COMPLEX ANALYSIS HW 13

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For each $\tau \in \mathbb{C}$ with $\Im(\tau) > 0$, define the lattice $\Lambda_\tau := \mathbb{Z} \times \tau\mathbb{Z} \subset \mathbb{C}$, and define the complex torus $X_\tau := \mathbb{C}/\Lambda_\tau$. For two such complex numbers $\tau_1, \tau_2 \in \mathbb{C}$ with $\Im(\tau_1) > 0$ and $\Im(\tau_2) > 0$, assume that there exist a holomorphic isomorphism $f : X_{\tau_1} \to X_{\tau_2}$ and an entire function $g : \mathbb{C} \to \mathbb{C}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{g} & \mathbb{C} \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ X_{\tau_1} & \xrightarrow[f]{} & X_{\tau_2} \end{array}$$

where each $p_k : \mathbb{C} \to X_{\tau_k} = \mathbb{C}/\Lambda_{\tau_k}$ is the canonical projection.

(322.1): Prove that there exist constants $c_0, c_1 \in \mathbb{C}$ such that $g(z) = c_0 + c_1 z$ for every $z \in \mathbb{C}$. Prove also that there exist integers $k, \ell, m, n \in \mathbb{Z}$ with $kn - \ell m = 1$ and such that $\tau_1 = (k\tau_2 + \ell)/(m\tau_2 + n)$.

Proof. We will make use of the following lemma

Lemma 0.1. *If $f_{i,j} : X_{\tau_i} \to X_{\tau_j}$ is holomorphic, then there exists entire $g_{i,j} : \mathbb{C} \to \mathbb{C}$ such that the following diagram commutes:*

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{g_{i,j}} & \mathbb{C} \\ {\scriptstyle p_i}\downarrow & & \downarrow{\scriptstyle p_j} \\ X_{\tau_i} & \xrightarrow[f_{i,j}]{} & X_{\tau_j} \end{array}$$

Proof. Since $p_j : \mathbb{C} \to X_{\tau_j}$ defines a covering and $f_{i,j} \circ p_i : \mathbb{C} \to X_{\tau_j}$ is continuous, there exists a lift $g_{i,j} = \widetilde{f_{i,j} \circ p_i} : \mathbb{C} \to \mathbb{C}$ such that $p_j \circ g_{i,j} = f_{i,j} \circ p_i$. Now, let $z \in \mathbb{C}$. Let U_i be a neighborhood of z such that $p_i : U_i \to p_i(U)$ is a homeomorphism (such a neighborhood exists since $p_i : \mathbb{C} \to X_{\tau_i}$ is a covering). Similarly, there exists $U_j \ni z$ such that $p_j : U_j \to p_j(U_j)$ is a homeomorphism. Furthermore, by the definition of the Riemann surface structure on a complex torus,

$(U_i, p_i\upharpoonright_{U_i}^{-1})$ and $(U_j, p_j\upharpoonright_{U_j}^{-1})$ are coordinate charts containing $p_i(z)$ and $p_j(z)$, respectively. Let $U = U_i \cap U_j$. Thus, since $f_{i,j}$ is holomorphic,

$$p_j\upharpoonright_U^{-1} \circ f_{i,j} \circ p_i\upharpoonright_U = p_j\upharpoonright_U^{-1} \circ f_{i,j} \circ \left(p_i\upharpoonright_U^{-1}\right)^{-1}$$

is holomorphic on U. On the other hand, since $p_j \circ g_{i,j} = f_{i,j} \circ p_i$, we see that the above map is just $g_{i,j}\upharpoonright_U$, so $g_{i,j}$ is holomorphic on U and, hence, at z. Since our choice of z was arbitrary, we see that $g_{i,j} \in \mathcal{H}(\mathbb{C})$. □

Now, with Lemma 0.1 in hand, we see that, since $f^{-1} : X_{\tau_2} \to X_{\tau_1}$ is also holomorphic, there exists $h : \mathbb{C} \to \mathbb{C}$ such that the following diagram commutes:

$$\begin{array}{ccccc} \mathbb{C} & \xrightarrow{g} & \mathbb{C} & \xrightarrow{h} & \mathbb{C} \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle p_1} \\ X_{\tau_1} & \xrightarrow[f]{} & X_{\tau_2} & \xrightarrow[f^{-1}]{} & X_{\tau_1}. \end{array}$$

Since g and h are both entire, $h \circ g$ is also entire. Furthermore, since the above diagram commutes, for any $z \in \mathbb{C}$,

$$p_1(z) = f^{-1} \circ f \circ p_1(z) = p_1 \circ h \circ g(z),$$

so $h \circ g(z) = z + a_z\tau_1 + b_z$ for some $a_z, b_z \in \mathbb{Z}$. Since $h \circ g$ is holomorphic, so is the function given by $z \mapsto h \circ g(z) - z = a_z\tau_1 + b_z$; since $a_z, b_z \in \mathbb{Z}$, we see that they don't depend on z, so they are constant and we can just label them by a and b. Now, define $h_1(z) = h(z) - a\tau_1 - b$. Then h_1 is also entire and

$$h_1 \circ g(z) = h_1(g(z)) = h(g(z)) - a\tau_1 - b = z + a\tau_1 + b - a\tau_1 - b = z.$$

Since this is true for all $z \in \mathbb{C}$, we see that $h_1 = g^{-1}$. Therefore, g must be bijective. Since the only bijective entire functions are linear, we see that $g(z) = c_0 + c_1 z$ for some constants c_0 and c_1. Note that a similar argument holds for h_1, so $h_1(z) = d_0 + d_1 z$ for some constants d_0, d_1.

Therefore, for some $k, \ell, m, n \in \mathbb{Z}$, since the above diagram commutes,

$$c_0 + k\tau_2 + \ell = g(0) + k\tau_2 + \ell = g(0 + \tau_1) = g(\tau_1) = c_0 + c_1\tau_1$$

and

$$c_0 + m\tau_2 + n = g(0) + m\tau_2 + n = g(0 + 1) = g(1) = c_0 + c_1.$$

Cancelling the c_0's, we see that $c_1\tau_1 = k\tau_2 + \ell$ and $c_1 = m\tau_2 + n$; dividing these two equations, we see that

$$\tau_1 = \frac{c_1\tau_1}{c_1} = \frac{k\tau_2 + \ell}{m\tau_2 + n}.$$

Similar arguments show that $d_1\tau_2 = k'\tau_1 + \ell'$ and $d_1 = m'\tau_1 + n'$ for $k', \ell', m', n' \in \mathbb{Z}$, so

$$\tau_2 = \frac{k'\tau_1 + \ell'}{m'\tau_1 + n'}.$$

Let $u_1(z) = \frac{kz+\ell}{mz+n}$ and $u_2(z) = \frac{k'z+\ell'}{m'z+n'}$. Then, since g and h_1 are inverses, $u_1 \circ u_2$ must define the identity transformation. In matrix form, this means

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} k & \ell \\ m & n \end{pmatrix} \begin{pmatrix} k' & \ell' \\ m' & n' \end{pmatrix}.$$

Since determinants are multiplicative and both matrices on the right must have integer determinants, we see that they must both have determinant ± 1. Since both τ_1 and τ_2 have positive imaginary part, these determinants are actually 1, so we see that $kn - \ell m = 1$. □

(322.2): Deduce that two complex tori X_{τ_1} and X_{τ_2} are holomorphically isomorphic if, and only if, there exist integers $k, \ell, m, n \in \mathbb{Z}$ such that $kn - \ell m = 1$ and such that $\tau_1 = (k\tau_2 + \ell)/(m\tau_2 + n)$.

Proof. By Lemma 0.1, if $f : X_{\tau_1} \to X_{\tau_2}$ is a holomorphic isomorphism, then there exists $g : \mathbb{C} \to \mathbb{C}$ such that the diagram given in the statement of the problem commutes. Therefore, by the result proved in (322.1) above, we see that there exist integers $k, \ell, m, n \in \mathbb{Z}$ such that $kn - \ell m = 1$ and such that $\tau_1 = (k\tau_2 + \ell)/(m\tau_2 + n)$.

On the other hand, suppose there exist such k, ℓ, m, n. Then $z \mapsto \frac{kz+\ell}{mz+n}$ defines a fractional linear transformation with inverse $z \mapsto \frac{k'z+\ell'}{m'z+n'}$ for some $k', \ell', m', n' \in \mathbb{Z}$; furthermore, since these maps are inverses, we know, by the properties of fractional linear transformations proved on HW 1, that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} k' & \ell' \\ m' & n' \end{pmatrix} \begin{pmatrix} k & \ell \\ m & n \end{pmatrix}.$$

Since determinant is multiplicative, we see that $k'n' - \ell'm' = 1$ as well. Now, since $\{\tau_1, 1\}$ are linearly independent, this is a basis for $\mathbb{C}$, so we can define the linear map $g : \mathbb{C} \to \mathbb{C}$ simply by requiring that

$$\tau_1 \mapsto k\tau_2 + \ell$$
$$1 \mapsto m\tau_2 + n$$

and extending linearly. Similarly, we can define $h : \mathbb{C} \to \mathbb{C}$ by

$$\tau_2 \mapsto k'\tau_1 + \ell'$$
$$1 \mapsto m'\tau_1 + n'$$

Then both g and h are entire; define $f = p_2 \circ g \circ p_1^{-1}$. Note that this is well-defined, since, for all $z \in \mathbb{C}$ and all $a, b \in \mathbb{Z}$,

$$g(z + a\tau_1 + b) = g(z) + g(a\tau_1) + g(b)$$

and $g(a\tau_1), g(b) \in \Lambda_{\tau_2}$, so $p_2 \circ g(z + a\tau_1 + b) = p_2 \circ g(z)$. Furthermore, if $z \in X_{\tau_1}$, let (U_1, ϕ_1) be a coordinate chart containing z and (U_2, ϕ_2) a coordinate chart containing $f(z)$. Then, restricted to $U_1 \cap f^{-1}(U_2)$,

$$\phi_2 \circ f \circ \phi_1^{-1} = p_2^{-1} \circ f \circ p_1 = p_2^{-1} \circ p_2 \circ g \circ p_1^{-1} \circ p_1 = g,$$

which is entire. Thus, f is holomorphic at z and therefore, since our choice of z was arbitrary, on all of X_{τ_1}. A parallel argument shows that $\widetilde{f} = p_1 \circ h \circ p_2^{-1} : X_{\tau_2} \to X_{\tau_1}$ is holomorphic. Now,

$$f \circ \widetilde{f} = (p_2 \circ g \circ p_1^{-1}) \circ (p_1 \circ h \circ p_2^{-1}) = p_2 \circ (g \circ h) \circ p_2^{-1} = p_2 \circ p_2^{-1} = id,$$

so we see that $\widetilde{f} = f^{-1}$, so f is an anlytic isomorphism. □

(322.3): Generalize the foregoing results to complex tori $X_{\omega_1,\omega_2} = \mathbb{C}/\Lambda_{\omega_1,\omega_2}$ with lattices $\Lambda_{\omega_1,\omega_2} = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ such that $\omega_1, \omega_2 \in \mathbb{C}$ are linearly independent over $\mathbb{R}$: under what conditions does f lift to g?

Proof. Note that the requirement that ω_1, ω_2 be linearly independent over $\mathbb{R}$ just means that they do not lie on the same line through the origin. Hence, $\frac{\omega_1}{\omega_2} \notin \mathbb{R}$. Hence, if we have the lattice $\Lambda_{\omega_1,\omega_2}$, with associated torus X_{ω_1,ω_2}, we would like to show that there exists a holomorphic isomorphism $f : X_{\omega_1,\omega_2} \to X_{\frac{\omega_1}{\omega_2},1}$. If we can show this for all such tori, then, by (322.2) above, we will have the result that X_{ω_1,ω_2} and $X_{\omega_1',\omega_2'}$ are holomorphically isomorphic if and only if there exist $k, \ell, m, n \in \mathbb{Z}$ with $kn - \ell m = 1$ and

$$\tau = \frac{k\tau' + \ell}{m\tau' + n}$$

where $\tau = \frac{\omega_1}{\omega_2}$, $\tau' = \frac{\omega_1'}{\omega_2'}$. □

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Consider the submanifold $S \subset \mathbb{C}^2$ defined by

$$S := \{(z, w) \in \mathbb{C}^2 : z^2 + w^2 = 1\}.$$

(325.1): Prove that S has a Riemann surface structure uniquely determined by the condition that the restrictions to S of the coordinate functions z and w are holomorphic.

Proof. Let $P(z, w) = z^2 + w^2 - 1$. Then

$$S = \{(z, w) \in \mathbb{C}^2 : P(z, w) = 0\}.$$

Furthermore, $\frac{\partial P}{\partial z}(z, w) = 2z$ and $\frac{\partial P}{\partial w}(z, w) = 2w$, so P_z and P_w are only simultaneously zero when $z = w = 0$. However, $P(0, 0) = -1 \neq 0$, so $(0, 0) \notin S$. Therefore, the problem of showing that S has a

Riemann surface structure in which the coordinate functions z and w are holomorphic is a special case of problem 326 below.

Now, to prove the uniqueness of this Riemann surface structure on S, suppose we consider two different Riemann surfaces, X and Y, both of whose points are given by the points of S, with coordinate charts $\{(U_i, \phi_i)\}$ and $\{(V_j, \psi_j)\}$, respectively. Let f and g denote the restrictions to S of the coordinate maps $(z, w) \mapsto z$ and $(z, w) \mapsto w$, respectively. Let $(z_0, w_0) \in S$ and let $\delta = \max\{|z_0|, |w_0|\}$. Let $B_\delta = \{(z, w) \in S : |z - z_0| < \delta, |w - w_0| < \delta\}$. Then B_δ is open in S and either $f\restriction_{B_\delta}$ or $g\restriction_{B_\delta}$ is injective. To see why this is so, suppose $f\restriction_{B_\delta}$ is not injective. Then there exist $(z, w_1), (z, w_2) \in B_\delta$ for some z, w_1, w_2 with $w_1 \neq w_2$. This implies that

$$z^2 + w_1^2 = 1 = z^2 + w_2^2,$$

so $w_1^2 = w_2^2$, meaning that $w_1 = -w_2$. Hence, $\delta = |z_0|$. However, this implies that, for any $(z', w'), (\widetilde{z}, \widetilde{w}) \in B_\delta$ with $(z', w') \neq (\widetilde{z}, \widetilde{w})$, $z', \widetilde{z} \in D_\delta(z_0)$, meaning $z' \neq -\widetilde{z}$. Therefore, since

$$(z')^2 + (w')^2 = 1 = (\widetilde{z})^2 + (\widetilde{w})^2,$$

$w' \neq \widetilde{w}$. Hence, g is injective on B_δ.

Thus, we see that either $f\restriction_{B_\delta}$ or $g\restriction_{B_\delta}$ is injective; suppose, without loss of generality, that $f\restriction_{B_\delta}$ is injective (this is legitimate because in all of the following we can simply substitute g for f if f is not injective on B_δ). Then, if (U_i, ϕ_i) is a coordinate chart in X containing (z_0, w_0),

$$f \circ \phi_i^{-1} : \phi_i(U_i \cap B_\delta) \to f(U_i \cap B_\delta)$$

is bijective. Since this map is also holomorphic, we see that it is an analytic isomorphism, which is to say

$$(f \circ \phi_i^{-1})^{-1} = \phi_i \circ f^{-1} : f(U_i \cap B_\delta) \to \phi_i(U_i \cap B_\delta)$$

is also holomorphic. Therefore, if (V_j, ψ_j) is a coordinate chart in Y containing (z_0, w_0), then, on $U_i \cap V_j \cap B_\delta$

$$\phi_i \circ \psi_j^{-1} = \phi_i \circ f\restriction_{B_\delta}^{-1} \circ f \circ \psi_j^{-1}.$$

Since $f \in \mathcal{H}(Y)$, $f \circ \psi_j^{-1}$ is holomorphic, so we see that $\phi_i \circ \psi_j^{-1}\restriction_{U_i \cap V_j \cap B_\delta}$ is the composition of holomorphic maps, and so is holomorphic (in a neighborhood of (z_0, w_0)). A completely parallel argument shows that $\psi_j \circ \phi_i^{-1}\restriction_{U_i \cap V_j \cap B_\delta}$ is also holomorphic in a neighborhood of (z_0, w_0). Since our choice of (z_0, w_0) was arbitrary, we see that the coordinate charts of X and Y are compatible. This in turn means that any function holomorphic on X is holomorphic on Y and *vice versa*, so we see that X and Y have the same Riemann surface structure. Thus, we conclude that the Riemann surface structure on S is uniquely determined by the requirement that the coordinate functions z and w are holomorphic. □

(325.2): Prove that the restriction to S of the function $(z, w) \mapsto z+iw$ gives a conformal equivalence of S with an open subset of the plane.

Proof. If $(z_0, w_0) \in S$ and (U_i, ϕ_i) is a coordinate chart containing (z_0, w_0), then, if f and g are as in (325.1) above,

$$f \circ \phi_i^{-1} : \phi_i(U_i) \to \mathbb{C}$$

and

$$g \circ \phi_i^{-1} : \phi_i(U_i) \to \mathbb{C}$$

are holomorphic maps. Now, if h denotes the map $(z, w) \mapsto z + iw$, then

$$h \circ \phi_i^{-1} = (f \circ \phi_i^{-1}) + i(g \circ \phi_i^{-1}) : \phi_i(U_i) \to \mathbb{C}$$

and, since multiplying a holomorphic map by i yields another holomorphic map and the sum of holomorphic maps is holomorphic, we see that $h \circ \phi_i^{-1} : \phi_i(U_i) \to \mathbb{C}$ is holomorphic. Since our choice of (z_0, w_0) was arbitrary, we see that h is holomorphic at all $(z, w) \in S$, so $h : S \to \mathbb{C}$ is holomorphic. Since $h(1) = 1 \neq -1 = h(-1)$ and $1, -1 \in S$, we see that h is non-constant so, by the open mapping principle, h is an open map. Therefore, since S is open in S, $h(S)$ is open in $\mathbb{C}$ (which is trivially a Riemann surface). Also, $h : S \to h(S)$ is certainly surjective.

Therefore, it remains only to show that h is injective. To demonstrate this, suppose $(z, w), (\widetilde{z}, \widetilde{w}) \in S$ such that $h(z, w) = h(\widetilde{z}, \widetilde{w})$. Then $z + iw = \widetilde{z} + i\widetilde{w}$ and so

$$(1)\quad 1 = z^2+w^2 = (z+iw)(z-iw) = (\widetilde{z}+i\widetilde{w})(z-iw) = z\widetilde{z}+w\widetilde{w}+i(z\widetilde{w}-\widetilde{z}w),$$

so $z\widetilde{w} - \widetilde{z}w = 0$. Hence, $z\widetilde{w} = \widetilde{z}w$. If $\widetilde{w} = 0$, then $\widetilde{z} = \pm 1$, so $z + iw = \pm 1$. Hence, if $z = z_1 + iz_2$ and $w = w_1 + iw_2$, then

$$\pm 1 = (z_1 + iz_2) + i(w_1 + iw_2) = (z_1 - w_2) + i(w_1 + z_2).$$

Hence $z_2 = -w_1$ and $z_1 = \pm 1 + w_2$. In turn, this implies that

$$\begin{aligned} 1 = z^2 + w^2 &= (z_1 + iz_2)^2 + (w_1 + iw_2)^2 \\ &= z_1^2 - z_2^2 + 2iz_1z_2 + w_1^2 - w_2^2 + 2iw_1w_2 \\ &= (\pm 1 + w_2)^2 - w_1^2 + 2i(\pm 1 + w_2)(w_1) + w_1^2 - w_2^2 + 2iw_1w_2 \\ &= 1 \pm 2w_2 \pm 2iw_1. \end{aligned}$$

Hence $w_1 = w_2 = 0$, so $z = \widetilde{z}$ and $w = \widetilde{w}$. On the other hand, if $\widetilde{w} \neq 0$, then $z = \frac{\widetilde{z}w}{\widetilde{w}}$. Also in the style of (**??**, we have

$$(2)\quad 1 = \widetilde{z}^2+\widetilde{w}^2 = (\widetilde{z}+i\widetilde{w})(\widetilde{z}-i\widetilde{w}) = (z+iw)(\widetilde{z}-i\widetilde{w}) = z\widetilde{z}+w\widetilde{w}+i(\widetilde{z}w-z\widetilde{w}).$$

Now, adding (1) and (2), we see that

$$2 = 2z\widetilde{z} + 2w\widetilde{w}.$$

Now, dividing by 2 and substituting $z = \frac{\widetilde{z}w}{\widetilde{w}}$, we see that

$$1 = \frac{\widetilde{z}w}{\widetilde{w}}\widetilde{z} + w\widetilde{w}.$$

Multiplying both sides by $\frac{\widetilde{w}}{w}$ yields,

$$\frac{\widetilde{w}}{w} = \widetilde{z}^2 + \widetilde{w}^2 = 1,$$

so $w = \widetilde{w}$. Therefore,

$$z = \frac{\widetilde{z}w}{\widetilde{w}} = \widetilde{z}\frac{w}{\widetilde{w}} = \widetilde{z}.$$

Therefore, we see that $h(z, w) = h(\widetilde{z}, \widetilde{w})$ implies that $(z, w) = (\widetilde{z}, \widetilde{w})$, so we see that h is injective. □

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Consider a polynomial $P \in \mathbb{C}[z, w]$ for which the set

$$Y := \big\{(z, w) \in \mathbb{C}^2 : P(z, w) = 0, D_1P(z, w) = 0, D_2P(z, w) = 0\big\}$$

is empty. Also, assume that

$$X := \big\{(z, w) \in \mathbb{C}^2 : P(z, w) = 0\big\}$$

is connected. Prove that X has the structure of a Riemann surface such that the functions $(z, w) \mapsto z$ and $(z, w) \mapsto w$ are holomorphic functions on X.

Proof. Let $U = \{(z, w) \in X : D_1P(z, w) \neq 0\}$ and let $V = \{(z, w) \in X : D_2P(z, w) \neq 0\}$. Since $D_iP : \mathbb{C}^2 \to \mathbb{C}$ is continuous for $i = 1, 2$, $(D_iP)^{-1}(0)$ is closed, which in turn means that $(D_iP)^{-1}(0) \cap X$ is closed in X. Hence, since U and V are the complements of $(D_1P)^{-1}(0) \cap X$ and $(D_2P)^{-1}(0) \cap X$ in X, we see that U and V are open.

Now, for each point $(z_0, w_0) \in U$, since $D_1P(z_0, w_0) \neq 0$, the implicit function theorem implies that there exists a neighborhood $\widetilde{U}_{(z_0,w_0)}$ of w_0 and a holomorphic map $f : \widetilde{U}_{(z_0,w_0)} \to U$ such that

$$P(f(w), w) = 0$$

for all $w \in \widetilde{U}_{(z_0,w_0)}$; that is, $(f(w), w) \in X$ for all $w \in \widetilde{U}_{(z_0,w_0)}$. Let $U_{(z_0,w_0)} = \{(f(w), w) \in X : w \in \widetilde{U}_{(z_0,w_0)}\}$. Then $\phi_{(z_0,w_0)} : (f(w), w) \mapsto w \in \widetilde{U}_{(z_0,w_0)}$ is certainly continuous and the inverse of f, so we see that f is a homeomorphism, meaning $U_{(z_0,w_0)}$ is open in X. In general, denote f by $\widetilde{\phi}_{(z_0,w_0)}$.

Similarly, for each point $(z_1, w_1) \in V$, since $D_2P(z_1, w_1) \neq 0$, the implicit function theorem implies that there exists a neighborhood $\widetilde{U}_{(z_1,w_1)}$ of w_1 and a holomorphic map $g : \widetilde{U}_{(z_0,w_0)} \to U$ such that

$$P(g(w), w) = 0$$

for all $w \in \widetilde{U}_{(z_1,w_1)}$; that is, $(f(w), w) \in X$ for all $w \in \widetilde{U}_{(z_1,w_1)}$. Let $U_{(z_1,w_1)} = \{(f(w), w) \in X : w \in \widetilde{U}_{(z_1,w_1)}\}$. Then $\phi_{(z_1,w_1)} : (f(w), w) \mapsto w \in \widetilde{U}_{(z_1,w_1)}$ is certainly continuous and the inverse of g, so we see that g is a homeomorphism, meaning $U_{(z_1,w_1)}$ is open in X. In general, denote g by $\widetilde{\phi}_{(z_1,w_1)}$.

Now, let us define what we hope will be a holomorphic structure on X by $\{(U_{(z,w)}, \phi_{(z,w)}) : (z, w) \in U\} \cup \{(V_{(z,w)}, \psi_{(z,w)}) : (z, w) \in V\}$. To see that this really endows X with a Riemann surface structure, we need to check the intersections. Now, if $U_{(z_0,w_0)} \cap U_{(z_1,w_1)} \neq \emptyset$, then, on the intersection, $\phi_{(z_0,w_0)} \circ \phi^{-1}_{(z_1,w_1)}$ is simply the identity, and *vice versa*. A similar argument holds where two of the V's intersect, so we see that the only potential problem is where a U and a V intersect.

To address this case, suppose $U_{(z_0,w_0)} \cap V_{(z_1,w_1)} \neq 0$ for some $(z_0, w_0), (z_1, w_1) \in X$; to simplify notation, define $(U_0, \phi_0) := (U_{(z_0,w_0)}, \phi_{(z_0,w_0)})$ and $(V_1, \psi_1) := (V_{(z_1,w_1)}, \psi_{(z_1,w_1)})$. Then, for $z \in \psi_1(U_0 \cap V_1)$,

$$\phi_0 \circ \psi_1^{-1}(z) = \phi_0(z, \widetilde{\psi}_1(z)) = \widetilde{\psi}_1(z);$$

that is, $\phi_0 \circ \psi_1^{-1} \restriction_{\psi_1(U_0 \cap V_1)} \equiv \widetilde{\psi}_1$, which is holomorphic on the intersection. Similarly, for $w \in \phi_0(U_0)$,

$$\psi_1 \circ \phi_0^{-1}(w) = \psi_1(\widetilde{\phi}_0(w), w) = \widetilde{\phi}_0(w),$$

so $\psi_1 \circ \phi_0^{-1} \restriction_{\phi_0(U_0 \cap V_1)} \equiv \widetilde{\phi}_0$, which is holomorphic.

Therefore, we see that the X is indeed a Riemann surface with coordinate charts as defined above. Finally, let f denote the map $(z, w) \mapsto z$ and let g denote the map $(z, w) \mapsto w$. Suppose $(z, w) \in X$ has coordinate chart (U_1, ϕ_1). Then, for $w \in \phi_1(U_1) \subset \mathbb{C}$,

$$f \circ \phi_1^{-1}(w) = f(\widetilde{\phi}_1(w), w) = \widetilde{\phi}_1(w),$$

so $f \circ \phi_1^{-1} \restriction_{\phi_1(U_1)} \equiv \widetilde{\phi}_1$, which is holomorphic. On the other hand, again for $w \in \phi_1(U_1)$,

$$g \circ \phi_1^{-1}(w) = g(\widetilde{\phi}_1(w), w) = w,$$

so $g \circ \phi_1^{-1} \restriction_{\phi_1(U_1)} \equiv id$, which is certainly holomorphic. Hence, we see that f and g are both holomorphic on U_1.

On the other hand, if $(z, w) \in X$ has coordinate chart (V_1, ψ_1), then, for $z \in \psi_1(V_1) \subset \mathbb{C}$,

$$f \circ \psi_1^{-1}(z) = f(z, \widetilde{\psi}_1(z)) = z$$

and

$$g \circ \psi_1^{-1}(z) = g(z, \widetilde{\psi}_1(z)) = \widetilde{\psi}_1(z),$$

so we see that f and g are holomorphic on V_1. Thus, having dispatched all possible points in X, we see that f and g are holomorphic at all points on X, so $(z, w) \mapsto z$ and $(z, w) \mapsto w$ are holomorphic functions on X. $\square$

5

Let ψ be a homeomorphism from the unit disk D_1 to an open subset Ω of $\mathbb{C}$. This homeomorphism defines a Riemann surface structure on the unit disk, as described in class. We denote the unit disk with this Riemann surface strucutre by X_ψ. Let X_{Id} denote the unit disk with its usual Riemann surface structure; find an analytic isomorphism $\phi : X_\psi \to X_{Id}$.

Answer: Note that $\Omega \neq \mathbb{C}$ (else $\phi \circ \psi^{-1}$ would define an entire, bounded, non-constant, holomorphic function). Since Ω is homeomorphic to D_1, Ω is simply connected. Thus, by the Riemann Mapping Theorem, there exists an analytic isomorphism $f : \Omega \to D_1$. Define $g = f \circ \psi : D_1 \to D_1$. Note that the coordinate chart on X_ψ is given by (D_1, ψ) and the coordinate chart on X_{Id} is given by (D_1, Id). Then, in coordinates,

$$Id \circ g \circ \psi^{-1} = Id \circ (f \circ \psi) \circ \psi^{-1} = Id \circ f \circ (\psi \circ \psi^{-1}) = Id \circ f \circ Id = f.$$

Therefore, $Id \circ g \circ \psi^{-1} = f : \Omega \to D_1$ is holomorphic, meaning that $g \in \mathcal{H}(X_\psi, X_{Id})$.

On the other hand, if we define $h = \psi^{-1} \circ f^{-1}$, then

$$\psi \circ h \circ Id^{-1} = \psi \circ h = \psi \circ (\psi^{-1} \circ f^{-1}) = (\psi \circ \psi^{-1}) \circ f^{-1} = f^{-1};$$

since $f^{-1} : D_1 \to \Omega$ is also holomorphic, $\psi \circ h \circ Id^{-1} = f^{-1} : D_1 \to \Omega$ is holomorphic, so we see that $h : X_{Id} \to X_\psi$ is a holomorphic map.

Finally,

$$g \circ h = (f \circ \psi) \circ (\psi^{-1} \circ f^{-1}) = f \circ (\psi \circ \psi^{-1}) \circ f^{-1} = f \circ f^{-1} = Id$$

and

$$h \circ g = (\psi^{-1} \circ f^{-1}) \circ (f \circ \psi) = \psi^{-1} \circ (f^{-1} \circ f) \circ \psi = \psi^{-1} \circ \psi = Id,$$

so we see that $h = g^{-1}$. Therefore, since g is holomorphic with holomorphic inverse, we see that g is an analytic isomorphism from X_ψ to X_{Id}.

♣

6

Let X and Y be Riemann surfaces and let $\psi : X \to Y$ be a continuous map. For an open subset U of X (V of Y) we let $\mathcal{H}_X(U)$ ($\mathcal{H}_Y(V)$) denote the holomorphic functions defined on U (V). For each $U \subset X$, the pullback operation

$$\psi^* f(p) = f(\psi(p))$$

defines a map $\psi^* : C^0_Y(\psi(U)) \to C^0_X(U)$. Show that ψ is holomorphic if and only if, for every open $U \subset X$, and every open subset $V \supset \psi(U)$, we have

$$\psi^* \mathcal{H}_Y(V)\restriction_U \subset \mathcal{H}_X(U).$$

Proof. ($\Rightarrow$) Suppose ψ is holomorphic. Let $U \subset X$ be open and let $V \supset \psi(U)$ be open. Let $f \in \mathcal{H}_Y(V)$, let $p \in U$, let (U_i, ρ_i) be a coordinate chart containing p and let (V_j, ϕ_j) be a coordinate chart containing $\psi(p)$. Then, since our choice of $p \in U$ was arbitrary, it suffices to show that $\psi^* f$ is holomorphic at p. In turn, this requires that we show that

$$\psi^* f \circ \rho_i^{-1} : \rho_i(U_i) \to \mathbb{C}$$

is holomorphic at $\rho_i(p)$. Note that, since $f \in \mathcal{H}_Y(V)$,

$$f \circ \phi_j^{-1} : \phi_j(V_j) \to \mathbb{C}$$

is holomorphic at $\phi_j(\psi(p))$. Also, since ψ is holomorphic,

$$\phi_j \circ \psi \circ \rho_i^{-1} : \rho_i(U_i) \to \phi_j(V_j)$$

is holomorphic on all of $\rho_i(U_i)$, including at $\rho_i(p)$. Now,

$$\psi^* f \circ \rho_i^{-1} = f \circ \psi \circ \rho_i^{-1} = f \circ \phi_j^{-1} \circ \phi_j \circ \psi \circ \rho_i^{-1}.$$

Since $f \circ \phi_j^{-1}$ is holomorphic at $\phi_j(\psi(p))$ and $\phi_j \circ \psi \circ \rho_i^{-1}$ is holomorphic at $\rho_i(p)$ and

$$\phi_j \circ \psi \circ \rho_i^{-1}(\rho_i(p)) = \phi_j(\psi(p)),$$

we see that $\psi^* f \circ \rho_i^{-1}$ is the composition of holomorphic functions at $\rho_i(p)$, and so $\psi^* f \circ \rho_i^{-1}$ is holomorphic at $\rho_i(p)$. Therefore, since our choice of p was arbitrary, we see that $\psi^* f$ is holomorphic on all of U. Since our choice of f was arbitrary, this in turn implies that

$$\psi^* \mathcal{H}_Y(V)\restriction_U \subset \mathcal{H}_X(U).$$

Finally, since our choices of U and V were arbitrary, we see that this holds for all such choices of U and V.

($\Leftarrow$) On the other hand, suppose that

$$\psi^* \mathcal{H}_Y(V)\restriction_U \subset \mathcal{H}_X(U).$$

for all open $U \subset X$ and all open $V \supset \psi(U)$. Let $p \in X$, let (U_i, ρ_i) be a coordinate neighborhood containing p and let (ϕ_j, V_j) be a coordinate neighborhood containing $\psi(p)$. Let $U = U_i \cap \psi^{-1}(V_j)$. Then certainly $\phi_j \in \mathcal{H}_Y(V_j)$ and $V_j \supset \psi(U)$ so, by hypothesis, $\psi^* \phi_j\restriction_U \in \mathcal{H}_X(U)$, which is to say

$$\psi^* \phi_j \circ \rho_i^{-1} : \rho_i(U) \to \mathbb{C}$$

is holomorphic. In particular, $\psi^* \phi_j \circ \rho_i^{-1}$ is holomorphic at $\rho_i(p)$. Hence,

$$\psi^* \phi_j \circ \rho_i^{-1} = \phi_j \circ \psi \circ \rho_i^{-1}$$

is holomorphic at $\rho_i(p)$, which is precisely the condition necessary to show that ψ is holomorphic at p. Therefore, since our choice of p was arbitrary, we see that ψ is holomorphic at every point in X, so $\psi \in \mathcal{H}(X, Y)$. □

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