COMPLEX ANALYSIS HW 5

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Prove that at each $c \in \mathbb{C}^*$ the complex square root admits exactly two germs, and hence exactly two analytic continuations along every continuous curve that does *not* pass through the origin in the plane.

Proof. Let f represent the principal branch of the complex square root; that is

$$f(z) = f(x+iy) = \sqrt{\frac{\sqrt{x^2+y^2}+x}{2}} + \frac{z}{|z|}\sqrt{\frac{\sqrt{x^2+y^2}-x}{2}}$$

for $z \in \mathbb{C}^*$. Let $g(z) = -f(z)$. Then for any $z \in \mathbb{C}^*$,

$$z = (f(z))^2 = (-f(z))^2 = (g(z))^2$$

and, moreover, $f(z) \neq -f(z) = g(z)$, since $0^2 = 0 \neq z$. Hence, f and g define different germs f_c and g_c for each $c \in \mathbb{C}^*$.

On the other hand, let h_c be a germ of the complex square root at $c \in \mathbb{C}^*$; that is, if (U, h) is a representative of h_c, then $(h(z))^2 = z$ for $z \in U$. Then for all $z \in U$,

$$(f(z) - h(z))(f(z) + h(z)) = (f(z))^2 - (h(z))^2 = z - z = 0.$$

Hence, since we can't multiply non-zero complex numbers and get zero, we see that either $f(z) - h(z) = 0$ or $f(z) + h(z) = 0$. In the former case, $h(z) = f(z)$; in the latter, $h(z) = -f(z) = g(z)$. Since our choice of $z \in U$ was arbitrary, we see that h must agree with either f or g at each point of U. Suppose, without loss of generality, that $h(c) = f(c)$. Now, since h is continuous on U, there exists $\delta_1 > 0$ such that, for $w \in U$ such that $|w - c| < \delta_1$,

$$|h(w) - h(c)| < |f(c)|.$$

Also, there exists $\delta_2 > 0$ such that $|w-c| < \delta_2$ implies that $|-f(w)+f(c)| < |f(c)|$. Let $\epsilon = \min\{\delta_1, \delta_2\}$. Then, for $|w_0 - c| < \epsilon$, $h(w_0) = f(w_0)$ or $h(w_0) = g(w_0)$; if $h(w_0) = g(w_0)$, then

$$|h(w_0) - h(c)| = |g(w_0) - f(c)| = |-f(w_0) - f(c)| > |c|$$

since $-f(w_0) \in D_{|f(c)|}(-f(c))$. Therefore, it must be the case that $h(w_0) = f(w_0)$; since our choice of w_0 was arbitrary, we see that $h(z) = f(z)$ for all $z \in D_\epsilon(c)$, so, by the analytic continuation principle, $h = f$ on U, and so $h_c = f_c$. □

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Prove that there exist germs of holomorphic functions with representatives (U, f) for which there does *not* exist any "largest" open set $V \subseteq \mathbb{C}$ and holomorphic function $g \in \mathcal{H}(V)$ such that $U \subseteq V$ and $g|_U = f$ on U.

Proof. Let $f(z) = \log z$ be the principal branch of the logarithm; that is,

$$f(re^{i\theta}) = \log |r| + i\theta$$

defined for $-\pi < \theta < \pi$. Let define

$$g(re^{i\theta}) = \log |r| + i\theta$$

defined on $-\pi/2 < \theta < 3\pi/2$. Then $f(1) = 0 = g(1)$ and, furthermore, for $-\pi/2 < \theta < \pi$,

$$f(re^{i\theta}) = \log |r| + i\theta = g(re^{i\theta}).$$

Since $\{re^{i\theta} | -\pi/2 < \theta < \pi\} \subset \mathbb{C}$ is an open set, we see that the germ f_1 of f at 1 is the same as the germ g_1 of g at 1; that is, $f_1 = g_1$. Furthermore, if we let $U = \{re^{i\theta} | -\pi < \theta < \pi\}$ and $V = \{re^{i\theta} | -\pi/2 < \theta < 3\pi/2\}$, then (U, f) and (V, g) are representatives of this germ. Now, suppose there exists an open set $W \subset \mathbb{C}$ and a function $h \in \mathcal{H}(W)$ such that $h|_U = f$ and $h|_V = g$. Then $W \supset U \cup V = \mathbb{C} \backslash \{0\}$. Also,

$$h(e^{i\pi}) = g(e^{i\pi}) = \log 1 + i\pi = i\pi.$$

Furthermore, since h is holomorphic and, therefore, continuous, on $W \supset \mathbb{C} \backslash \{0\}$ and $e^{i\pi} = -1 = e^{-i\pi}$

$$i\pi = h(e^{i\pi}) = \lim_{n\to\infty} h(e^{i\left(-\pi + \frac{1}{n}\right)}) = \lim_{n\to\infty} \log 1 + i\left(-\pi + \frac{1}{n}\right) = \lim_{n\to\infty} -i\pi + \frac{i}{n} = -i\pi.$$

From this contradiction, then, we conclude that there is no such pair (W, h). $\square$

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(166.1): Prove that the germs defined by $z \mapsto z$ and $z \mapsto 1/z$ at $z := 1$ lie in different connected components of $\mathcal{O}$.

Proof. Let f_1 denote the germ defined by $f : z \mapsto z$ at $z = 1$ and let g_1 denote the germ defined by $g : z \mapsto 1/z$ at $z = 1$. Then, since f is entire, $N(\mathbb{C}, f)$ is an open set in $\mathcal{O}$ containing f_1; we claim that $N(\mathbb{C}, f)$ is the connected component of $\mathcal{O}$ containing f_1. More generally, we prove the following lemma:

Lemma 0.1. *If $f \in \mathcal{H}(\mathbb{C})$, then $N(\mathbb{C}, f)$ is the connected component of the germ f_c of f at $c \in \mathbb{C}$.*

Proof. If $p : \mathcal{O} \to \mathbb{C}$ is the standard projection, then $p|_{N(\mathbb{C},f)} : N(\mathbb{C}, f) \to \mathbb{C}$ is certainly surjective and continuous. Also, since elements of $N(\mathbb{C}, f)$ are simply of the form f_a for $a \in \mathbb{C}$, $p(f_a) = p(f_b)$ if and only if $a = b$, so $p|_{N(\mathbb{C},f)}$ is injective. Finally, since p is a local homeomorphism, p is an open map; a bijective continuous open map is a homeomorphism, so $p|_{N(\mathbb{C},f)} : N(\mathbb{C}, f) \to \mathbb{C}$ is a homeomorphism. Hence, since $\mathbb{C}$ is connected, we see that $N(\mathbb{C}, f)$ is connected.

Thus, the only obstacle to our claim is if $N(\mathbb{C}, f) \subsetneq \mathcal{N}$ where $\mathcal{N}$ is connected. Now, $\mathcal{N} = N(\mathbb{C}, f) \cup (\mathcal{N} \backslash N(\mathbb{C}, f)) = N(\mathbb{C}, f) \cup [\mathcal{N} \cap (\mathcal{O} \backslash N(\mathbb{C}, f)]$, so if we can demonstrate that $\mathcal{O} \backslash N(\mathbb{C}, f)$ is open, this will suffice to show that any such $\mathcal{N}$ is disconnected. To that end, suppose $h_a \in \mathcal{O} \backslash N(\mathbb{C}, f)$ and let (U, h) be a representative of h_a. Then elements of $N(U, h)$ are of the form h_z for $z \in U$. Suppose $h_z \in N(\mathbb{C}, f)$ for some z. Then $h_z = f_z$ and so there exists an open set V containing z such that $f|_V = h|_V$. Specifically, if $W = U \cap V$, then $f|_W = h|_W$. Since $W \subset U \cap \mathbb{C}$, this implies, by the identity theorem, that $f = h$ on all of $U \cap \mathbb{C} = U$. However, this in turn implies that $(U, h)|_a = f_a$, meaning that $h_a = f_a$, so $h_a \in N(\mathbb{C}, f)$, contradicting our assumption. Therefore, we conclude that $N(U, h) \subset \mathcal{O} \backslash N(\mathbb{C}, f)$, so $\mathcal{O} \backslash N(\mathbb{C}, f)$ is open. Thus, indeed, $N(\mathbb{C}, f)$ is the connected component of $\mathcal{O}$ containing f_c. □

Now, it remains only to show that $g_1 \notin N(\mathbb{C}, f)$. Let U be an open set containing 1; then (U, g) is a representative of g_1. Since the only point in $\mathbb{C}$ at which $\frac{1}{z} = z$ is 1, we see that for all $z \in U$ such that $z \neq 1$,

$$f_z(z) = f(z) = z \neq \frac{1}{z} = g(z) = g_z(z).$$

Hence, there is no open subset $W \subset U = U \cap \mathbb{C}$ containing 1 such that $f|_W = g|_W$, and so

$$g_1 = (U, g)_1 \neq (U, f)_1 = f_1,$$

meaning $g_1 \notin N(\mathbb{C}, f)$. Therefore, since $N(\mathbb{C}, f)$ is the connected component of $\mathcal{O}$ containing f_1, we conclude that f_1 and g_1 lie in different connected components. □

(166.2): For all $f, g \in \mathcal{H}(\mathbb{C})$, prove that if there exists $c \in \mathbb{C}$ such that the germs f_c and g_c lie in the same connected component of $\mathcal{O}$, then $f = g$.

Proof. By Lemma 0.1 proved above, $N(\mathbb{C}, f)$ is the connected component of f_c in $\mathcal{O}$. Hence, if g_c lies in the same connected component as f_c, then $g_c \in N(\mathbb{C}, f)$. Hence, $g_c = f_c$ so there exists an open set U containing c such that $f|_U = g|_U$. Hence, $(f - g)|_U = 0$, so, by the principle of analytic continuation, since $f - g \in \mathcal{H}(\mathbb{C})$, $f - g = 0$ on all of $\mathbb{C}$, meaning $f = g$. □

(166.3): For each $f \in \mathcal{H}(\mathbb{C})$, prove that $N(\mathbb{C}, f)$ is the connected component of the germ f_0 of f at 0 in $\mathbb{C}$.

Proof. This is a special case of Lemma 0.1 above. □

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Verify that Γ is holomorphic at every z with $0 < \Re(z)$.

Proof. First, suppose $f(x) = \exp(ax \log b)$. Then

$$f'(x) = \exp(ax \log b) \cdot a \log b = a \exp(ax \log b) \log b.$$

Note, specifically, that $f'(0) = a \exp(0) \log b = a \log b$. Now, suppose $h \in \mathbb{R}$ is small and that $z \in \mathbb{C}$ such that $\Re(z) > 0$; then

$$\begin{aligned}
\lim_{h\to 0} \frac{\Gamma(z+h) - \Gamma(z)}{h} &= \lim_{h\to 0} \frac{\int_0^\infty \exp[(z+h-1)\log t - t]dt - \int_0^\infty \exp[(z-1)\log t - t]dt}{h} \\
&= \lim_{h\to 0} \frac{1}{h} \int_0^\infty \exp[(z+h-1)\log t] \exp(-t) - \exp[(z-1)\log t] \exp(-t) dt \\
&= \lim_{h\to 0} \frac{1}{h} \int_0^\infty \exp(-t) \exp[(z-1)\log t] \left[\exp(h \log t) - 1\right] dt \\
&= \lim_{h\to 0} \int_0^\infty \exp[(z-1)\log t - t] \frac{\exp(h \log t) - \exp(0 \cdot \log t)}{h} dt.
\end{aligned}$$

Now,

$$\frac{\exp((h+0)\log t) - \exp(0 \log t)}{h}$$

is just the derivative of $x \exp(h \log t)$ at 0, which we calculated above to be $\log t$. Hence,

$$\frac{\partial \Gamma}{\partial x}|_z = \lim_{h\to 0} \frac{\Gamma(z+h) - \Gamma(z)}{h} = \int_0^\infty \exp[(z-1)\log t - t] \log t dt.$$

On the other hand, if $h \in \mathbb{R}$ is still small,

$$\begin{aligned}
\lim_{h\to 0} \frac{\Gamma(z+ih) - \Gamma(z)}{h} &= \lim_{h\to 0} \frac{\int_0^\infty \exp[(z+ih-1)\log t - t]dt - \int_0^\infty \exp[(z-1)\log t - t]dt}{h} \\
&= \lim_{h\to 0} \frac{1}{h} \int_0^\infty \exp[(z+ih-1)\log t] \exp(-t) - \exp[(z-1)\log t] \exp(-t) dt \\
&= \lim_{h\to 0} \frac{1}{h} \int_0^\infty \exp(-t) \exp[(z-1)\log t] \left[\exp(ih \log t) - 1\right] dt \\
&= \lim_{h\to 0} \int_0^\infty \exp[(z-1)\log t - t] \frac{\exp(ih \log t) - \exp(0 \cdot \log t)}{h} dt.
\end{aligned}$$

Now,

$$\frac{\exp((ih+0)\log t) - \exp(0 \log t)}{h}$$

is just the derivative of $ix \exp(h \log t)$ at 0, which we calculated above to be $i \log t$. Hence,

$$\frac{\partial \Gamma}{\partial y}|_z = \lim_{h\to 0} \frac{\Gamma(z+ih) - \Gamma(z)}{h} = \int_0^\infty \exp[(z-1)\log t - t] i \log t dt = i \frac{\partial \Gamma}{\partial x},$$

so Γ satisfies the Cauchy-Riemann equations and, therefore, is holomorphic wherever it is defined; that is, Γ is holomorphic on the region $\Re(z) > 0$. $\square$

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Verify that $\Gamma(z+1) = z\Gamma(z)$ for every z with $0 < \Re(z)$.

Proof. First, note that if $v = \frac{\exp(z \log t)}{z}$, then

$$\frac{dv}{dt} = \frac{\exp(z\log t)\frac{z}{t}}{z} = \frac{\exp(z\log t)}{t} = \exp(z\log t)\exp(-\log t) = \exp[(z-1)\log t].$$

Hence, if $\Re(z) > 0$,

$$\Gamma(z) = \int_0^\infty \exp[(z-1)\log t - t]dt = \int_0^\infty \exp[(z-1)\log t]\exp(-t)dt;$$

integrating by parts yields:

$$\begin{aligned}
\Gamma(z) &= \left[\frac{\exp(z\log t)\exp(-t)}{z}\right]_0^\infty + \int_0^\infty \frac{\exp(z\log t)\exp(-t)}{z}dt \\
&= \left[\frac{\exp(z\log t)}{z\exp(t)}\right]_0^\infty + \frac{1}{z}\int_0^\infty \exp[((z+1)-1)\log t]\exp(-t)dt \\
&= 0 + \frac{1}{z}\int_0^\infty \exp[((z+1)-1)\log t - t]dt \\
&= \frac{1}{z}\Gamma(z+1).
\end{aligned}$$

Hence, for $\Re(z) > 0$, $\Gamma(z+1) = z\Gamma(z)$. $\square$

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From $\Gamma(z+1) = z\Gamma(z)$, define $\Gamma(z) := \Gamma(z+1)/z$ for every $z \in \mathbb{C}^*$ with $-1 < \Re(z)$.

(170.1): Verify that the definition $\Gamma(z) := \Gamma(z+1)/z$ extends Γ to a function holomorphic at every $c \in \mathbb{C}^*$ with $-1 < \Re(z)$.

Proof. If $\Re(z) > 0$, then we know that Γ is holomorphic at z by our work in problem 168 above. Now, if $1 < \Re(z) \leq 0$ with $z \neq 0$, then

$$\Gamma(z) = \Gamma(z+1)/z.$$

Now, $\Re(z+1) > 0$, so Γ is holomorphic at $z+1$. Furthermore, since $z \neq 0$ and $z \mapsto \frac{1}{z}$ is holomorphic on $\mathbb{C}^*$, $z \mapsto \frac{1}{z-1}$ is holomorphic on $\mathbb{C}\backslash\{1\}$ and so is holomorphic at $z+1$. Hence, the product

$$\frac{1}{(z+1)-1}\Gamma(z+1) = \frac{1}{z}\Gamma(z+1)$$

is holomorphic for all $z \in \mathbb{C}$ such that $\Re(z) > -1$. $\square$

(170.2): Verify that the extended function Γ has a simple pole at the origin.

Proof. Note, first off, that

$$\Gamma(1) = \int_0^\infty t^{1-1}e^{-t}dt = \int_0^\infty e^{-t}dt = \left[-\frac{1}{e^t}\right]_0^\infty = 1.$$

Now, let $D_1(1)$ be the open disc of radius 1 centered at 1. Then we've seen that $\Gamma \in \mathcal{H}(D_1(1))$, so Γ has a power series expansion

$$\Gamma(z) = \sum_{n=0}^\infty c_n(z-1)^n$$

in $D_1(1)$, where $c_0 = \Gamma(1) = 1$. Now, let D^* be the punctured disc of radius 1 centered at the origin. Then in D^*, $\Gamma(z) = \Gamma(z+1)/z$; since $z+1 \in D_1(1)$,

$$\Gamma(z) = \frac{1}{z}\Gamma(z+1) = \frac{1}{z}\sum_{n=0}^\infty c_n((z+1)-1)^n = \frac{1}{z}\sum_{n=0}^\infty c_n z^n.$$

Since $\sum c_n(z-1)^n$ converges absolutely on $D_1(1)$ and uniformly on any compact subset thereof, we pull the $1/z$ inside to see that

$$\Gamma(z) = \sum_{n=-1}^\infty c_{n+1}z^n$$

converges absolutely and uniformly on any compact subset of D^*, and so, by the uniqueness of the Laurent expansion, is the Laurent expansion of Γ in D^*. Since the coefficient on z^{-1} is non-zero (specifically, $c_0 = 1$) and this is the first non-zero coefficient in the Laurent expansion, we see that Γ has a simple pole at $z = 0$. □

(170.3): Prove that Γ extends to a meromorphic function on $\mathbb{C}$ with simple poles at the non-positive integers $\mathbb{Z}_-$ and holomorphic on $\mathbb{C}\backslash\mathbb{Z}_-$.

Proof. Note that, for $k \in \mathbb{N}$, and $z \in \mathbb{C}^*$ such that $-(k+1) < \Re(z)$,

$$\begin{aligned}\Gamma(z+k+1) &= (z+k)\Gamma(z+k)\\ &= (z+k)(z+(k-1))\Gamma(z+(k-1))\\ &\vdots\\ &= (z+k)\cdots z\Gamma(z).\end{aligned}$$

Hence, for any such k, the natural extension of Γ would be:

$$\Gamma(z) = \frac{1}{z(z+1)\cdots(z+k)}\Gamma(z+k+1)$$

for $-(k+1) < \Re(z)$. Now, since $z \mapsto \frac{1}{z(z+1)\cdots(z+k)}$ is holomorphic on all $\mathbb{C}\backslash\{0,-1,-2,\dots,-k\}$ and Γ is holomorphic on the right half plane, note that $\Re(z+k) > 0$ for $\Re(z) > -(k+1)$, so the extended function $\Gamma(z) = \frac{1}{z(z+1)\cdots(z+k)}\Gamma(z+k+1)$ is holomorphic on $\{z \in$

$\mathbb{C}|\Re(z) > -(k+1), z \neq 0, z \neq -1, \ldots, z \neq -k\}$. Also, since $z \mapsto \Gamma(z+k+1)$ is holomorphic on $\{z \in \mathbb{C}|\Re z > -(k+1)\}$ and $z \mapsto z(z+1)\cdots(z+k)$ is an entire function, we see that

$$\Gamma(z) = \frac{\Gamma(z+k+1)}{z(z+1)\cdots(z+k)}$$

is meromorphic on $\{z \in \mathbb{C}|\Re z > -(k+1)\}$. Hence, we see that Γ extends to a meromorphic function on $\mathbb{C}$ that is holomorphic on $\mathbb{C}\backslash\mathbb{Z}_-$.

Now, as in (170.2) above, let $\sum_{n=0}^{\infty} c_n(z-1)^n$ be the power series of Γ on $D_1(1)$. Let $z \in D_1(-k)\backslash\{-k\}$ for some $k \in \mathbb{N}$. Then

$$\begin{aligned}
\Gamma(z) &= \frac{1}{z(z+1)\cdots(z+k)}\Gamma(z+(k+1)) \\
&= \frac{1}{z(z+1)\cdots(z+k)}\sum_{n=0}^{\infty} c_n((z+k+1)-1)^n \\
&= \frac{1}{z(z+1)\cdots(z+k)}\sum_{n=0}^{\infty} c_n(z+k)^n \\
&= \frac{1}{z(z+1)\cdots(z+(k-1))}\sum_{n=-1}^{\infty} c_{n+1}(z+k)^n.
\end{aligned}$$

Now, since $\frac{1}{z(z+1)\cdots(z+(k-1))}$ is holomorphic on $D_1(-k)\backslash\{-k\}$, so it has a power series representation $\sum_{n=0}^{\infty} a_n(z+k)^n$ in this disc. Hence,

$$\Gamma(z) = \left(\sum_{n=0}^{\infty} a_n(z+k)^n\right)\left(\sum_{n=-1}^{\infty} c_{n+1}(z+k)^n\right) = \frac{a_0c_0}{z+k} + \sum_{n=0}^{\infty} d_n(z+k)^n$$

for some $d_0, d_1, \ldots \in \mathbb{C}$. Since both series converge absolutely and uniformly on any compact subset of $D_1(-k)\backslash\{-k\}$, this series converges and so is the Laurent series expansion of Γ on $D_1(-k)\backslash\{-k\}$. Hence, we see that Γ has a simple pole at $-k$. Since our choice of $k \in \mathbb{N}$ was arbitrary, we conclude that our extended Γ has simple poles at the non-negative integers. $\square$

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Consider the reciprocal function

$$f : \mathbb{C}^* \to \mathbb{C}, \quad z \mapsto \frac{1}{z}.$$

Also consider the following parametrization of the unit circle

$$\gamma : \mathbb{R} \to S^1 \subset \mathbb{C}, \quad t \mapsto \exp(it),$$

Moreover, consider the canonical projection,

$$p : \mathcal{O} \to \mathbb{C}, \quad g_z \mapsto z,$$

and let $\widetilde{\gamma}$ represent the lifting of γ with respect to p, passing through the germs of f so that

$$\widetilde{\gamma} : \mathbb{R} \to \mathcal{O}, \quad t \mapsto f_{\gamma(t)}.$$

Finally, consider the covering map defined by differentiation,

$$d : \mathcal{O} \to \mathcal{O}, \quad g_z \mapsto (g')_z.$$

Let Γ denote a lifting of $\widetilde{\gamma}$ with respect to d. Thus, $\Gamma(t)$ is the germ $F_{\gamma(t)}$ at the point $\gamma(t)$ of a primitive F of the function f.

Prove that the image of Γ, that is, the set $\Gamma(\mathbb{R}) \subset \mathcal{O}$ with its relative topology induced from that of $\mathcal{O}$, is a one-dimensional real manifold homeomorphic to $\mathbb{R}$. In particular, the image of Γ does not intersect itself, even though the image of γ intersects itself infinitely many times.

Finally, verify that the restriction of $p \circ d$ to Γ projects Γ to γ as does the projection of a right circular helix onto a circle.

Proof. Since $\Gamma : \mathbb{R} \to \mathcal{O}$ is a lifting of a continuous map relative to a covering map, it is necessarily continuous. Now, we want to show that Γ is actually bijective. To that end, suppose, to the contrary, that $F^1_{\gamma(t_1)} = \Gamma(t_1) = \Gamma(t_2) = F^2_{\gamma(t_2)}$ for some $t_1, t_2 \in \mathbb{R}$ such that $t_1 \neq t_2$. First, note that $\gamma(t_1)$ must equal $\gamma(t_2)$; since γ is 2π-periodic, we see that $t_2 = t_1 + 2\pi n$ for some $n \in \mathbb{Z}$. Suppose, without loss of generality, that $n > 0$. Define the map $r : [t_1, t_2] \to [0, 1]$ by

$$t \mapsto \frac{t - t_0}{t_1 - t_0}.$$

Then r is a homeomorphism; define $\rho : [0, 1] \to \mathbb{C}$ by

$$t \mapsto e^{2\pi i n t}.$$

Then, by construction, $\gamma|_{[t_0,t_0]} = \rho \circ r$. Hence,

$$\begin{aligned} F^2_{\gamma(t_2)}(\gamma(t_2)) - F^1_{\gamma(t_1)}(\gamma(t_1)) &= \int_\rho f(z)dz \\ &= \int_\rho \frac{1}{z} dz \\ &= n \int_\phi \frac{1}{z} dz \end{aligned}$$

where $\phi : [0, 1] \to \mathbb{C}$ is given by $t \mapsto e^{2\pi i t}$; the standard parametrization of the unit circle traversed once counter-clockwise. Then, by Cauchy's integral formula, since the constant function 1 is entire,

$$n \int_\phi \frac{1}{z} dz = n 2\pi i.$$

Hence, $F^2_{\gamma(t_2)}(\gamma(t_2)) - F^1_{\gamma(t_1)}(\gamma(t_1)) = 2\pi i n \neq 0$ since $n > 0$. Therefore, $F^1_{\gamma(t-1)} \neq F^2_{\gamma(t_2)}$, and so $\Gamma(t_1) \neq \Gamma(t_2)$. Therefore, we see that Γ is injective.

On the other hand, $\Gamma : \mathbb{R} \to \Gamma(\mathbb{R})$ is surjective by definition, so it remains only to show that Γ is an open map.

If (t_1, t_2) is a basic open set of $\mathbb{R}$, then let $F_{\gamma(t)} \in \Gamma((t_1, t_2))$. Then $t \in (t_1, t_2)$. Then $f \in \mathcal{H}(D_1(e^{it}))$ so any primitive of f is analytic on $D_1(e^{it})$; specifically, F is analytic on $D_1(e^{it})$. Now let $a = \min\{|t - t_1|, |t_2 - t|\}$, let $a' = |e^{it}(e^{id} - 1)|$ and let $D' = D_{a'}(e^{it})$. Then either $D_1(e^{it}) \subset D'$ or *vice versa*; either way, let D be the smaller of the two. Then F is analytic on D and $D \cap \gamma((t_1, t_2))$ is an open subset of $\gamma((t_1, t_2))$. Hence, $N(D, F) \cap \Gamma((t_1, t_2))$ is an open subset of $\Gamma((t_1, t_2))$ containing $F_{\gamma(t)}$, so we see that $\Gamma((t_1, t_2))$ is an open subset of $\Gamma(\mathbb{R})$, and so Γ is an open map. Since Γ is an open, continuous, bijective map, $\Gamma : \mathbb{R} \to \Gamma(\mathbb{R})$ is a homeomorphism, and so we see that $\Gamma(\mathbb{R})$ is a one-dimensional real manifold homeomorphic to $\mathbb{R}$.

Suppose $t \in \mathbb{R}$. Note that, for $F_{\gamma(t)} \in \Gamma(\mathbb{R})$,

$$p \circ d(F_{\gamma(t)} = p(F'_{\gamma(t)}) = p(f_{\gamma(t)}) = \gamma(t);$$

since our choice of t was arbitrary, we conclude that $p \circ d$ projects Γ onto γ. $\square$

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