COMPLEX ANALYSIS HW 6

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177

Prove that for each function $f : \Omega \to \mathbb{C}$ holomorphic on an open set $\Omega \subseteq \mathbb{C}$, and for any two *closed* curves γ_0 and γ_1 mutually homotopic by closed curves in Ω, the equality $\oint_{\gamma_0} f(z)dz = \oint_{\gamma_1} f(z)dz$ holds, regardless of whether γ_0 and γ_1 have any point in common. Deduce from this an alternate proof of the existence of the Laurent series for functions holomorphic on an annulus.

Proof. Let $a = \gamma_0(0) = \gamma_0(1)$ and let $b = \gamma_1(0) = \gamma_1(1)$. Let $\{\gamma_u\}_{u\in I}$ be a homotopy between γ_0 and γ_1. Let $\eta : [0,1] \to \Omega$ be given by

$$t \mapsto \gamma_t(0).$$

Then η is a continuous path from a to b, since $\gamma_0(0) = a$ and $\gamma_1(0) = b$ and γ_u is continuous in u. Now, define $\sigma : [0,1] \times [0,1] \to \Omega$ by

$$(u,t) \mapsto \begin{cases} \eta(4t) & 0 \le t \le \frac{1-u}{4} \\ \gamma_{1-u}\left(\frac{2}{1+u}\left(t - \frac{1-u}{4}\right)\right) & \frac{1-u}{4} \le t \le \frac{3+u}{4} \\ \eta(4(1-t)) & \frac{3+u}{4} \le t \le 1. \end{cases}$$

Then each of the pieces of σ is continuous since η and γ_u are continuous. Also,

$$\eta\left(4\frac{1-u}{4}\right) = \eta(1-u) = \gamma_{1-u}(0) = \gamma_{1-u}\left(\frac{2}{1+u}\left(\frac{1-u}{4} - \frac{1-u}{4}\right)\right)$$

and

$$\begin{aligned}\gamma_{1-u}\left(\frac{2}{1+u}\left(\frac{3+u}{4} - \frac{1-u}{4}\right)\right) &= \gamma_{1-u}\left(\frac{2}{1+u}\frac{1+u}{2}\right) \\ &= \gamma_{1-u}(1) \\ &= \gamma_{1-u}(0) \\ &= \eta(1-u) \\ &= \eta\left(4\left(1 - \frac{3+u}{4}\right)\right)\end{aligned}$$

so we see that the pieces agree at their endpoints, so σ is continuous. Denote $\sigma(u,t)$ by $\sigma_u(t)$. Hence, $\{\sigma_u\}_{u\in I}$ is a homotopy from σ_1 to σ_0. Furthermore, for any $u \in I$,

$$\sigma_u(0) = \eta(0) = \gamma_0(0) = a$$

and

$$\sigma_u(1) = \eta(0) = \gamma_0(0) = a,$$

so each σ_u is a closed loop based at a and $\{\sigma_u\}_{u\in I}$ is, in fact, a homotopy from σ_1 to σ_0 fixing endpoints. Finally, note that since γ_u and η are all contained in Ω, each σ_u is contained in Ω. Therefore, by the Homotopy Form of Cauchy's Theorem,

$$\int_{\sigma_1} f(z)dz = \int_{\sigma_0} f(z)dz. \tag{1}$$

Now, since $\frac{1-1}{4} = 0$ and $\frac{3+1}{4} = 1$, $\sigma_1 = \gamma_0$. On the other hand,

$$\int_{\sigma_0} f(z)dz = \int_\eta f(z)dz + \int_{\gamma_1} f(z)dz - \int_\eta f(z)dz = \int_{\gamma_1} f(z)dz.$$

Therefore, (1) reduces to

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz.$$

Now, suppose $A(a, r_1, r_2)$ is an annulus centered at a. Suppose $r_1 < \rho_0 < \rho_1 < r_2$. Then we can parametrize the circle given by $|z - a| = \rho_0$ by $\gamma_0(t) = a + \rho_1 e^{2\pi i t}$ and the circle given by $|z-a| = \rho_1$ by $\gamma_1(t) = a+\rho_1 e^{2\pi i t}$. Define $\gamma : [0,1] \times [0,1] \to A$ by

$$(u,t) \mapsto (1-u)\gamma_0(t) + u\gamma_1(t).$$

Then γ is certainly continuous and, moreover, $\gamma(0,t) = \gamma_0(t)$ and $\gamma(1,t) = \gamma_1(t)$. Hence, it makes sense to denote $\gamma(u,t)$ by $\gamma_u(t)$, since, in this notation, $\{\gamma_u\}_{u\in I}$ is a homotopy from γ_0 to γ_1.

Therefore, by the result proved above,

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz$$

for any $f \in \mathcal{H}[A(a, r_1, r_2)]$. Since our choice of ρ_1 and ρ_2 were arbitrary, we see that

$$\int_{|z-a|=r} f(z)dz$$

is independent of $r_1 < r < r_2$. Thus, we've provided an alternative proof of a key lemma used to prove the existence of the Laurent expansion. □

182

Prove that for each compact subset $K \subset \mathbb{C}$ the complement $\mathbb{C}\backslash K$ has exactly one unbounded connected component.

Proof. Since K is compact, K is closed and bounded. That is, there exists an open disc $D_r(0)$ of radius r centered at 0 such that, if $|z| \geq r$, $z \notin K$. Now, consider $\mathbb{C}\backslash D_r(0)$. Then $\mathbb{C}\backslash D_r(0) \subset \mathbb{C}\backslash K$. Let $z, w \in \mathbb{C}\backslash D_r(0)$. Define $\gamma_1 : [0,1] \to \mathbb{C}\backslash D_r(0)$ by

$$t \mapsto (1-t)z + t\frac{r}{|z|}z,$$

let $\gamma_2 : [0,1] \to \mathbb{C}\backslash D_r(0)$ be the path given by traversing the circle $\partial D_r(0)$ counterclockwise from $r\frac{z}{|z|}$ to $r\frac{w}{|w|}$ and define $\gamma_3 : [0,1] \to \mathbb{C}\backslash D_r(0)$ by

$$t \mapsto (1-t)\frac{rw}{|w|} + tw.$$

Then γ_1 and γ_3 are just straight lines and γ_2 is an arc of a circle, so each is continuous. Furthermore, $\gamma_1(1) = r\frac{z}{|z|} = \gamma_2(0)$ and $\gamma_2(1) = r\frac{w}{|w|} = \gamma_3(0)$ so, by pasting them together, we get $\gamma : [0,1] \to \mathbb{C}\backslash D_r(0)$ given by, say

$$t \mapsto \begin{cases} \gamma_1(4t) & 0 \le t \le 1/4 \\ \gamma_2(2(t-1/4)) & 1/4 \le t \le 3/4 \\ \gamma_3(4t-3) & 3/4 \le t \le 1 \end{cases}$$

which is continuous. Now, $\gamma(0) = \gamma_1(0) = z$ and $\gamma(1) = \gamma_3(1) = w$, so γ is a path in $\mathbb{C}\backslash D_r(0)$ from z to w. Since our choice of z and w was arbitrary, we see that there is such a connecting path for any 2 points in $\mathbb{C}\backslash D_r(0)$, so $\mathbb{C}\backslash D_r(0)$ is path-connected and, therefore, connected. Hence, $\mathbb{C}\backslash D_r(0)$ is contained in a single connected component of $\mathbb{C}\backslash K$, so $\mathbb{C}\backslash K$ has at least one unbounded connected component.

On the other hand, $D_r(0)$ is bounded and since every connected component of $\mathbb{C}\backslash K$ except the one containing $\mathbb{C}\backslash D_r(0)$ must be contained in $D_r(0)$, we see that every other connected component of $\mathbb{C}\backslash K$ must be bounded, so we conclude that $\mathbb{C}\backslash K$ has exactly one unbounded connected component. □

185

Let $\Omega \subset \mathbb{C}$ denote a non-empty bounded open subset with piecewise differentiable boundary $\partial\Omega$. Assume Ω *convex*. Prove that for *each* point $c \in \Omega$,

$$\oint_{\partial\Omega} \frac{1}{w-c} dw = 2\pi i.$$

Proof. Let $\partial\Omega$ be parametrized by a closed curve $\gamma : [0,1] \to \mathbb{C}$ such that $\gamma(0) = \gamma(1) = c + r$ for some $r \in \mathbb{R}$. Let $c \in \Omega$ and let D_c be an open disc of radius r_c centered at c contained in Ω (since Ω is open, we know there exists such a disc). Let $\gamma' : [0,1] \to \Omega$ be given by

$$t \mapsto c + r_c e^{2\pi i t}.$$

Now, define $\widetilde{\gamma} : [0,1] \times [0,1] \to \mathbb{C}$ by

$$(u,t) \mapsto (1-u)\gamma(t) + u\left(r_c\frac{\gamma(t)-c}{|\gamma(t)-c|} + c\right).$$

Note, first, that since $c \notin \partial\Omega$, $\widetilde{\gamma}$ is continuous. Furthermore,

$$\left|\left(r_c \frac{\gamma(t)-c}{|\gamma(t)-c|}+c\right)-c\right| = r_c \left|\frac{\gamma(t)-c}{|\gamma(t)-c|}\right| = r_c,$$

so $r_c \frac{\gamma(t)-c}{|\gamma(t)-c|}+c \in \partial D_c$. Denote $\widetilde{\gamma}(u,t)$ by γ_u. Then

$$\gamma_0(t) = \gamma(t)$$

and

$$\gamma_1(t) \in \partial D_c,$$

so $\{\gamma_u\}_{u\in I}$ is a homotopy from γ to γ_1. Further, since Ω is convex, each γ_u lies entirely in $\Omega\backslash\{c\}$. Hence, since $\frac{1}{w-c}$ is holomorphic on all of $\mathbb{C}\backslash\{c\}$, the result proved in problem 177 above yields

$$\int_{\partial\Omega} \frac{1}{w-c}dw = \int_\gamma \frac{1}{w-c}dw = \int_{\gamma_1} \frac{1}{w-c}dw.$$

Since Ω is convex, $\gamma_1(t) \neq \gamma_1(s)$ unless $t = s$, since this would require that the line passing through $\gamma_1(t)$ and $\gamma(t)$ also passes through $\gamma(s)$ and, hence, that the line segment from $\gamma(t)$ to $\gamma(s)$ not be contained in Ω. Therefore, up to a reparametrization, γ_1 *is* γ', and so, by Cauchy's integral formula,

$$\int_{\partial\Omega} \frac{1}{w-c}dw = \int_{\gamma_1} \frac{1}{w-c}dw = \int_{\gamma'} \frac{1}{w-c}dw = 2\pi i.$$

□

188

Prove that for each non-empty open subset $\Omega \subseteq \mathbb{C}$, and for all points $a, b \in \mathbb{C}\backslash\Omega$ in the same connected component of $\mathbb{C}\backslash\Omega$, there exists a complex square root of $z \mapsto (z-a)(z-b)$, in other words, a holomorphic function $f : \Omega \to \mathbb{C}$ such that $[f(z)]^2 = (z-a)(z-b)$ for every $z \in \Omega$.

Proof. Let $g(z) = \frac{z-a}{z-b}$. Then $g \in \mathcal{H}(\Omega)$ since $a, b \notin \Omega$. We want to let h be a primitive of $\frac{g'}{g}$; to see that such a primitive exists, we need only show that $\int_\gamma \frac{g'}{g} = 0$ for all closed curves $\gamma : [0,1] \to \Omega$. To that end, let γ be such a closed curve. Note that $g'(z) = \frac{(z-b)-(z-a)}{(z-b)^2} = \frac{a-b}{(z-b)^2}$. Hence,

$$\begin{aligned}
\int_\gamma \frac{g'(z)}{g(z)}dz = \int_\gamma \frac{\frac{a-b}{(z-b)^2}}{\frac{z-a}{z-b}}dz &= \int_\gamma \frac{a-b}{(z-a)(z-b)}dz \\
&= \int_\gamma \frac{(z-b)-(z-a)}{(z-a)(z-b)}dz \\
&= \int_\gamma \left[\frac{1}{z-a} - \frac{1}{z-b}\right]dz \\
&= 2\pi i\left[n(\gamma,a) - n(\gamma,b)\right].
\end{aligned}$$

Now, since a and b lie in the same connected component of $\mathbb{C}\backslash\Omega$ and γ is a curve in Ω, it's certainly the case that a and b lie in the same connected component of $\mathbb{C}\backslash\gamma$, so $n(\gamma, a) = n(\gamma, b)$, meaning that

$$\int_\gamma \frac{g'(z)}{g(z)} dz = 0.$$

Since our choice of γ was arbitrary, we see that this holds for all such curves, so $\frac{g'}{g}$ does indeed have a primitive; call this primitive h. Then, since $h' = \frac{g'}{g}$,

$$\begin{aligned}\frac{d}{dz}\left[e^{-h(z)}g(z)\right] = -e^{-h(z)}h'(z)g(z) + e^{-h(z)}g'(z) &= -e^{-h(z)}\frac{g'(z)}{g(z)}g(z) + e^{-h(z)}g'(z) \\ &= e^{-h(z)}g'(z) + e^{-h(z)}g'(z) \\ &= 0,\end{aligned}$$

so $e^{-h(z)}g(z) = c$ for some constant $c \in \mathbb{C}$; hence, $g(z) = ce^{h(z)}$. If $\alpha \in \mathbb{C}$ such that $e^\alpha = c$, this in turn implies that $g(z) = e^{\alpha + h(z)}$. Let $f(z) = \alpha + h(z)$.

Now, since h is holomorphic on Ω, so is f and, hence, $f(z)/2$. Since composition of holomorphic functions is holomorphic, this in turn implies that $e^{f(z)/2}$ is holomorphic on Ω. Now, $z - b$ is holomorphic on all of $\mathbb{C}$, so the product

$$\phi(z) = e^{f(z)/2}(z - b) \in \mathcal{H}(\Omega).$$

Now, we note that

$$(\phi(z))^2 = \left(e^{f(z)/2}(z-b)\right)^2 = e^{f(z)}(z-b)^2 = \frac{z-a}{z-b}(z-b)^2 = (z-a)(z-b) = g(z),$$

so ϕ is our desired complex square root. □

191

For each injective $f \in \mathcal{H}[D(0,1)]$ with $f(0) = 0$, prove that there exists some $g \in \mathcal{H}[D(0,1)]$ such that $[g(z)]^2 = f(z^2)$ for every $z \in D(0,1)$, and that g is also injective.

Proof. Let $\sum c_n z^n$ be the Taylor series of f centered at 0. Note that, since $f(0) = 0$, $c_0 = 0$; however, since f is injective, $f'(z) \neq 0$ for any $z \in D(0,1)$; specifically, $c_1 = f'(0) \neq 0$. Let $h(z) = f(z^2) = \sum c_n z^{2n}$. Then h is holomorphic on $D(0,1)$. Furthermore, although $h(0) = 0$ and $h'(0) = 0$, $h''(0) = 2c_1 \neq 0$, so we see that $\text{ord}_0(h) = 2$. Therefore, so $h(z) = z^2\phi(z)$ for some ϕ such that $\phi(0) \neq 0$ and $\phi \in \mathcal{H}[D(0,1)]$. Now, since $h(z) = f(z^2) \neq 0$ for $z \neq 0$, $\phi(z) \neq 0$ for all $z \in D(0,1)$. Therefore, since $\phi(z) \neq 0$ on the simply connected open set $D(0,1)$, there exists a complex square root ψ of ϕ; that is, for $z \in D(0,1)$, $(\psi(z))^2 = \phi(z)$ and ψ holomorphic on the disc. Now, let

$$g(z) = z\psi(z).$$

Then $g \in \mathcal{H}[D(0,1)]$ and, moreover,

$$(g(z))^2 = (z\psi(z))^2 = z^2(\psi(z))^2 = z^2\phi(z) = h(z) = f(z^2).$$

$\square$

6

Suppose that $\gamma : [0,1] \to \mathbb{C}\backslash\{a\}$ is a closed curve. Show that γ is homotopic to the curve

$$\eta(t) = a + e^{2\pi imt}$$

if and only if $n(\gamma, a) = m$.

Proof. Suppose γ is homotopic to η. Then, by the result proved in problem 177 above,

$$\int_\gamma f(z)dz = \int_\eta f(z)dz$$

for any $f \in \mathcal{H}(\mathbb{C}\backslash\{a\})$. Let $\eta_1(t) = a + e^{2\pi it}$. Note that $f(z) = \frac{1}{z-a}$ is holomorphic on all of $\mathbb{C}\backslash\{a\}$, so

$$n(\gamma, a) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a} = \frac{1}{2\pi i}\int_\eta \frac{dz}{z-a} = \frac{m}{2\pi i}\int_{\eta_1} \frac{dz}{z-a} = \frac{m}{2\pi i}2\pi i = m,$$

where the second-to-last equality in the above is due to Cauchy's integral formula.

On the other hand, suppose $n(\gamma, a) = m$. Suppose, up to a reparametrization, that $\gamma(0) = a + r_1$ for some $r_1 \in \mathbb{R}$. We will show that γ is homotopic to η by induction on m. Suppose $m = 1$. Define $\widetilde{\gamma} : [0,1] \times [0,1] \to \mathbb{C}$ by

$$(u,t) \mapsto (1-u)\gamma(t) + u\left(\frac{\gamma(t)-a}{|\gamma(t)-a|} + a\right).$$

Then $\widetilde{\gamma}$ is certainly continuous. Furthermore,

$$\widetilde{\gamma}(0,t) = \gamma(t) \quad \widetilde{\gamma}(1,t) = \frac{\gamma(t)-a}{|\gamma(t)-a|} + a;$$

Since $\left|\left(\frac{\gamma(t)-a}{|\gamma(t)-a|} + a\right) - a\right| = \left|\frac{\gamma(t)-a}{|\gamma(t)-a|}\right| = 1$, we see that $\gamma(1,t)$ is contained in the circle defined by η. Hence, if we denote $\widetilde{\gamma}(u,t)$ by $\gamma_u(t)$, then $\{\gamma_u\}_{u\in I}$ is a homotopy from γ to γ_1. Note that, since γ and γ_1 are homotopic, $1 = n(\gamma, a) = n(\gamma_1, a)$. Since γ_1 is a continuous map into the circle, we can parametrize γ_1 by $\gamma_1(t) = a + re^{2\pi i f(z)}$ for some $f : [0,1] \to \mathbb{R}$ such that $f(0) = 0$.

Let γ' be a lifting of γ_1 with respect to $z \mapsto a + e^z$ and let η' be a lifting of η with respect to this map such that $\gamma'(0) = \eta'(0)$. Then $1 = n(\gamma_1, a) = \frac{1}{2\pi i}(\gamma'(1) - \gamma'(0))$ and $n(\eta, a) = \frac{1}{2\pi i}(\eta'(1) - \eta'(0))$. On the other hand,

$$n(\eta, a) = \frac{1}{2\pi i}\int_\eta \frac{dz}{z-a} = \frac{1}{2\pi i}\int_{|z-a|=1} \frac{dz}{z-a} = 1,$$

so we see that $\gamma'(1) - \gamma'(0) = \eta'(1) - \eta'(0)$; since $\gamma'(0) = \eta'(0)$, this in turn means that $\gamma'(1) = \eta'(1)$. Therefore, $f(1) = 1$. Now, define the map $H : [0,1] \times [0,1] \to \mathbb{R}$ by

$$(u,t) \mapsto (1-u)f(t) + ut.$$

Then H is continuous and, moreover, $H(0,t) = f(t)$ for all t, $H(1,t) = t$ for all t and

$$H(u,0) = (1-u)f(0) + u(0) = 0; \quad H(u,1) = (1-1)f(1) + 1 = 1$$

for all u. Hence, H is a homotopy with fixed endpoints from f to the identity map on the interval $[0,1]$.

Now, define $\Gamma : [0,1] \times [0,1] \to \partial D_r(a)$ by

$$(u,t) \mapsto c + re^{2\pi i H(u,t)}.$$

Then Γ is continuous; denote $\Gamma(u,t)$ by $\Gamma_u(t)$. Then

$$\Gamma_0(t) = c + re^{2\pi i H(0,t)} = c + re^{2\pi i f(t)} = \gamma_1(t)$$

and

$$\Gamma_1(t) = c + re^{2\pi i H(1,t)} = c + re^{2\pi i t} = \eta(t).$$

Furthermore, $\Gamma_u(0) = c + re^{2\pi i H(u,0)} = c + re^0 = c + r$ and $\Gamma_u(1) = c + re^{2\pi i H(u,1)} = c + re^{2\pi i} = c + r$, so we see that $\{\Gamma_u\}_{u \in I}$ is a homotopy from γ_1 to η with fixed endpoints.

Therefore, since γ is homotopic to γ_1 by $\{\gamma_u\}_{u\in I}$ and γ_1 is homotopic to η by $\{\Gamma_u\}_{u\in I}$, we can form the concatenation

$$\widetilde{\Gamma}_u(t) = \begin{cases} \gamma_u(2t) & 0 \le t \le 1/2 \\ \Gamma_u(2t-1) & 1/2 \le t \le 1, \end{cases}$$

which is a homotopy from γ to η.

Thus, having shown the base case, suppose $n(\gamma, a) = k-1$ implies that γ is homotopic to $\eta_{k-1}(t) = a + e^{2\pi i(k-1)t}$. Then suppose $n(\gamma, a) = k$. For each $t \in I$, let γ^t denote the curve given by traversing γ from $\gamma(0)$ to $\gamma(t)$. Let t_0 be the minimum value of $t \in I$ such that $\int_{\gamma^t} \frac{dz}{z-a} dz = 1$ and $\gamma(t_0) = a + s$ for $s \in \mathbb{R}$. Let β denote the straight-line path from $\gamma(t_0)$ to $\gamma(0)$ and let γ^* be the curve given by traversing γ^{t_0} followed by β. Then γ^* is a curve with winding number 1 and so, by the base case just proved, is homotopic to $\eta_1(t) = a + re^{2\pi i t}$ by some homotopy H_u^*. On the other hand, if γ' denotes the path given by traversing β^{-1} and then γ from $\gamma(t_0)$ to $\gamma(1)$, then

$$\int_\gamma \frac{dz}{z-a} = \int_{\gamma^*} \frac{dz}{z-a} + \int_{\gamma'} \frac{dz}{z-a},$$

so $n(\gamma', a) = k-1$. Therefore, by the induction hypothesis, γ' is homotopic to η_{k-1} by some homotopy H_u'. Since the base point of both γ^* and γ' is $\gamma(0)$ and the homotopies we've described just map $\gamma(0)$ in a straight-line path to $a + r$, we can reparemetrize these homotopies, if necessary, so that for all u,

$$H_u^*(1) = H_u'(0).$$

Thus, the concatenation

$$\widetilde{H}_u(t) = \begin{cases} H_u^*(2t) & 0 \leq t \leq 1/2 \\ H_u'(t) & 1/2 \leq t \leq 1 \end{cases}$$

is continuous and is a homotopy of $\gamma^* \cdot \gamma'$, the closed curve given by traversing γ^* followed by γ', with $\eta_k(t) = a + re^{2\pi imt}$. This curve is almost γ; in fact, it is homotopic to γ simply by retracting $\beta \cdot \beta^{-1}$ down to the point $\gamma(t_0)$.

Therefore, we conclude, by induction, that γ with $n(\gamma, a) = m$ is homotopic to η_m for any $m \in \mathbb{N}$. Furthermore, an exactly parallel argument demonstrates that the same holds for $m < 0$. If $m = 0$, then γ is defined on some simply connected open set Ω not containing a, so γ is contractible to a point and, therefore, homotopic to the constant map $\eta_0(t) = a + re^{2\pi i0(t)} = a + r$. □

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