COMPLEX ANALYSIS HW 4

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Consider a meromorphic function $f : \mathbb{C} \to \mathbb{C}$ with Laurent series $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$ at the origin. Find a necessary and sufficient condition on the complex coefficients (c_n) so that $f(\mathbb{R}) \subseteq \mathbb{R}$.

Claim: If $f : \mathbb{C} \to \mathbb{C}$ is a meromorphic function with Laurent series $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$ at the origin, then $f(\mathbb{R}) \subseteq \mathbb{R}$ if and only if $c_n \in \mathbb{R}$ for all $n \in \mathbb{Z}$.

Proof. If $c_n \in \mathbb{R}$ for all n, then certainly $f(\mathbb{R}) \subseteq \mathbb{R}$, so we need only prove that the coefficients being in $\mathbb{R}$ is a necessary condition for $f(\mathbb{R}) \subseteq \mathbb{R}$. To that end, suppose $f(\mathbb{R}) \subseteq \mathbb{R}$. Since f is meromorphic, there can only be finitely many negative terms in the Laurent expansion, so there exists $N \in \mathbb{N}$ such that $f(z) = \sum_{n=-N}^{\infty} c_n z^n$ and $c_{-N} \neq 0$. Thus, we can re-write f as

$$f(z) = \sum_{n=-N}^{\infty} c_n z^n = \frac{1}{z^N} \sum_{n=0}^{\infty} c_{n-N} z^n.$$

Let $g(z) = \sum_{n=0}^{\infty} c_{n-N} z^n$. Then $z^N f(z) = g(z)$. Now, if $r \in \mathbb{R}$, then $r^N \in \mathbb{R}$ and $f(r) \in \mathbb{R}$; since the product of two reals is real, this implies that $g(r) \in \mathbb{R}$. Since our choice of r was arbitrary, we see that $g(\mathbb{R}) \subseteq \mathbb{R}$. To deal with this situation, let us first prove a small lemma:

Lemma 0.1. *If $h \in \mathcal{H}(\mathbb{C})$ such that $h(\mathbb{R}) \subseteq \mathbb{R}$, then $h'(\mathbb{R}) \subseteq \mathbb{R}$.*

Proof. By definition of the derivative,

$$h'(z) = \lim_{\zeta \to 0} \frac{h(z+\zeta) - h(z)}{\zeta}$$

for any $z \in \mathbb{C}$. Now, if $z \in \mathbb{R}$ and $\zeta = k \in \mathbb{R}$, then $h(z+k) \in \mathbb{R}$ and $h(z) \in \mathbb{R}$, so $h(z+k) - h(z) \in \mathbb{R}$. Therefore, since the derivative is well-defined,

$$h'(z) = \lim_{k \to 0} \frac{h(z+k) - h(z)}{k} \in \mathbb{R}.$$

Since our choice of $z \in \mathbb{R}$ was arbitrary, we conclude that $h'(\mathbb{R}) \subseteq \mathbb{R}$. □

First, note that

$$g(0) = \sum_{n=0}^{\infty} c_{n-N} 0^n = c_{-N} \in \mathbb{R}.$$

Now, since g is holomorphic and $g(\mathbb{R}) \subseteq \mathbb{R}$, we can apply the above lemma to conclude that $g'(\mathbb{R}) \subseteq \mathbb{R}$. Now,

$$g'(z) = \sum_{n=1}^{\infty} n c_{n-N} z^{n-1},$$

so

$$g'(0) = \sum_{n=1}^{\infty} n c_{n-N} 0^{n-1} = c_{1-N} \in \mathbb{R}.$$

g' also satisfies the hypotheses of the lemma, so $g''(\mathbb{R}) \in \mathbb{R}$, and so

$$g''(0) = \sum_{n=2}^{\infty} n(n-1) c_{n-N} 0^{n-2} = 2c_{2-N} \in \mathbb{R}$$

so $c_{2-N} \in \mathbb{R}$. Iterating this process, we see that at the jth stage,

$$g^{(j)}(0) = \sum_{n=j}^{\infty} n(n-1)\cdots(n-(j+1)) c_{n-N} 0^{n-j} = j(j-1)\cdots 2 \cdot 1 c_{j-N} \in \mathbb{R}$$

and so $c_{j-N} \in \mathbb{R}$. Therefore, we conclude that $c_n \in \mathbb{R}$ for all n. □

2

Let ζ be a point in the unit disk and $f \in \mathcal{H}(\overline{D} \backslash \{\zeta\})$. Show that for all sufficiently small $\epsilon > 0$

$$\int_{|z|=1} f(z)dz = \int_{|z-\zeta|=\epsilon} f(z)dz.$$

Be careful, use only results that have been established in this course! If $E = \{\zeta_1, \ldots, \zeta_N\}$ are points in D, and $f \in \mathcal{H}(\overline{D} \backslash E)$, then show that, for sufficiently small $\epsilon > 0$,

$$\int_{|z|=1} f(z)dz = \sum_{j=1}^{N} \int_{|z-\zeta_j|=\epsilon} f(z)dz.$$

Proof. Since f is holomorphic on the unit disk except at ζ, if $\rho = \min_{|z|=1}\{|\zeta - z|\}$, then f has a Laurent expansion in $D_\rho(\zeta)\backslash\{\zeta\}$ given by

$$\sum_{n=-\infty}^{\infty} c_n (z-\zeta)^n.$$

Furthermore, this converges uniformly on any compact subset of $D_\rho(\zeta)\backslash\{\zeta\}$. In fact, as $|z-\zeta|$ gets larger, the sum

$$g(z) = \sum_{n=-\infty}^{-1} c_n (z-\zeta)^n \tag{1}$$

converges more and more rapidly, so in fact $g \in \mathcal{H}(\mathbb{C}\backslash\{\zeta\})$. Now, let

$$h(z) = f(z) - g(z).$$

Then, since $f \in (\overline{D}\backslash\{\zeta\})$ and $g \in \mathcal{H}(\mathbb{C}\backslash\{\zeta\})$, h is holomorphic on $\overline{D}\backslash\{\zeta\}$. Now, ζ is a removable singularity of h, since in $D_\rho(\zeta)\backslash\{\zeta\}$

$$f(z) = \sum_{n=0}^{\infty} c_n(z-\zeta)^n + g(z)$$

and so

$$h(z) = f(z) - g(z) = \sum_{n=0}^{\infty} c_n(z-\zeta)^n$$

in this neighborhood, meaning we can analytically continue h at ζ using this power series. In doing so, we now have $h \in \mathcal{H}(\overline{D})$; hence, by Cauchy-Goursat,

$$\int_\gamma h(z)dz = 0$$

for any closed curve $\gamma \in \overline{D}$. Since $f(z) = g(z) + h(z)$ away from ζ,

$$\int_\gamma f(z)dz = \int_\gamma g(z)dz + \int_\gamma h(z)dz = \int_\gamma g(z)$$

for any γ that doesn't pass through ζ. Since the sum in (1) converges uniformly outside any small disc centered at ζ, it converges uniformly on γ and so

$$\int_\gamma g(z)dz = \sum_{n=-\infty}^{-1} \int_\gamma c_n(z-\zeta)^n dz = \sum_{n=1}^{\infty} \int_\gamma \frac{c_{-n}}{(z-\zeta)^n} dz.$$

Now, if $n > 1$, then

$$\frac{1}{(z-\zeta)^n} = \frac{d}{dz}\left[\frac{-1}{(n-1)(z-\zeta)^{n-1}}\right]$$

so, by the result proved in problem 1 of homework 2,

$$\int_\gamma \frac{c_{-n}}{(z-\zeta)^n} dz = \frac{-c_{-n}}{(n-1)(\gamma(1)-\zeta)^{n-1}} - \frac{-c_{-n}}{(n-1)(\gamma(0)-\zeta)^{n-1}} = 0$$

since γ is a closed curve and so $\gamma(1) = \gamma(0)$. Therefore,

$$\int_\gamma f(z)dz = \int_\gamma g(z)dz = \int_\gamma \frac{c_{-1}}{z-\zeta} dz = c_{-1} \int_\gamma \frac{1}{z-\zeta} dz.$$

Since this is true for any closed $\gamma \in \overline{D}\backslash\{\zeta\}$, the above result holds for γ denoting the circle $|z| = 1 - \delta$ for any $\delta > 0$, and so we see that

(2) $$\int_{|z|=1} f(z)dz = c_{-1} \int_{|z|=1} \frac{1}{z-\zeta} dz.$$

Now,

$$\int_{|z|=1} \frac{1}{z-\zeta} dz = 2\pi i$$

by Cauchy's Integral Formula, since the function 1 is holomorphic in D.

Now, recall that c_{-1} is defined to be $\frac{1}{2\pi i}\int_{|z-\zeta|=\epsilon} f(z)dz$ for $0 < \epsilon < \rho$. Hence, plugging this data into (2), we see that

$$\int_{|z|=1} f(z)dz = \frac{1}{2\pi i}\int_{|z-\zeta|=\epsilon} f(z)dz 2\pi i = \int_{|z-\zeta|=\epsilon} f(z)dz.$$

To prove the case where $f \in \mathcal{H}(\overline{D}\backslash E)$ where $E = \{\zeta_1, \dots, \zeta_N\}$, let

$$\rho_0 = \min_{|z|=1}\{|\zeta_1 - z|, \dots, |\zeta_N - z|\},$$

let

$$\rho_i = \min_{j\neq i}\{|\zeta_i - \zeta_j|\}$$

and let

$$\rho = \min_{0\leq i\leq N}\{\rho_i\}.$$

Then for each $i = 1, \dots, N$, f is holomorphic on the punctured disc of radius ρ centered at ζ_i and so, for each i,

$$f(z) = \sum_{n_i=-\infty}^{\infty} c_{n_i} z^{n_i}$$

on $D_\rho(\zeta_i)\backslash\{\zeta_i\}$. Let

$$g_i(z) = \sum_{n=-\infty}^{-1} c_{n_i} z^{n_i} \tag{3}$$

and let

$$g(z) = \sum_{i=1}^{N} g_i(z)$$

Then, by the same argument for the g defined above, g is holomorphic on $\mathbb{C}\backslash E$ and so

$$h(z) = f(z) - g(z)$$

is holomorphic on $\overline{D}\backslash E$. Now, in $D_\rho(\zeta_i)$,

$$h(z) = f(z) - g(z) = \sum_{n_i=0}^{\infty} c_{n_i}(z-\zeta_i)^{n_i},$$

so each ζ_i is a removable singularity of h, which we can, thus, extend to a holomorphic function on all of $\overline{D}$. Thus, by Cauchy-Goursat,

$$\int_\gamma h(z)dz = 0$$

for any closed curve γ in $\overline{D}$. Since $f(z) = g(z) + h(z)$ away from ζ_i,

$$\int_\gamma f(z)dz = \int_\gamma g(z)dz + \int_\gamma h(z)dz = \int_\gamma g(z)$$

for any γ that doesn't pass through any of the ζ_i. Since $g = \sum g_i$ is a finite sum, we can swap sum and integral to see that

$$\int_\gamma f(z)dz = \sum_{i=1}^{N} \int_\gamma g_i(z). \tag{4}$$

Furthermore, since γ does not pass through any of the ζ_i, there is some $\delta > 0$ such that $D_\delta(\zeta_i)$ does not intersect γ for all i; since the sum in (3) converges uniformly outside $D_\delta(\zeta_i)$, at ζ, it converges uniformly on γ and so

$$\int_\gamma g_i(z)dz = \sum_{n_i=-\infty}^{-1} \int_\gamma c_{n_i}(z-\zeta)^{n_i}dz = \sum_{n_i=1}^{\infty} \int_\gamma \frac{c_{-n_i}}{(z-\zeta)^{n_i}}dz.$$

Now, if $n_i > 1$, then

$$\frac{1}{(z-\zeta)^{n_i}} = \frac{d}{dz}\left[\frac{-1}{(n_i-1)(z-\zeta)^{n_i-1}}\right]$$

so, by the result proved in problem 1 of homework 2,

$$\int_\gamma \frac{c_{-n_i}}{(z-\zeta)^{n_i}}dz = \frac{-c_{-n_i}}{(n_i-1)(\gamma(1)-\zeta)^{n_i-1}} - \frac{-c_{-n_i}}{(n_i-1)(\gamma(0)-\zeta)^{n_i-1}} = 0$$

since γ is a closed curve and so $\gamma(1) = \gamma(0)$. Therefore,

$$\int_\gamma g_i(z)dz = \int_\gamma \frac{c_{-1_i}}{z-\zeta}dz = c_{-1_i}\int_\gamma \frac{1}{z-\zeta}dz.$$

Since this is true for any closed $\gamma \in \overline{D}\backslash E$, the above result holds for γ denoting the circle $|z| = 1-\delta$ for any $\rho > \delta > 0$, and so we see that

$$\int_{|z|=1} g_i(z)dz = c_{-1_i}\int_{|z|=1} \frac{1}{z-\zeta}dz. \tag{5}$$

Now,

$$\int_{|z|=1} \frac{1}{z-\zeta_i}dz = 2\pi i$$

by Cauchy's Integral Formula, since the function 1 is holomorphic in D.

Now, recall that c_{-1_i} is defined to be $\frac{1}{2\pi i}\int_{|z-\zeta_i|=\epsilon} f(z)dz$ for $0 < \epsilon < \rho$. Hence, plugging this data into (5), we see that

$$\int_{|z|=1} g_i(z)dz = \frac{1}{2\pi i}\int_{|z-\zeta_i|=\epsilon} f(z)dz 2\pi i = \int_{|z-\zeta_i|=\epsilon} f(z)dz.$$

Therefore, by (4),

$$\int_{|z|=1} f(z)dz = \sum_{i=1}^{N}\int_{|z|=1} g_i(z)dz = \sum_{i=1}^{N}\int_{|z-\zeta_i|=\epsilon} f(z)dz$$

for any $\epsilon < \rho$. □

3

Compute the following integrals

$$\int_{|z|=1} e^{\left(z+\frac{1}{z}\right)} dz$$

$$\int_{|z|=2} \frac{dz}{1+z^2}$$

Answer: The Laurent series for $f(z) = e^{z+1/z}$ in $\mathbb{C}\backslash\{0\}$ is of the form

$$\sum_{n=-\infty}^{\infty} c_n z^n$$

for

$$c_n = \frac{1}{2\pi i} \int_{|z|=\rho} f(z) z^{-(n+1)} dz$$

for $\rho > 0$ and all n. Specifically, if we let $\rho = 1$,

$$c_{-1} = \int_{|z|=1} f(z) dz.$$

By the uniqueness of the Laurent expansion, if we can find a series of the above form that converges locally uniformly to f, then the coefficient on $\frac{1}{z}$ must be $\int_{|z|=1} f(z)dz$. To that end, note that

$$e^w = \sum_{n=0}^{\infty} \frac{w^n}{n!}$$

for any $w \in \mathbb{C}$ and that this series converges uniformly. Hence, for any $z \neq 0$,

$$e^{z+1/z} = \sum_{n=0}^{\infty} \frac{(z+1/z)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{k=0}^{n} \binom{n}{k} \frac{z^k}{z^{n-k}} \right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{k=0}^{n} \binom{n}{k} z^{2k-n} \right)$$

which converges locally uniformly on $\mathbb{C}\backslash\{0\}$. Hence, this is the Laurent expansion of $e^{z+1/z}$, so we need to determine the coefficient on $\frac{1}{z}$. It's clear that, since the power on z is $2k-n$, only odd n's will contribute to this coefficient; specifically, for each odd n, we will pick up a

$$\frac{1}{n!} \binom{n}{\frac{n-1}{2}} = \frac{1}{n!} \frac{n!}{\left(\frac{n-1}{2}\right)! \left(\frac{n+1}{2}\right)!} = \frac{1}{\left(\frac{n-1}{2}\right)! \left(\frac{n+1}{2}\right)!}$$

Therefore, the coefficient on $\frac{1}{z}$ is

$$\sum_{n=0}^{\infty} \frac{1}{\left(\frac{(2n+1)-1}{2}\right)! \left(\frac{(2n+1)+1}{2}\right)!} = \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!}.$$

Hence, we conclude that

$$\int_{|z|=1} e^{z+1/z} dz = \sum_{n=0}^{\infty} \frac{1}{(n+1)!n!}.$$

Turning to the second integral, note that $1+z^2 = (z-i)(z+i)$, so $\frac{1}{1+z^2}$ is holomorphic on $D_2(0)$ except at $\pm i$. The result proved in problem 2 above did not depend on the size of the disc, so that result tells us that

$$\int_{|z|=2} \frac{dz}{1+z^2} = \int_{|z-i|=1/2} \frac{dz}{1+z^2} + \int_{|z+i|=1/2} \frac{dz}{1+z^2}.$$

Now, on $D_{1/2}(i)$, the function $g(z) = \frac{1}{z+i}$ is holomorphic, so, by Cauchy's Integral Formula,

$$\frac{1}{2i} = g(i) = \frac{1}{2\pi i} \int_{|z-i|=1/2} \frac{g(z)}{z-i} dz = \frac{1}{\pi i} \int_{|z-i|=1/2} \frac{1}{(z+i)(z-i)} dz = \frac{1}{2\pi i} \int_{|z-i|=1/2} \frac{dz}{1+z^2},$$

so $\int_{|z-i|=1/2} \frac{dz}{1+z^2} = \frac{2\pi i}{2i} = \pi$.

On the other hand, $h(z) = \frac{1}{z-i}$ is holomorphic on $D_{1/2}(-i)$, so Cauchy's Integral Formula tells us

$$\frac{1}{-2i} = h(-i) = \frac{1}{2\pi i} \int_{|z+i|=1/2} \frac{h(z)}{z+i} dz = \frac{1}{2\pi i} \int_{|z+i|=1/2} \frac{dz}{1+z^2},$$

so $\int_{|z+i|=1/2} \frac{dz}{1+z^2} = \frac{2\pi i}{-2i} = -\pi$. Hence,

$$\int_{|z|=2} \frac{dz}{1+z^2} = \int_{|z-i|=1/2} \frac{dz}{1+z^2} + \int_{|z+i|=1/2} \frac{dz}{1+z^2} = \pi - \pi = 0.$$

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4

The function $f(z) = \frac{1}{(z-2)(z-1)}$ is meromorphic. Locate its poles. This function can be represented by three different Laurent series centered on $z = 0$. Find them and give their domains of convergence.

Answer: f is certainly holomorphic except at $z = 2$ and $z = 1$, which are the poles of f. Note that

$$\frac{1}{(z-2)(z-1)} = \frac{1}{z-2} - \frac{1}{z-1} = \frac{\frac{-1}{2}}{1-\frac{z}{2}} - \frac{-1}{1-z}. \tag{6}$$

Now, $\frac{\frac{-1}{2}}{1-z/2}$ is holomorphic on $D_2(0)$ and has power series given by the geometric series:

$$\frac{-1}{2} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right) = -\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}.$$

On the other hand, $\frac{-1}{1-z}$ is holomorphic on $D_1(0)$ and has power series given by the geometric series:

$$-\sum_{n=0}^{\infty} z^n.$$

Hence, applying these power series to (6), we see that

$$f(z) = -\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} + \sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} z^n - \frac{z^n}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{2^{n+1}-1}{2^{n+1}} z^n,$$

which is a Laurent (actually Taylor) series expansion of f in $D_1(0)$.

Now, the Laurent expansion of f in $A(0,1,2)$, i.e. the annulus given by $1 < |z| < 2$ is defined to be

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$$

where

$$c_n = \frac{1}{2\pi i} \int_{|z|=\rho} f(z) z^{-(n+1)} = \frac{1}{2\pi i} \int_{|z|=\rho} \frac{z^{-(n+1)}}{(z-2)(z-1)}$$

and $1 < \rho < 2$. Now, since the result proved in problem 2 above did not depend on the size of the circle, and the poles of f are at 1 and 2, we can use that result to see that

(7)

$$c_n = \frac{1}{2\pi i} \int_{|z|=\rho} \frac{z^{-(n+1)}}{(z-2)(z-1)} = \frac{1}{2\pi i} \int_{|z|=1/4} \frac{z^{-(n+1)}}{(z-2)(z-1)} + \frac{1}{2\pi i} \int_{|z-1|=1/4} \frac{z^{-(n+1)}}{(z-2)(z-1)}$$

if $n \geq 0$ and (since then $z^{-(n+1)}$ has a pole at 0)

$$c_n = \frac{1}{2\pi i} \int_{|z|=\rho} \frac{z^{-(n+1)}}{(z-2)(z-1)} = \frac{1}{2\pi i} \int_{|z-1|=1/4} \frac{z^{-(n+1)}}{(z-2)(z-1)} \tag{8}$$

otherwise. Now, by the Cauchy Formula,

$$\begin{aligned} -1 = \frac{(1)^{-(n+1)}}{1-2} &= \frac{1}{2\pi i} \int_{|z-1|=1/4} \frac{\frac{z^{-(n+1)}}{z-2}}{z-1} dz \\ &= \frac{1}{2\pi i} \int_{|z-1|=1/4} \frac{z^{-(n+1)}}{(z-2)(z-1)} dz. \end{aligned}$$

On the other hand, if $n \geq 0$, then

$$\frac{1}{2\pi i} \int_{|z|=1/4} \frac{z^{-(n+1)}}{(z-2)(z-1)} dz = \frac{1}{2\pi i} \int_{|z|=1/4} \frac{\frac{1}{(z-2)(z-1)}}{z^{n+1}} dz$$

is just the nth coefficient in the power series expansion of $\frac{1}{(z-2)(z-1)}$, which we calculated above to be $\frac{2^{n+1}-1}{2^{n+1}}$. Hence, if $n \geq 0$, (7) tells us that

$$c_n = \frac{2^{n+1}-1}{2^{n+1}} - 1 = \frac{-1}{2^{n+1}}$$

and if $n < 0$, then (8) tells us that

$$c_n = -1.$$

Hence,

$$f(z) = \sum_{n=-\infty}^{-1} -z^n + \sum_{n=0}^{\infty} \frac{-1}{2^{n+1}} z^n,$$

for $z \in A(0, 1, 2)$.

Finally, if we consider the annulus $A(0, 2, \infty)$ then, on this annulus, f has a Laurent expansion as before except with $\rho > 2$. Hence, if $n \geq 0$,

(9)

$$c_n = \frac{1}{2\pi i} \int_{|z|=1/4} \frac{z^{-(n+1)}}{(z-2)(z-1)} dz + \frac{1}{2\pi i} \int_{|z-1|=1/4} \frac{z^{-(n+1)}}{(z-2)(z-1)} dz + \frac{1}{2\pi i} \int_{|z-2|=1/4} \frac{z^{-(n+1)}}{(z-2)(z-1)} dz$$

and

$$(10) \quad c_n = \frac{1}{2\pi i} \int_{|z-1|=1/4} \frac{z^{-(n+1)}}{(z-2)(z-1)} dz + \frac{1}{2\pi i} \int_{|z-2|=1/4} \frac{z^{-(n+1)}}{(z-2)(z-1)} dz$$

otherwise.

Now, we've already calculated all these terms except

$$\begin{aligned} 2^{-(n+1)} = \frac{2^{-(n+1)}}{(2-1)} &= \frac{1}{2\pi i} \int_{|z-2|=1/4} \frac{\frac{z^{-(n+1)}}{z-1}}{z-2} dz \\ &= \frac{1}{2\pi i} \int_{|z-2|=1/4} \frac{z^{-(n+1)}}{(z-2)(z-1)} dz, \end{aligned}$$

Hence, if $n \geq 0$,

$$c_n = \frac{2^{n+1}-1}{2^{n+1}} - 1 + 2^{-(n+1)} = \frac{-1}{2^{n+1}} + \frac{1}{2^{n+1}} = 0.$$

If $n < 0$, then

$$c_n = -1 + 2^{-(n+1)}.$$

Hence, plugging into (9) and (10), we see that

$$f(z) = \sum_{n=-\infty}^{-1} (-1 + 2^{-(n+1)}) z^n = \sum_{n=1}^{\infty} \frac{-1 + \frac{1}{2^{n-1}}}{z^n}$$

outside the circle of radius 2.

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5

Let $f \in \mathcal{H}(D_r(0)^c)$, that is f is holomorphic outside of the disk of radius r. The function $F(z) = f(\frac{1}{z})$ is then holomorphic in a punctured disk, centered at zero. We say that:

- **(a):** f has a pole at infinity, if F has a pole at zero.
- **(b):** f has an essential singularity at infinity, if F has an essential singularity at zero.
- **(c):** f is analytic at infinity, if F has a removable singularity at zero.

Prove that a nonconstant, entire function either has a pole or an essential singularity at infinity. Give examples of entire functions with each of these types of singularities. Give an example of a function with a removable singularity at infinity.

Proof. Suppose f is a non-constant entire function. Clearly, f cannot be bounded, else, by Liouville's Theorem, f would be constant. Furthermore, since f is entire, it is continuous on all of $\mathbb{C}$ and so is bounded on any compact subset of $\mathbb{C}$. Since any closed disc centered at the origin is compact, we see that, for $|z| \leq N$, $|f(z)|$ is bounded, so $|f(z)|$ is unbounded as $|z| \to \infty$. This leaves two possibilities: either $|f(z)| \to \infty$ as $|z| \to \infty$, or $|f(z)|$ has no limit as $|z| \to \infty$.

In the former case, this implies that $F(z) = f(\frac{1}{z})$ is such that $|F(z)| \to \infty$ as $z \to 0$, meaning that F has a pole at zero and so f has a pole at infinity. In the latter case, we see that $F(z) = f(\frac{1}{z})$ is unbounded as $z \to 0$ and yet $|F(z)| \not\to \infty$ as $z \to 0$, meaning that F is not meromorphic at 0. Since having an essential singularity is equivalent to not being meromorphic, this in turn means that F has an essential singularity at 0, so f has an essential singularity at infinity. □

Examples: First, consider $f(z) = z^2$. Then f is its own power series and thus is certainly entire. Now, for $|z| > 1$, $|z|^2 > |z|$, so $|f(z)| = |z^2| = |z|^2 > |z|$. Hence, $|f(z)| \to \infty$ as $|z| \to \infty$, which is equivalent to saying that $|F(z)| \to \infty$ as $z \to 0$ for $F(z) = f(\frac{1}{z})$, so F has a pole at 0 and so f has a pole at infinity.

Now, consider $f(z) = e^z$. Note first that f is an entire function. Then, if $r \in \mathbb{R}$ such that $r > 0$, then $e^r > r$, so $|f(r)| = |e^r| = e^r \to +\infty$ as $r \to +\infty$. Hence, f is certainly unbounded. However,

$$e^{-r} = \frac{1}{e^r} \to 0$$

as $r \to +\infty$, so we see that $|f(z)|$ cannot have a limit at infinity. Hence, f does not have a pole at infinity and so, by the above argument, must have an essential singularity at infinity.

Finally, consider $f(z) = z \sin\left(\frac{1}{z}\right)$. Since $\sin z$ is entire, $\sin\left(\frac{1}{z}\right) \in \mathcal{H}(\mathbb{C}\backslash\{0\})$ and so, since z is entire, $f \in \mathcal{H}(\mathbb{C}\backslash\{0\})$. Now, consider $F(z) = f\left(\frac{1}{z}\right) = \frac{\sin z}{z}$, which is holomorphic on $\mathbb{C}\backslash\{0\}$. Now, for $z \neq 0$,

$$zF(z) = z\frac{\sin z}{z} = \sin z \to 0$$

as $z \to 0$ (since $\sin 0 = 0$), so, by Riemann's Removable Singularities Theorem, F has a removable singularity at 0, meaning that f has a removable singularity at infinity.

♣

6

Show that if $f \in \mathcal{H}(D_0\backslash\{0\})$ has a nontrivial pole at 0, then e^f has an essential singularity at 0. A pole is nontrivial if it is not a removable singularity, i.e. some coefficient of a negative degree term in the Laurent expansion is non-zero.

Proof. Suppose f has a nontrivial pole at 0. Then f has a Laurent series

$$f(z) = \sum_{n=-N}^{\infty} c_n z^n$$

where $N > 0$, $c_{-N} \neq 0$ and this series converges in some punctured disc D^* about the origin. Now, $g(z) = e^z$ has a uniformly convergent power series

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Hence, for any $z \in D^*$,

$$e^{f(z)} = \sum_{n=0}^{\infty} \frac{(f(z))^n}{n!}$$

where this series converges uniformly for all $z \neq 0$. Hence,

$$e^{f(z)} = \sum_{n=0}^{\infty} \frac{\left(\sum_{j=-N}^{\infty} c_j z^j\right)^n}{n!}$$

in D^*. If $\epsilon > 0$, then for any $z \in D^*$ such that $|z| > \epsilon$, both sums converge uniformly, so we can swap the sums:

$$e^{f(z)} = \sum_{j=-N}^{\infty} \sum_{n=0}^{\infty} \frac{1}{n!} c_j^n z^{nj}.$$

Since our choice of ϵ was arbitrary, we see that this is a Laurent series for e^f. If we restrict our attention to the term given by $j = -N$, note that $n(-N) \to -\infty$ as $n \to \infty$, so we see that there are terms in this Laurent expansion of arbitrarily negative orders. By the uniqueness of the Laurent expansion, this is the Laurent expansion of e^f centered at 0, so we see that e^f has infinitely many negative terms in its Laurent expansion, which is precisely what it means for e^f to have an essential singularity at 0. □

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