COMPLEX ANALYSIS HW 2

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Prove that if $f : \mathcal{D} \to \mathbb{C}$ has a complex derivative that is continuous on the open set $\mathcal{D} \subseteq \mathbb{C}$, and if $\gamma : [0,1] \to \mathcal{D}$ is a differentiable curve from $z_0 := \gamma(0)$ to $z_1 := \gamma(1)$, then

$$f(z_1) - f(z_0) = \int_\gamma f'(\zeta)d\zeta$$

and that

$$|f(z_1) - f(z_0)| \leq \text{Length}(\gamma) \cdot \max\{|f'(z)| : z \in \gamma([0,1])\}$$

Proof. Since f' is continuous and using the result proved in problem 41 of last week's homework,

$$\begin{aligned}
\int_\gamma f'(\zeta)d\zeta &= \int_0^1 f'(\gamma(t))\gamma'(t)dt \\
&= \int_0^1 (f \circ \gamma)'(t)dt \\
&= (f \circ \gamma)(1) - (f \circ \gamma)(0) \\
&= f(z_1) - f(z_0).
\end{aligned}$$

Therefore,

$$\begin{aligned}
|f(z_1) - f(z_0)| &= \left| \int_\gamma f'(\zeta)d\zeta \right| \\
&\leq \int_\gamma |f'(\zeta)||d\zeta| \\
&\leq \int_\gamma \max_{z \in \gamma([0,1])}\{|f'(z)|\}|d\zeta| \\
&= \max_{z \in \gamma([0,1])}\{|f'(z)|\} \int_\gamma |d\zeta| \\
&= \text{Length}(\gamma) \cdot \max_{z \in \gamma([0,1])}\{|f'(z)|\}.
\end{aligned}$$

$\square$

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For each non-zero integer $n \in \mathbb{Z}\backslash\{0\}$, consider the function f defined by $f(z) = z^n$; calculate $\mathrm{Deg}(f, 0)$.

Answer: Let $n \in \mathbb{Z}\backslash\{0\}$ and let $f(z) = z^n$. Let $\sigma : [0,1] \to S^1 \subseteq \mathbb{C}$ be given by $\sigma(t) = e^{2\pi i t}$. Then

$$\begin{aligned}
\mathrm{Deg}(f,0) &= n(f \circ \sigma, 0) \\
&= \frac{1}{2\pi i} \oint_{f\circ\sigma} \frac{1}{z-0} dz \\
&= \frac{1}{2\pi i} \int_0^1 \frac{1}{f \circ \sigma(t)} (f \circ \sigma)'(t) dt \\
&= \frac{1}{2\pi i} \int_0^1 \frac{1}{f \circ \sigma(t)} f'(\sigma(t))\sigma'(t) dt \\
&= \frac{1}{2\pi i} \int_0^1 \frac{1}{(e^{2\pi i t})^n} n \left(e^{2\pi i t}\right)^{n-1} 2\pi i e^{2\pi i t} dt \\
&= \int_0^1 n dt \\
&= n.
\end{aligned}$$

Since this result only depended on the fact that $n \neq 0$, we see that, for all $n \in \mathbb{Z}\backslash\{0\}$, if $f_n(z) = z^n$, then

$$\mathrm{Deg}(f_n, 0) = n.$$

♣

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Verify that the proof of Morera's Theorem also proves the following version of the Fundamental Theorem of Calculus: if $f : D(0,R) \to \mathbb{C}$ is continuous and $\oint_{\partial\Gamma} f(z)dz = 0$ for every rectangle $\Gamma \subset D(0,R)$ with sides parallel to the axes, then the function $F : D(0,R) \to \mathbb{C}$ defined by $F(z) := \int_0^z f(\zeta)d\zeta$ (along the segments $[0,(x,0)]$ and $[(x,0),(x,y)]$) has a complex derivative, and $F'(z) = f(z)$ for every $z = (x,y) \in D(0,R)$.

Proof. For $z \in D(0,R)$, $z = x + iy$ for $x, y \in \mathbb{R}$. Let $z' = x$ and let $\gamma_z = \overrightarrow{0z'} \cdot \overrightarrow{z'z}$. Then we define

$$F(z) = \int_{\gamma_z} f(w)dw.$$

Let $h \neq 0$ be small real number. Then

$$\begin{aligned}\lim_{h\to 0}\frac{F(z+ih)-F(z)}{h} &= \lim_{h\to 0}\frac{\int_{\gamma_{z+ih}} f(w)dw - \int_{\gamma_z} f(w)dw}{h}\\ &= \lim_{h\to 0}\frac{1}{h}\int_{\overrightarrow{z\cdot(z+ih)}} f(w)dw\\ &= \lim_{h\to 0}\frac{i}{h}\int_y^{y+h} f(x+it)dt.\end{aligned}$$

Hence, $\partial F/\partial y$ exists and is equal to if on $D(0,R)$.

Now, let $z'' = iy$ and let $\Gamma_z = \overrightarrow{0z''}\cdot\overrightarrow{z''z}$. Then we claim that

$$\int_{\Gamma_z} f(w)dw = \int_{\gamma_z} f(w)dw.$$

If $z \neq 0$, then

$$\int_{\gamma_z} f(w)dw - \int_{\Gamma_z} f(w)dw = \int_{\partial R} f(w)dw = 0$$

where R is the rectangle with vertices $0, z', z, z''$ and the last equality is by hypothesis. On the other hand, if $x = 0$, then $z' = 0$ and $z'' = z$, so

$$\int_{\gamma_z} f(w)dw - \int_{\Gamma_z} f(w)dw = \int_{\overrightarrow{0z}} f(w)dw - \int_{\overrightarrow{0z}} f(w)dw = 0,$$

so equality does indeed hold in this case. A similar argument shows that $\int_{\gamma_z} f(w)dw = \int_{\Gamma_z} f(w)dw$ when $y = 0$.

Now, let $h \neq 0$ be a small real number. Then

$$\begin{aligned}\lim_{h\to 0}\frac{F(z+h)-F(z)}{h} & \\ &= \lim_{h\to 0}\frac{1}{h}\int_{\gamma_{z+h}} f(w)dw - \int_{\gamma_z} f(w)dw\\ &= \lim_{h\to 0}\frac{1}{h}\int_{\Gamma_{z+h}} f(w)dw - \int_{\Gamma_z} f(w)dw\\ &= \lim_{h\to 0}\frac{1}{h}\int_{\overrightarrow{z\cdot(z+h)}} f(w)dw\\ &= \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t+iy)dt.\end{aligned}$$

Hence, $\partial F/\partial x$ exists and equals f. Since f and if are both continuous, we see that $\partial F/\partial x$ and $\partial F/\partial y$ are continuous and

$$\frac{\partial F}{\partial x} = f = -i(if) = -i\frac{\partial F}{\partial y}.$$

Therefore, F is $\mathbb{C}$-differentiable and $F'(z) = f(z)$ for all $z \in D(0,R)$. $\square$

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Prove that if a sequence (f_n) of continuous functions $f_n : \mathcal{D} \to \mathbb{C}$ converges uniformly on each compact subset of the open set $\mathcal{D} \subset \mathbb{C}$ to a function $f : \mathcal{D} \to \mathbb{C}$, then for each differentiable curve $\gamma : [0,1] \to \mathcal{D}$,

$$\lim_{n\to\infty}\left[\int_\gamma f_n(z)dz\right] = \int_\gamma \left\{\lim_{n\to\infty}[f_n(z)]\right\} dz.$$

Proof. Let $\gamma : [0,1] \to \mathcal{D}$ be a differentiable curve and let $\epsilon > 0$. Then, since $f_n \to f$ uniformly, there exists $N \in \mathbb{N}$ such that, if $n \geq N$,

$$|f_n(z) - f(z)| < \frac{\epsilon}{\text{Length}(\gamma)}.$$

Let $n > N$. Then

$$\begin{aligned}
\left|\int_\gamma f_n(z)dz - \int_\gamma f(z)dz\right| &= \left|\int_\gamma [f_n(z) - f(z)]dz\right| \\
&\leq \int_\gamma |f_n(z) - f(z)||dz| \\
&\leq \int_\gamma \frac{\epsilon}{\text{Length}(\gamma)}|dz| \\
&= \frac{\epsilon}{\text{Length}(\gamma)}\int_\gamma |dz| \\
&= \frac{\epsilon}{\text{Length}(\gamma)} \cdot \text{Length}(\gamma) \\
&= \epsilon.
\end{aligned}$$

Since our choice of $\epsilon > 0$ was arbitrary, we see that, for any differentiable γ,

$$\lim_{n\to\infty}\left[\int_\gamma f_n(z)dz\right] = \int_\gamma \left\{\lim_{n\to\infty}[f_n(z)]\right\} dz.$$

□

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Verify that the Mean Value Theorem for derivatives does *not* generalize to complex functions.

Answer: The natural generalization of the Mean Value Theorem for derivatives would be that, for $z_1, z_2 \in \mathbb{C}$ and f holomorphic on a region containing the line segment connecting z_1 and z_2, there exists z_0 on the line segment connecting z_1 and z_2 such that

$$f'(z_0) = \frac{f(z_2) - f(z_1)}{z_2 - z_1}$$

Another generalization might make the same conclusion for any path γ connecting z_1 and z_2 on which f is holomorphic, but we see that the first generalization is a special case of this more general one, so if we can show the first does not hold, then the second also does not hold.

To that end, consider the exponential function $f(z) = e^z$, which is holomorphic on all of $\mathbb{C}$. Let $z_1 = 2\pi i$ and let $z_2 = \frac{5\pi}{2} i$. Then

$$\frac{f(z_2) - f(z_1)}{z_2 - z_1} = \frac{e^{\frac{5\pi}{2} i} - e^{2\pi i}}{\frac{5\pi}{2} i - 2\pi i} = \frac{i - 1}{\frac{\pi}{2} i} = \frac{2i - 2}{\pi i} = \frac{2i + 2}{\pi}.$$

Now, $f'(z) = e^z$ and, for any ai on the line segment connecting $2\pi i$ and $\frac{5\pi}{2} i$, e^{ai} lies on the unit circle and, hence, has modulus 1. However,

$$\left| \frac{2i + 2}{\pi} \right| = \left| \frac{1}{\pi} \right| |2i + 2| = \frac{1}{\pi} \sqrt{8} = \frac{2\sqrt{2}}{\pi} < 1,$$

so there is no $a \in \left[2\pi, \frac{5\pi}{2}\right]$ such that $f'(ai) = \frac{2i+2}{\pi}$. Hence, we conclude that the Mean Value Theorem for derivatives does not generalize to complex functions.

♣

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Derive a complex integral formula for $R_{f,N,z_0}(z)$.

Answer: Let r be the radius of convergence of the Taylor series for f centered at z_0 and let $0 < \rho < r$. Let γ be the circle $t \mapsto z_0 + \rho e^{2\pi i t}$. Then, by Cauchy's Integral Formula, if $z \in D(z_0, \rho)$,

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta.$$

Now,

$$\frac{1}{\zeta - z} = \frac{1}{z - z_0} \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = \sum_{k=0}^{\infty} \frac{(z - z_0)^k}{(\zeta - z_0)^{k+1}},$$

which converges uniformly for $|\zeta - z_0| < \rho$ if $|z - z_0| < \rho$. Hence,

$$f(z) = \sum_{k=0}^{\infty} \left[\frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} d\zeta \right] (z - z_0)^k$$

and

$$T_{f,N,z_0} = \sum_{k=0}^{N} \left[\frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} d\zeta \right] (z - z_0)^k.$$

(So far, we've just rehashed the proof of Taylor's Theorem). Now, by definition,

$$\begin{aligned}
R_{f,N,z_0}(z) &= f(z) - T_{f,N,z_0}(z) \\
&= f(z) - \sum_{k=0}^{N} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k \\
&= \sum_{k=0}^{\infty}\left[\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-z_0)^{k+1}}d\zeta\right](z-z_0)^k - \sum_{k=0}^{N}\left[\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)(z-z_0)^k}{(\zeta-z_0)^{k+1}}d\zeta\right] \\
&= \frac{1}{2\pi i}\int_\gamma \left[\sum_{k=0}^{\infty}\frac{f(\zeta)(z-z_0)^k}{(\zeta-z_0)^{k+1}}\right]d\zeta - \frac{1}{2\pi i}\int_\gamma \left[\sum_{k=0}^{N}\frac{f(\zeta)(z-z_0)^k}{(\zeta-z_0)^{k+1}}\right]d\zeta \\
&= \frac{1}{2\pi i}\int_\gamma \left[\sum_{k=N+1}^{\infty}\frac{f(\zeta)(z-z_0)^k}{(\zeta-z_0)^{k+1}}\right]d\zeta, \\
&= \frac{1}{2\pi i}\int_\gamma \left[\sum_{k=N+1}^{\infty}\frac{f(\zeta)}{\zeta-z_0}\left(\frac{z-z_0}{\zeta-z_0}\right)^k\right]d\zeta,
\end{aligned}$$

where interchanging sums and integrals was acceptable because this sum converges absolutely. Now, suppose $a, b \in \mathbb{C}$ and $S_n = a + ab + ab^2 + \ldots + ab^n$. Then

$$S_n(1-b) = S_n - bS_n = (a+ab+\ldots+ab^n)-(ab+ab^2+\ldots+ab^{n+1}) = a - ab^{n+1} = a(1-b^{n+1}),$$

so

$$S_n = \frac{a(1-b^{n+1})}{1-b}.$$

Note that

$$\sum_{k=N+1}^{\infty} ab^k = \sum_{k=0}^{\infty} ab^k - S_N = \frac{a}{1-b} - \frac{a(1-b^{N+1})}{1-b} = \frac{ab^{N+1}}{1-b}.$$

Therefore, in the case where $a = \frac{f(\zeta)}{\zeta - z_0}$ and $b = \frac{z-z_0}{\zeta-z_0}$,

$$\sum_{k=N+1}^{\infty} \frac{f(\zeta)}{\zeta-z_0}\left(\frac{z-z_0}{\zeta-z_0}\right)^k = \frac{f(\zeta)\left(\frac{z-z_0}{\zeta-z_0}\right)^{N+1}}{(\zeta-z_0)\left(1-\frac{z-z_0}{\zeta-z_0}\right)} = f(\zeta)\frac{(z-z_0)^{N+1}}{(\zeta-z_0)^{N+1}(\zeta-z)}.$$

Therefore, substituting into equation (1), we see that

$$R_{f,N,z_0}(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)(z-z_0)^{N+1}}{(\zeta-z)(\zeta-z_0)^{N+1}}d\zeta.$$

♣

97.2

Prove an extension of Schwarz's Reflection principle with $\Omega \cap \mathbb{R}$ replaced by the intersection of Ω with any straight line and with real values replaced by values on any straight line in the plane.

Proof. Suppose Ω is open, connected and symmetric about the line defined by $r(\cos\theta + i\sin\theta)$ $(r \in \mathbb{R})$. Let us introduce the rotation

$$R_1(z) = (\cos\theta + i\sin\theta)z.$$

Note that R_1 is a fractional linear transformation, and so is holomorphic on all of $\mathbb{C}$ and has a well-defined inverse R_1^{-1}. Also, note that the line $r(\cos\theta + i\sin\theta)$ is simply $R_1(\mathbb{R})$. Let $R_1^{-1}(\Omega) = \Omega'$. Then Ω being symmetric means that, if $z \in \Omega$, $R_1(\overline{R_1^{-1}(z)}) \in \Omega$. Now, suppose $f : R_1(\Omega' \cap H_+) \to \mathbb{C}$ is holomorphic on $R_1(\Omega' \cap H_+^o)$, continuous on $(R_1(\Omega' \cap H_+^o)) \cup (\Omega \cap R_1(\mathbb{R}))$ and that, for $z \in \Omega \cap R_1(\mathbb{R})$, $f(z)$ lies on a line ℓ in the plain. Then we want to show that there exists an extension $F : \Omega \to \mathbb{C}$ of f that is holomorphic on Ω.

Note that $\ell = R_2 \circ T(\mathbb{R})$, where R_2 is a rotation and T is a translation; namely,

$$T(z) = z + f(R_1(0))$$

and

$$R_2 = (\cos\psi + i\sin\psi)z$$

where ψ is the angle ℓ makes with $\mathbb{R}$. Note that R_2 and T are fractional linear transformations, and so are holomorphic on $\mathbb{C}$ and have fractional linear inverses. Furthermore, $\mathbb{R} = (R_2 \circ T)^{-1}(\ell) = T^{-1} \circ R_2^{-1}(\ell)$. Define $g : \Omega' \cap H_+ \to \mathbb{C}$ by

$$g(z) = (T^{-1} \circ R_2^{-1} \circ f \circ R_1)(z) = (\cos -\psi + i\sin -\psi)f((\cos\phi + i\sin\phi)z).$$

Since R_1, R_2, T are holomorphic on $\mathbb{C}$ and f is continuous on $R_1(\Omega' \cap H_+)$ and holomorphic on $R_1(\Omega' \cap H_+^o)$, g is continuous on $\Omega' \cap H_+$ and holomorphic on $\Omega' \cap H_+^o$. Therefore, by the version of Schwarz's Reflection Principle proved in class,

$$G(z) = \begin{cases} g(z), & \text{if } z \in \Omega' \cap H_+ \\ \overline{g(\bar{z})} & \text{otherwise} \end{cases}$$

is holomorphic on Ω'. Define

$$F(z) = R_2 \circ T \circ G \circ R_1^{-1}(z).$$

Then, since $R_1^{-1}(\Omega) = \Omega'$ and G holomorphic on Ω', F is holomorphic on Ω. Furthermore, if $z \in R_1(\Omega' \cap H_+)$, $R_1^{-1}(z) \in \Omega' \cap H_+$, so

$$F(z) = R_2 \circ T \circ G \circ R_1^{-1}(z) = R_2(T(g(R_1^{-1}(z)))) = R_2(T(T^{-1}(R_2^{-1}(f(R_1(R_1^{-1}(z))))))) = f(z),$$

so F is the desired extension. $\square$

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Prove that for each differentiable curve $\gamma : [0,1] \to \mathbb{C}$ that does not pass through the origin, there exist *differentiable* functions ρ and θ such that $\gamma(t) = \rho(t)\exp[i\theta(t)]$ for every $t \in [0,1]$.

Proof. By the result from problem 101, we know that, for each t, $\gamma(t) = \rho_t(\cos\theta_t + i\sin\theta_t)$ for unique ρ_t and θ_t since γ does not pass through the origin. Hence, if we define $\rho(t) = \rho_t$ and $\theta(t) = \theta_t$, $\rho : [0,1] \to \mathbb{R}^+$ and $\theta : [0,1] \to (-\pi, \pi]$ are well-defined. Now,

$$\begin{aligned}
\cos\theta(t) + i\sin\theta(t) &= \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!}\theta^{2m} + i\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\theta^{2m+1} \\
&= \sum_{m=0}^{\infty} \left[\frac{(-1)^m}{(2m)!}\theta^{2m} + i\frac{(-1)^m}{(2m+1)!}\theta^{2m+1}\right] \\
&= \sum_{m=0}^{\infty} \frac{i^m}{m!}\theta^m \\
&= \sum_{m=0}^{\infty} \frac{(i\theta)^m}{m!} \\
&= \exp(i\theta(t)),
\end{aligned}$$

so we can express $\gamma(t) = \rho(t)\exp(i\theta(t))$. Now, since $|\exp(it)| = 1$ for all $t \in \mathbb{R}$ (problem 99),

$$|\gamma(t)| = |\rho(t)||\exp(i\theta(t))| = |\rho(t)| = \rho(t)$$

since $\rho_t > 0$ for all $t \in [0,1]$. Since γ is differentiable and $|\cdot|$ is differentiable away from the origin, we see that ρ, as the composition of differentiable functions, is differentiable. Furthermore, since $\rho > 0$, the quotient $\frac{\gamma(t)}{\rho(t)} = \exp(i\theta(t))$ is also differentiable, with derivative

$$\left(\frac{\gamma}{\rho}\right)'(t) = \lim_{h\to 0} \frac{\exp(i\theta(t+h)) - \exp(i\theta(t))}{h}.$$

Now, if $\theta(t+h) \neq \theta(t)$ for all h sufficiently close to 0, then

$$\begin{aligned}
\left(\frac{\gamma}{\rho}\right)'(t) &= \lim_{h\to 0} \frac{\exp(i\theta(t+h)) - \exp(i\theta(t))}{h} \\
&= \lim_{h\to 0} \frac{\exp(i\theta(t+h)) - \exp(i\theta(t))}{i\theta(t+h) - i\theta(t)} \cdot \frac{i\theta(t+h) - i\theta(t)}{h} \\
&= \lim_{h\to 0} \frac{\exp(i\theta(t+h)) - \exp(i\theta(t))}{i\theta(t+h) - i\theta(t)} \lim_{h\to 0} \frac{i\theta(t+h) - i\theta(t)}{h} \\
&= \exp'(i\theta(t)) \cdot \lim_{h\to 0} \frac{i\theta(t+h) - i\theta(t)}{h}
\end{aligned}$$

since exp is $\mathbb{C}$-differentiable. Therefore,

$$\lim_{h\to 0} \frac{i\theta(t+h) - i\theta(t)}{h} = \frac{\left(\frac{\gamma}{\rho}\right)'(t)}{\exp'(i\theta(t))}$$

exists (since $\exp(it) \neq 0$ for all $t \in \mathbb{R}$), so θ is differentiable.

On the other hand, suppose there exist h arbitrarily close to zero such that $\theta(t+h) = \theta(t)$. Then, since θ is continuous, $\theta(t+h) = \theta(t)$ for all h sufficiently close to 0, so

$$\lim_{h\to 0} \frac{\theta(t+h) - \theta(t)}{h} = \frac{\theta(t) - \theta(t)}{h} = 0$$

and, again, θ is differentiable. $\square$

112

Prove that if $P : \mathbb{C} \to \mathbb{C}$ is a polynomial, if $f : \mathbb{C} \to \mathbb{C}$ is holomorphic on all of $\mathbb{C}$, and if there exists a real $C > 0$ such that $|f(z)| \leq C \cdot |P(z)|$ for every $z \in \mathbb{C}$, then $f = c \cdot P$ for some $c \in \mathbb{C}$. Is there an analogous statement with P replaced by an arbitrary holomorphic function on all of $\mathbb{C}$?

Proof. By the fundamental theorem of algebra, there exists $z_0 \in \mathbb{C}$ such that $P(z_0) = 0$. Hence, we can factor out the linear term $z - z_0$ from P to get

$$P(z) = (z - z_0)P_0(z)$$

where P_0 is a polynomial. Since f is entire, we can represent f by its power series centered at z_0:

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n.$$

Since $|f(z_0)| \leq C \cdot |P(z_0)| = 0$, we see that

$$0 = f(z_0) = \sum_{n=0}^{\infty} c_n (z_0 - z_0)^n = c_0.$$

Hence, we can drop this first term in the power series, yielding

$$f(z) = \sum_{n=1}^{\infty} c_n (z - z_0)^n = (z - z_0) \sum_{n=0}^{\infty} c_{n+1} (z - z_0)^n.$$

Let $f_0(z) = \sum_{n=0}^{\infty} c_{n+1}(z - z_0)^n$. Note that this power series converges on the same region (namely all of $\mathbb{C}$) as the original power series representation of f. To see this, we use Hadamard's Formula:

$$R_0 = \frac{1}{\limsup(\sqrt[n]{|c_{n+1}|})} \geq \frac{1}{\limsup(\sqrt[n]{|c_n|})} = \infty.$$

Hence we can divide both sides of the inequality $|f(z)| \leq C \cdot |P(z)|$ by $|z - z_0|$ to get

$$|f_0(z)| = \frac{|z - z_0||f_0(z)|}{|z - z_0|} = \frac{|f(z)|}{|z - z_0|} \leq C \cdot \frac{|P(z)|}{|z - z_0|} = C \cdot \frac{|z - z_0||P_0(z)|}{|z - z_0|} = C \cdot |P_0(z)|.$$

Now, P_0 is also a polynomial, and so, by the fundamental theorem of algebra, has a zero z_1 and so can be factored as $(z-z_1)P_1(z)$ for some polynomial P_1. Furthermore, since we defined f_0 as a power series convergent on all of $\mathbb{C}$, f_0 is entire and so we can represent it by a power series centered at z_1. The same process outlined above yields:

$$|f_1(z)| \leq C \cdot |P_1(z)|$$

for an entire function f_1. Note that $f_1(z) = \frac{f(z)}{(z-z_0)(z-z_1)}$ and $P_1(z) = \frac{P(z)}{(z-z_0)(z-z_1)}$. Now, when we iterate this process through all the zeros of P, we will be left with

$$|f_{n-1}(z)| \leq C \cdot |p_n|$$

where $f_{n-1}(z) = \frac{f(z)}{(z-z_0)\cdots(z-z_{n-1})}$ is entire and $p_n = P_{n-1}(z) = \frac{P(z)}{(z-z_0)\cdots(z-z_{n-1})}$. Therefore,

$$P(z) = p_n(z-z_0)\cdots(z-z_{n-1}).$$

Hence, $\frac{f(z)}{P(z)} = \frac{1}{p_n} f_{n-1}(z)$ is entire and

$$\left|\frac{f(z)}{P(z)}\right| = \frac{1}{|p_n|}\,|f_{n-1}(z)| \leq C.$$

Thus, by Liouville's Theorem, $f(z)/P(z)$ is constant. Therefore, we conclude that f is a constant multiple of P. $\square$

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Consider a complex polynomial p with $p(z) = p_0 + p_1 z + \cdots + p_{n-1}z^{n-1} + p_n z^n$. Find a necessary and sufficient condition on the complex coefficients $p_0, \ldots, p_n$ so that $p(\mathbb{R}) \subseteq \mathbb{R}$.

Lemma 0.1. *If $p(z) = p_0 + p_1 z + \ldots + p_n z^n$ is a complex polynomial such that $p(\mathbb{R}) \subseteq \mathbb{R}$, then $p'(\mathbb{R}) \subseteq \mathbb{R}$.*

Proof. By the definition of the derivative,

$$p'(z) = \lim_{\zeta \to 0} \frac{p(z+\zeta) - p(z)}{\zeta}.$$

Now, suppose $z \in \mathbb{R}$, and $\zeta = h \in \mathbb{R}$. Then, since $z + h \in \mathbb{R}$, $p(z+h) \in \mathbb{R}$ and so $p(z+h) - p(z) \in \mathbb{R}$ for all such h. Hence, since the derivative is well-defined,

$$p'(z) = \lim_{h \to 0} \frac{p(z+h) - p(z)}{h} \in \mathbb{R}.$$

$\square$

Now, we are ready to prove the following claim:
Claim: $p(\mathbb{R}) \subseteq \mathbb{R}$ if and only if $p_i \in \mathbb{R}$ for all $i = 0, \ldots, n$.

Proof. $p_0, \ldots, p_n \in \mathbb{R}$ is certainly a sufficient condition to ensure that $p(\mathbb{R}) \subseteq \mathbb{R}$. To prove that it is necessary, suppose p is a polynomial such that $p(\mathbb{R}) \in \mathbb{R}$. First, note that $p(0) \in \mathbb{R}$, and so

$$p_0 = p_0 + p_1 \cdot 0 + \ldots + p_n \cdot 0^n = p(0) \in \mathbb{R}.$$

Now, by our lemma, since $p(\mathbb{R}) \subseteq \mathbb{R}$, $p'(\mathbb{R}) \subseteq \mathbb{R}$. Since p is entire and is its own power series expansion at 0, we know that

$$p'(z) = p_1 + 2p_2 z + \ldots + np_n z^{n-1}.$$

Hence,

$$p_1 = p_1 + 2p_2 \cdot 0 + \ldots + np_n \cdot o^{n-1} = p'(0) \in \mathbb{R}.$$

Now, p' also satisfies the hypotheses of the above lemma, so its derivative $p''(\mathbb{R}) \subseteq \mathbb{R}$. Hence,

$$2p_2 = 2p_2 + 3 \cdot 2p_3 \cdot 0 + \ldots + n(n-1)p_n \cdot 0^{n-2} = p''(0) \in \mathbb{R},$$

which implies that $p_2 = p''(0)/2 \in \mathbb{R}$. Iterating this process, we see that, at the ith stage,

$$i!p_i = i!p_i + (i+1)(i) \cdots (2)p_{i+1} \cdot 0 + \ldots + n(n-1) \cdots (n-i+1)p_n \cdot 0^{n-i} = p^{(i)}(0) \in \mathbb{R},$$

which implies $p_i = p^{(i)}(0)/i! \in \mathbb{R}$.

Therefore, we conclude that, for all $i = 0, \ldots, n$, $p_i \in \mathbb{R}$. □

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