative at z_0 given by

$$f'(z_0) = z_0^{\alpha-1}(1-z_0)^{\beta-1},$$

which is a positive real. Since this is true for all z_0 between 0 and 1, we see that $f(z) > 0$ for $0 < z < 1$, so the side $c \subset [0, \infty)$. Hence, since we know the length of c and since

$$f(0) = \int_0^0 \zeta^{\alpha-1}(1-\zeta)^{\beta-1} d\zeta = 0,$$

we see that $f(1) = \frac{\sin \pi(1-\alpha-\beta)}{\pi}\Gamma(\alpha)\Gamma(\beta)\Gamma(1-\alpha-\beta)$. Now, we also know that the length of b is given by

$$\frac{\sin \pi\beta}{\pi}\Gamma(\alpha)\Gamma(\beta)\Gamma(1-\alpha-\beta);$$

since this is the side connecting $f(0)$ and $f(\infty)$ and the angle at $f(0) = 0$ is $\pi\alpha$, we need merely find the line segment of length b in the direction of $e^{i\pi\alpha}$ to determine $f(\infty)$. Specifically,

$$f(\infty) = e^{i\pi\alpha}\frac{\sin \pi\beta}{\pi}\Gamma(\alpha)\Gamma(\beta)\Gamma(1-\alpha-\beta).$$

Since two adjacent sides and their adjoining angle completely determine a triangle, we see that the vertices $f(0)$, $f(1)$ and $f(\infty)$ determined above are indeed the vertices of the triangle given by f.

♣

DRL 3E3A, University of Pennsylvania
E-mail address: `shonkwil@math.upenn.edu`