COMPLEX ANALYSIS HW 3

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Justify the swap of limit and integral carefully to prove that for each function $f : \mathcal{D} \to \mathbb{C}$ continuous on an open set $\mathcal{D} \subseteq \mathbb{C}$, and for each differentiable curve $\gamma : [0,1] \to \mathcal{D}$, the function g defined on $\mathbb{C}\backslash\gamma([0,1])$ defined by

$$g(z) := \int_\gamma \frac{f(w)}{w-z}dw$$

is holomorphic, with derivatives

$$g^{(n)}(z) = n! \int_\gamma \frac{f(w)}{(w-z)^{n+1}}dw.$$

Proof. Let $\epsilon > 0$. Let $z \in \mathbb{C}\backslash\gamma([0,1])$. Let d be the positive distance from z to the curve γ and let $\delta = d/2$. Let $l(\gamma)$ denote the length of γ and let M be the maximum of f on $\gamma([0,1])$ (such a maximum exists since the image of a compact set, like $[0,1]$, is compact). Let $\xi \in \mathbb{C}$ such that $|\xi| \leq \min\left\{\frac{\epsilon\delta^3}{Ml(\gamma)}, \delta\right\}$. Then, for any $w \in \gamma[(0,1)]$,

$$\begin{aligned}|w-(z+\xi)| = |w-z+z-(z+\xi)| &= |(w-z)-((z+\xi)-z)| \\ &\geq |w-z| - |z+\xi-z| \\ &= |w-z| - |\xi| \\ &\geq 2\delta - \delta \\ &= \delta.\end{aligned}$$

Now, we want to show that

$$\lim_{\xi\to 0}\left[\frac{g(z+\xi)-g(z)}{\xi} - \int_\gamma \frac{f(w)}{w-z}dw\right] = 0.$$

First, note that

$$\begin{aligned}\frac{g(z+\xi)-g(z)}{\xi} &= \frac{1}{\xi}\left[\int_\gamma \frac{f(w)}{w-(z+\xi)}dw - \int_\gamma \frac{f(w)}{w-z}dw\right]\\ &= \frac{1}{\xi}\left[\int_\gamma \frac{f(w)(w-z)-f(w)(w-(z+\xi))}{(w-z)(w-(z+\xi))}dw\right]\\ &= \frac{1}{\xi}\int_\gamma \frac{f(w)\xi}{(w-z)(w-(z+\xi))}dw\\ &= \int_\gamma \frac{f(w)}{(w-z)(w-(z+\xi))}dw\end{aligned}$$

and so

$$\begin{aligned}\frac{g(z+\xi)-g(z)}{\xi} - \int_\gamma \frac{f(w)}{(w-z)^2}dw &= \int_\gamma \left[\frac{f(w)}{(w-z)(w-(z+\xi))} - \frac{f(w)}{(w-z)^2}\right]dw\\ &= \int_\gamma \frac{f(w)(w-z)-f(w)(w-(z+\xi))}{(w-z)^2(w-(z+\xi))}dw\\ &= \int_\gamma \frac{f(w)\xi}{(w-z)^2(w-(z+\xi))}dw\end{aligned}$$

Now, since $|\xi| < \delta$, $|(w-z)^2(w-(z+\xi))| = |w-z|^2|w-(z+\xi)| \geq \delta^2 \cdot \delta$. Hence,

$$\begin{aligned}\left|\int_\gamma \frac{f(w)\xi}{(w-z)^2(w-(z+\xi))}dw\right| &\leq \int_\gamma \left|\frac{f(w)\xi}{(w-z)^2(w-(z+\xi))}\right||dw|\\ &\leq |\xi|\int_\gamma \frac{M}{\delta^3}|dw|\\ &= |\xi|\frac{M}{\delta^3}l(\gamma)\\ &\leq \frac{\epsilon\delta^3}{Ml(\gamma)}\frac{M}{\delta^3}l(\gamma)\\ &= \epsilon.\end{aligned}$$

Since our choice of ϵ was arbitrary, we see that

$$\lim_{\xi\to 0}\left|\frac{g(z+\xi)-g(z)}{\xi} - \int_\gamma \frac{f(w)}{(w-z)^2}dw\right| = 0,$$

so g is $\mathbb{C}$-differentiable at z with $g'(z) = \int_\gamma \frac{f(w)}{(w-z)^2}dw$. Since our choice of $z \in \mathbb{C}\backslash\gamma([0,1])$ was arbitrary, we see that g is holomorphic on $\mathbb{C}\backslash\gamma([0,1])$.

Now, suppose, for the sake of induction, that if f is continuous on Ω, and

$$g_f = \int_\gamma \frac{f(w)}{z-w}dw$$

then

$$g^{(k)} = k!\int_\gamma \frac{f(w)}{(w-z)^{k+1}}dw$$

for all $k \leq n-1$. Then, returning to our original g, we want to show that

$$lim_{\xi \to 0} \frac{g^{(n-1)}(z+\xi) - g^{(n-1)}(z)}{\xi} = n! \int_\gamma \frac{f(w)}{(w-z)^{n+1}} dw.$$

Let $z\mathbb{C}\backslash\gamma([0,1])$. Note that

$$\begin{aligned}
& \frac{1}{(w-z)^{n-1}(w-(z+\xi))} - \frac{\xi}{(w-z)^n(w-(z+\xi))} \\
= \quad & \frac{w-z}{(w-z)^n(w-(z+\xi))} - \frac{\xi}{(w-z)^n(w-(z+\xi))} \\
= \quad & \frac{w-(z+\xi)}{(w-z)^n(w-(z+\xi))} \\
= \quad & \frac{1}{(w-z)^n}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{g^{(n-1)}(z+\xi) - g^{(n-1)}(z)}{\xi} &= \frac{(n-1)!}{\xi} \int_\gamma \frac{f(w)}{(w-(z+\xi))^n} - \frac{(n-1)!}{\xi} \int_\gamma \frac{f(w)}{(w-z)^n} \\
&= \frac{(n-1)!}{\xi} \left[\int_\gamma \frac{f(w)}{(w-(z+\xi))^n} - \int_\gamma \frac{f(w)}{(w-z)^{n-1}(w-(z+\xi))} \right] \qquad (1) \\
&\quad + (n-1)! \int_\gamma \frac{f(w)}{(w-z)^n(w-(z+\xi))}
\end{aligned}$$

Now, if we let $h(w) = \frac{f(w)}{w-(z+\xi)}$, then, since $z+\xi \notin \gamma([0,1])$ for all sufficiently small ξ (i.e. $|\xi| < \delta$), h is continuous on Ω so we can apply the induction hypothesis to conclude that

$$\begin{aligned}
(n-1)! \int_\gamma \frac{h(w)}{(w-z)^n} dw &= g_h^{(n-1)}(z) \\
&= \lim_{\xi \to 0} \frac{(n-2)!}{\xi} \left[\int_\gamma \frac{f(w)/(w-(z+\xi))}{(w-(z+\xi))^{n-1}} dw - \int_\gamma \frac{f(w)/(w-(z+\xi))}{(w-z)^{n-1}} dw \right],
\end{aligned}$$

so the first term in the right side of (1) above is thus simply

$$(n-1)(n-1)! \int_\gamma \frac{h(w)}{(w-z)^n} dw = (n-1)(n-1)! \int_\gamma \frac{f(w)}{(w-z)^n(w-(z+\xi))} dw.$$

As $\xi \to 0$, $w-(z+\xi) \to w-z$, so this converges to

$$(n-1)(n-1)! \int_\gamma \frac{f(w)}{(w-z)^{n+1}} dw.$$

Similarly, the second term in (1),

$$(n-1)! \int_\gamma \frac{f(w)}{(w-z)^n(w-(z+\xi))} dw \to (n-1)! \int_\gamma \frac{f(w)}{(w-z)^{n+1}} dw,$$

so (1) reduces to

$$(n-1)(n-1)!\int_\gamma \frac{f(w)}{(w-z)^n(w-(z+\xi))}dw+(n-1)!\int_\gamma \frac{f(w)}{(w-z)^{n+1}}dw = n!\int_\gamma \frac{f(w)}{(w-z)^{n+1}}$$

for ξ sufficiently small. Thus,

$$g^{(n)}(z) = \lim_{\xi\to 0}\frac{g^{(n-1)}(z+\xi)-g^{(n-1)}(z)}{\xi} = n!\int_\gamma \frac{f(w)}{(w-z)^{n+1}}dw.$$

Therefore, we conclude, by induction, that, the desired result does indeed hold for all $n \in \mathbb{N}$. $\square$

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For each $c \in \mathbb{C}$, for all $r_1, r_2 \in \overline{\mathbb{R}}$, and for each $f \in \mathcal{H}[A(c, r_1, r_2)]$, prove that there exist $f_1 \in \mathcal{H}[A(c, r_1, \infty)]$ and $f_2 \in \mathcal{H}[D(c, r_2)]$ such that $f = f_1 - f_2$. Also prove that there exist only one pair of such functions f_1 and f_2 with $\lim_{|z|\to\infty}|f_1(z)| = 0$.

Proof. First, note that it suffices to show this for the case where $c = 0$, because if $c \neq 0$, then we can form the function $h(z) = f(z-c)$ which will be holomorphic on $A(0, r_1, r_2)$. Now, because $f \in \mathcal{H}[A(0, r_1, r_2)]$, f has a Laurent expansion

$$f(w) = \sum_{n=-\infty}^{\infty} c_n w^n$$

where

$$c_n = \frac{1}{2\pi i}\int_{|z|=\rho}\frac{f(z)}{z^{n+1}}dz.$$

Now, define

$$f_1(w) = -\sum_{n=-\infty}^{-1} c_n w^n$$

and

$$f_2(w) = -\sum_{n=0}^{\infty} c_n w^n.$$

To see why these are holomorphic in the appropriate regions, recall that

$$f(z) = \frac{1}{2\pi i}\int_{|z|=\rho_1}\frac{f(z)}{z-w}dz - \frac{1}{2\pi i}\int_{|z|=\rho_2}\frac{f(z)}{z-w}dz$$

where $r_1 < \rho_1 < |w| < \rho_2 < r_2$.

Now, for $|w| < \rho_2$ and $|z| = \rho_2$,

$$\frac{1}{z-w} = \frac{1}{z}\frac{1}{1-\frac{w}{z}} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{w}{z}\right)^n = \sum_{n=0}^{\infty}\frac{w^n}{z^{n+1}}$$

where the convergence is uniform since $\left|\frac{w}{z}\right| < 1$. Hence,

$$\begin{aligned}
\frac{1}{2\pi i}\int_{|z|=\rho_2}\frac{f(z)}{z-w}dz &= \frac{1}{2\pi i}\int_{|z|=\rho_2}\left[\sum_{n=0}^{\infty}\frac{f(z)w^n}{z^{n+1}}\right]dz\\
&= \frac{1}{2\pi i}\sum_{n=0}^{\infty}\left[\int_{|z|=\rho_2}\frac{f(z)}{z^{n+1}}dz\right]w^n\\
&= \frac{1}{2\pi i}\sum_{n=0}^{\infty}c_n w^n
\end{aligned}$$

which is just $-f_2$, where we can swap the sum and the integral since convergence is uniform and $c_n = \int_{|z|=\rho_2}\frac{f(z)}{z^{n+1}}dz$ holds because $\frac{f(z)}{z^{n+1}} \in \mathcal{H}[A(0,r_1,r_2)]$. Since our only condition for uniform convergence was that $|w| < \rho_2$, we see that $f_2 \in \mathcal{H}[D(0,\rho_2)]$. As we let $\rho_2 \to r_2$ the result still holds, so f_2 is, in fact, holomorphic on $D(0,r_2)$.

Similarly, if $|w| > \rho_1$ and $|z| = \rho_1$, then

$$\frac{1}{z-w} = \frac{1}{w}\frac{1}{\frac{z}{w}-1} = -\frac{1}{w}\frac{1}{1-\frac{z}{w}} = -\frac{1}{w}\sum_{n=0}^{\infty}\left(\frac{z}{w}\right)^n = -\sum_{n=0}^{\infty}\frac{z^n}{w^{n+1}}$$

where the convergence of the series is uniform since $\left|\frac{z}{w}\right| < 1$. Hence,

$$\begin{aligned}
\frac{1}{2\pi i}\int_{|z|=\rho_1}\frac{f(z)}{z-w}dz &= -\frac{1}{2\pi i}\int_{|z|=\rho_1}\left[\sum_{n=0}^{\infty}\frac{f(z)z^n}{w^{n+1}}\right]dz\\
&= -\frac{1}{2\pi i}\sum_{n=0}^{\infty}\left[\int_{|z|=\rho_1}f(z)z^n dz\right]w^{-n-1}\\
&= -\frac{1}{2\pi i}\sum_{n=-\infty}^{1}c_n w^n
\end{aligned}$$

which is just f_1, where we can swap the sum and integral since convergence is uniform and substituting c_n for the integral is appropriate as above. Our only condition for uniform convergence was that $|w| > \rho_1$, so we see that $f_1 \in \mathcal{H}[A(0,\rho_1,\infty)]$. As we let $\rho_1 \to r_1$ the result still holds, so f_1 is holomorphic on $A(0,r_1,\infty)$. Finally, note that, in $A(0,r_1,r_2)$,

$$\begin{aligned}
f(w) &= \frac{1}{2\pi i}\int_{|z|=\rho_2}\frac{f(z)}{z-w}dz - \frac{1}{2\pi i}\int_{|z|=\rho_1}\frac{f(z)}{z-w}dz\\
&= \frac{1}{2\pi i}\sum_{n=0}^{\infty}c_n w^n + \frac{1}{2\pi i}\sum_{n=-\infty}^{-1}c_n w^n\\
&= -f_2(w) + f_1(w)\\
&= f_1(w) - f_2(w).
\end{aligned}$$

Note that, since c_n is independent of w,

$$\lim_{|w|\to\infty} |f_1(w)| = 0.$$

Now, suppose there exist $g_1 \in \mathcal{H}[A(0,r_1,\infty)]$ and $g_2 \in \mathcal{H}[D(0,r_2)]$ such that $f = g_1 - g_2$ and $\lim_{|z|\to\infty} |g_1(z)| = 0$. Let $r_1 < \rho_1 < \rho_2 < r_2$ and define

$$h(z) = \begin{cases} f_1(z) - g_1(z) & \text{if } |z| \geq \rho_2 \\ \frac{|z|-|\rho_1|}{|\rho_2|-|\rho_1|}\left(f_1(z) - g_1(z)\right) + \frac{|\rho_2|-|z|}{|\rho_2|-|\rho_1|}\left(f_2(z) - g_2(z)\right) & \text{if } \rho_1 \leq |z| \leq \rho_2 \\ f_2(z) - g_2(z) & \text{if } |z| \leq \rho_1 \end{cases}$$

Then, for $|z| \geq \rho_2$, h is certainly holomorphic, since it is just the difference of two holomorphic functions. For $|z| \leq \rho_1$, h is also holomorphic, since it is, again, the difference of two holomorphic functions. Finally, for $\rho_1 < |z| < \rho_2$, h is holomorphic since $\frac{|z|-|\rho_1|}{|\rho_2|-|\rho_1|}$ and $\frac{|\rho_2|-|z|}{|\rho_2|-|\rho_1|}$ are holomorphic on this region, as are the differences $f_1 - g_1$ and $f_2 - g_2$. Since the cases agree on the boundaries of these regions, we conclude that h is entire.

Now, let $\epsilon > 0$. Since $\lim_{|z|\to\infty} |f_1(z)| = 0 = \lim_{|z|\to\infty} |g_1(z)|$, there exist N_1 and N_2 such that, if $|z| \geq N_1$, then $|f_1(z)| < \epsilon/2$ and, if $|z| \geq N_2$, $|g_1(z)| \leq \epsilon/2$. Let $N = \max\{N_1, N_2\}$. Then, $D_N(0) = \{z \in \mathbb{C} : |z| \leq N\}$ is compact and so, since h is holomorphic on this region, h is bounded on this region. Furthermore, if $|z| > N$,

$$|h(z)| = |f_1(z) - g_1(z)| \leq |f_1(z)| + |g_1(z)| = \epsilon/2 + \epsilon/2 = \epsilon,$$

so h is certainly bounded outside $D_N(0)$. Therefore, we conclude that h is an entire function that is bounded, and so, by Liouville's Theorem, h is constant. Since $h(z) \to 0$ as $|z| \to \infty$, it must, in fact, be the case that $h \equiv 0$. Therefore, for $|z| \geq rho_2$, $f_1(z) = g_1(z)$ and for $|z| \leq \rho_1$, $f_2(z) = g_2(z)$. Therefore, the power series expansions of f_2 and g_2 in $D(0,r_2)$ must be equal (since the power series is completely determined by the behavior of the functions at 0), and so $f_2 = g_2$ everywhere they are defined. Thus, for $\rho_1 < |z| < \rho_2$,

$$\begin{aligned} 0 = h(z) &= \frac{|z| - |\rho_1|}{|\rho_2| - |\rho_1|}\left(f_1(z) - g_1(z)\right) + \frac{|\rho_2| - |z|}{|\rho_2| - |\rho_1|}\left(f_2(z) - g_2(z)\right) \\ &= \frac{|z| - |\rho_1|}{|\rho_2| - |\rho_1|}\left(f_1(z) - g_1(z)\right), \end{aligned}$$

so $f_1 = g_1$ inside the annulus as well. From all this, then, we conclude that the f_1 and f_2 we found above are unique. □

4

Deduce Weierstrass' theorem that the uniform limit of holomorphic functions is holomorphic, from Morera's theorem.

Proof. Let $\langle f_n \rangle \subset \mathcal{H}(\Omega)$ be a sequence of holomorphic functions on some open set Ω such that $f_n \to f$ locally uniformly. Now, if $z \in \Omega$, there exists

$D_r(z) \subset \Omega$ since Ω is open, and so $\overline{D_{r/2}(z)} \subset \Omega$. Now, $\overline{D_{r/2}(z)}$ is compact and relatively compact in Ω, so $f_n \to f$ uniformly on $\overline{D_{r/2}(z)}$. Hence, $f|_{\overline{D_{r/2}(z)}}$ is continuous; specifically, f is continuous at z. Since our choice of z was arbitrary, we conclude that f is continuous on all of Ω. Now, let $\gamma : [0,1] \to \Omega$ be a closed path in Ω. Then, since each f_n is holomorphic on Ω, we know, by the Cauchy-Goursat Theorem, that

$$\int_\gamma f_n dz = 0.$$

Now, since $\gamma([0,1])$ is compact in Ω, $f_n \to f$ uniformly on γ. Hence, we are justified in swapping the limit and integral in the below:

$$\int_\gamma f dz = \int_\gamma \left[\lim_{n\to\infty} f_n\right] dz = \lim_{n\to\infty} \int_\gamma f_n dz = 0.$$

Since we saw above that f is continuous on Ω, the hypotheses of Morera's Theorem are satisfied, and so we conclude that f is holomorphic on Ω. $\square$

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The trigonometric functions $\sin z$ and $\cos z$ are defined by

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

Find formulae for the real and imaginary parts of $\sin(x+iy)$ and $\cos(x+iy)$, as explicit functions of (x, y). What are the zeros of $\sin z$ and $\cos z$ in the complex plane?

Answer: Let $z = x + iy \in \mathbb{C}$. Then

$$\begin{aligned}
\sin z = \sin(x+iy) &= \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i} \\
&= \frac{e^{-y+ix} - e^{y-ix}}{2i} \\
&= \frac{e^{-y}e^{ix} - e^{y}e^{-ix}}{2i} \\
&= \frac{e^{-y}\left(\cos x + i\sin x\right) - e^{y}\left(\cos(-x) + i\sin(-x)\right)}{2i} \\
&= \frac{e^{-y}\left(\cos x + i\sin x\right) - e^{y}\left(\cos x - i\sin x\right)}{2i} \\
&= \frac{\sin x}{2}\left(e^{-y} + e^{y}\right) + \frac{\cos x}{2i}\left(e^{-y} - e^{y}\right) \\
&= \frac{\sin x}{2}\left(e^{-y} + e^{y}\right) + i\frac{\cos x}{2}\left(e^{y} - e^{-y}\right).
\end{aligned}$$

Now, note that, since $e^r > 0$ for all $r \in \mathbb{R}$, $e^{-y} + e^y > 0$ for all $y \in \mathbb{R}$. Hence, if $\sin(x+iy) = 0$, then it must be the case that $\sin x = 0$, so $x = n\pi$ for some $n \in \mathbb{Z}$. Of course, for sin to have a zero, the imaginary part must also be 0; since $\cos n\pi = \pm 1$ for all $n \in \mathbb{Z}$, this implies that $e^y - e^{-y} = 0$, or $e^y = e^{-y}$; since the real exponential is monotonic, this only occurs when $y = -y$, which is to say when $y = 0$. Hence, the zeroes of the sin function are exactly those real numbers of the form $n\pi$ for $n \in \mathbb{Z}$.

Turning to cos, we see that

$$\begin{aligned}
\cos z = \cos(x+iy) &= \frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2} \\
&= \frac{e^{-y+ix} + e^{y-ix}}{2} \\
&= \frac{e^{-y}e^{ix} + e^{y}e^{-ix}}{2} \\
&= \frac{e^{-y}(\cos x + i\sin x) + e^{y}(\cos(-x) + i\sin(-x))}{2} \\
&= \frac{e^{-y}(\cos x + i\sin x) + e^{y}(\cos x - i\sin x)}{2} \\
&= \frac{\cos x}{2}\left(e^{-y} + e^{y}\right) + i\frac{\sin x}{2}\left(e^{-y} - e^{y}\right).
\end{aligned}$$

Again, $e^{-y} + e^{y} > 0$ for all y, so $\cos z$ has a zero only when $\cos x = 0$; namely, when $x = \frac{(2n+1)\pi}{2}$. Of course, the imaginary part must also be zero; since $\sin\frac{(2n+1)\pi}{2} = \pm 1$, this occurs only when $e^{-y} - e^{y} = 0$, which, as we saw above, only occurs when $y = 0$. Hence, the zeroes of $\cos z$ are exactly those real numbers of the form $\frac{(2n+1)\pi}{2}$ for $n \in \mathbb{Z}$.

♣

6

The function $\tan z = \frac{\sin z}{\cos z}$. Find the differential equation satisfied by its inverse functions $\arctan w$. As $\tan 0 = 0$, there is a unique branch of arctan defined in a neighborhood of 0, which satisfies $\arctan 0 = 0$. Find the power series for this branch of arctan.

Answer: First, note that

$$\frac{d\sin z}{dz} = \frac{d}{dz}\left(\frac{e^{iz} - e^{-iz}}{2i}\right) = \frac{1}{2i}\left(ie^{iz} + ie^{-iz}\right) = \frac{e^{iz} + e^{-iz}}{2} = \cos z$$

and

$$\frac{d\cos z}{dz} = \frac{d}{dz}\left(\frac{e^{iz} + e^{-iz}}{2}\right) = \frac{1}{2}\left(ie^{iz} - ie^{-iz}\right) = \frac{e^{iz} - e^{-iz}}{2i} = \sin z.$$

Hence,

$$\frac{d\tan z}{dz} = \frac{d}{dz}\left(\frac{\sin z}{\cos z}\right) = \cos z\frac{1}{\cos z} + \sin z\frac{1}{\cos^2 z}\sin z = 1 + \frac{\sin^2 z}{\cos^2 z} = \frac{1}{\cos^2 z}.$$

Therefore, since $\tan 0 = \frac{\sin 0}{\cos 0} = 0$ and

$$\frac{d\tan z}{dz}\Big|_{z=0} = \frac{1}{\cos^2 0} = 1,$$

we know, by the inverse function theorem, that tan has a local inverse, called arctan, in a neighborhood Ω of 0 such that $\arctan 0 = 0$. Furthermore, for $z \in \Omega$, $\tan(\arctan z) = z$, and so

$$1 = \frac{dz}{dz} = \frac{d\tan(\arctan z)}{dz} = \frac{1}{\cos^2(\arctan z)}\frac{d\arctan z}{dz}.$$

Note that, for any $w \in \mathbb{C}$, such that $\cos w \neq 0$ (i.e. where $\Re(w) \neq \frac{(2n+1)\pi}{2}$),

$$\tan^2 w + 1 = \frac{\sin^2 w}{\cos^2 w} + \frac{\cos^2 w}{\cos^2 w} = \frac{\sin^2 w + \cos^2 w}{\cos^2 w} = \frac{1}{\cos^2 w}.$$

Hence, the above reduces to

$$1 = (\tan^2(\arctan z) + 1)\frac{d\arctan z}{dz} = (z^2+1)\frac{d\arctan z}{dz},$$

so

$$\frac{d\arctan z}{dz} = \frac{1}{1+z^2}.$$

Now, for $|z| < 1$,

$$\frac{d\arctan z}{dz} = \frac{1}{1+z^2} = \sum_{n=0}^{\infty}(-z^2)^n = \sum_{n=0}^{\infty}(-1)^n z^{2n},$$

which converges absolutely for $|z| < 1$. Therefore, we can integrate term-by-term, so we see that the power series for $\arctan z$ on $D_1(0)$ is

$$\arctan z = \sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{2n+1}$$

♣

7

Let f be an entire function that satisfies $f(z+1) = f(z)$ and $f(z+i) = f(z)$ for every $z \in \mathbb{C}$. Prove that f is a constant.

Proof. Note that, if $n \in \mathbb{Z}$, then

$$f(z+n) = f(z+(n-1)+1) = f(z+(n-1)) = f(z+(n-2)+1) = f(z+(n-2)) = \cdots = f(z)$$

and

$$f(z+in) = f(z+i(n-1)+i) = f(z+i(n-1)) = f(z+i(n-2)+i) = f(z+i(n-2)) = \cdots = f(z).$$

Now, let $\lfloor\cdot\rfloor$ denote the floor function (i.e., for $x \in \mathbb{R}$, $\lfloor x\rfloor = \max\{a \in \mathbb{Z} | a \leq x\}$). Then, for any $z = x+iy$, note that $\lfloor x\rfloor, \lfloor y\rfloor \in \mathbb{Z}$ and so

$$\begin{aligned} f(z) = f(x+iy) &= f(\lfloor x\rfloor + (x-\lfloor x\rfloor) + i(\lfloor y\rfloor + (y - \lfloor y\rfloor))) \\ &= f((x-\lfloor x\rfloor) + i\lfloor y\rfloor + i(y-\lfloor y\rfloor)) \\ &= f((x-\lfloor x\rfloor) + i(y-\lfloor y\rfloor)) \end{aligned}$$

Given how $\lfloor \cdot \rfloor$ is defined, it must be the case that $x - \lfloor x \rfloor < 1$ and $y - \lfloor y \rfloor < 1$. Let $K = \{x + iy \in \mathbb{C} | 0 \leq x \leq 1, 0 \leq y \leq 1\}$. Hence,

$$\sup_{z \in \mathbb{C}} |f(z)| = \sup_{z \in K} |f(z)|.$$

Now, since K is compact and f is entire, $f|_K$ is a continuous function on a compact set and so has both a maximum and a minimum. Specifically, $|f(z)| \leq M$ for some $M \geq 0$ and for all $z \in K$. However, since we just saw that any maximum values of f must occur in K, this means that $|f(z)| \leq M$ for all $z \in \mathbb{C}$.

Therefore, f is an entire, bounded function and so, by Liouville's Theorem, must be constant. □

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