COMPLEX ANALYSIS HW 7

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Let f be a meromorphic function on $\mathbb{C}$ with period π; thus, $f(z+\pi) = f(z)$ for every $z \in \mathbb{C}$. Suppose that in the strip where $0 < \Re(z) \leq \pi$ the function f has only finitely many poles $w_1, \ldots, w_n$ and that they are all simple, with residues $c_1, \ldots, c_n$, respectively. Suppose also that there exist complex numbers p and q such that

$$\lim_{y \to +\infty} f(x+iy) = p,$$
$$\lim_{y \to +\infty} f(x-iy) = q,$$

uniformly with respect to $x \in [0, \pi]$. Prove that

$$f(z) = \frac{p+q}{2} + \sum_{k=1}^{n} c_k \cdot \cot(z - w_k),$$
$$q - p = 2i \sum_{k=1}^{n} c_k.$$

Proof. Let $\epsilon > 0$ be small enough such that $\Re(w_k) > \epsilon$ for all $k = 1, \ldots, n$. Let R_N be the rectangle determined by the points $\epsilon \pm iN$ and $\pi + \epsilon \pm iN$. Then f is holomorphic on all of R_N (since none of the poles of f lie on R_N) and R_N is homotopic in the strip $0 < \Re(z) \leq \pi + \epsilon$ to a circle.

Now, define $h(z) = f(z) - \sum_{k=1}^{n} c_k \cot(z - w_k)$. We claim that this function is holomorphic on the entire strip $0 < \Re(z) \leq \pi$. To see why, first note that $\cot(z - w_k) = \frac{\cos(z-w_k)}{\sin(z-w_k)}$ and the derivative of $\sin(z - w_k)$ is $\cos(z - w_k)$. Hence, $\cot(z - w_k)$ is of the form g'/g where $g(z) = \sin(z - w_k)$. Note that the only pole of $\cot(z - w_k)$ in the strip is at w_k. Thus, by the argument principle, $\cot(z - w_k)$ has a simple pole at w_k and $\text{res}(\cot(z - w_k), w_k) = \text{ord}_{w_k}(\sin(z - w_k)) = 1$, since $\cos(0) = 1$. Therefore, $c_k \cot(z - w_k)$ has residue c_k at w_k and, hence, since both f and $\cot(z - w_k)$ have simple poles at w_k with residue c_k,

$$f(z) - c_k \cot(z - w_k)$$

is holomorphic in a neighborhood of w_k. Since the w_k comprise all of the poles of f in this region, this implies that

$$h(z) = f(z) - \sum_{k=1}^{n} c_k \cot(z - w_k)$$

is holomorphic in the entire strip. Thus, by Cauchy's Theorem,

$$f(a) - \sum_{k=1}^{n} c_k \cot(a - w_k) = h(a) = \frac{1}{2\pi i}\int_{R_N} \frac{h(z)}{z-a}dz.$$

Therefore...(?)

Turning to the second equality, recall that R_N is homotopic in the strip $0 < \Re(z) \leq \pi + \epsilon$ to a circle. Thus, for $0 < \Re(z) \leq \pi$ with $|\Im(a)| < N$, $n(R_N, a) = 1$. Hence,

$$\frac{1}{2\pi i}\int_{R_N} f(z)dz = \sum_{k=1}^{n} c_k.$$

Let L_1 and L_2 denote the vertical lines in R_N and let L_3 and L_4 denote the horizontal lines, with L_1 corresponding to the line $\Re(z) = \pi + \epsilon$ and L_3 corresponding to the line $\Im(z) = N$. We will integrate in the usual counterclockwise direction. Now, since f is π-periodic, f behaves exactly the same on both L_1 and L_2; since L_1 and L_2 are traversed in opposite directions, we see that

$$\int_{L_1} f(z)dz + \int_{L_2} f(z)dz = 0.$$

Now, for N sufficiently large, f is arbitrarily close to the constant functions p and q on the respective horizontal segments, L_3 and L_4. Hence, for large N,

$$\int_{L_3} f(z)dz + \int_{L_4} f(z)dz \to \int_{L_3} pdz + \int_{L_4} qdz = \int_{\pi+\epsilon}^{\epsilon} pdz + \int_{\epsilon}^{\pi+\epsilon} qdz = -\pi p + \pi q.$$

Hence, as $N \to \infty$,

$$\begin{aligned}\sum_{k=1}^{n} c_k = \frac{1}{2\pi i}\int_{R_N} f(z)dz &= \frac{1}{2\pi i}\left[\int_{L_1} f + \int_{L_2} f + \int_{L_3} f + \int_{L_4} f\right] \\ &\to \frac{1}{2\pi i}(-\pi p + \pi q) \\ &= \frac{q-p}{2i}.\end{aligned}$$

Thus, we see that $q - p = 2i\sum_{k=1}^{n} c_k$. □

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Consider open subsets $U \subseteq \mathbb{C}$ and $V \subseteq \mathbb{C}$ with $0 \in U$ and $0 \in V$, and let $f : U \to V$ be a holomorphic isomorphism with holomorphic inverse $g : V \to U$, with $f(0) = 0 = g(0)$.

(200.1): For each real $r > 0$ such that $\overline{D(0,r)} \subset U$, with boundary $C := \partial D(0, r)$, prove that there exists a real $\rho > 0$ such that for every $w \in D(0,\rho)$,

$$g'(w) = \frac{1}{2\pi i}\oint_C \frac{dz}{f(z)-w}.$$

Proof. Let $\gamma : [0,1] \to C$ be the curve given by $t \mapsto re^{2\pi i t}$. Then, since f is injective, $f \circ \gamma$ is a simple closed curve. Now, let ρ be such that $D(0,\rho) \subset U \cap V$ (we know there is such a ρ since 0 and, hence, some neighborhood of 0, is contained in both U and V). Then, by Cauchy's Theorem, since $f\circ\gamma$ is a simple closed curve, $n(f\circ\gamma, 0) = 1$, and so, for $w \in D(0,\rho)$,

$$g'(w) = \frac{1}{2\pi i}\int_{f\circ\gamma} \frac{g'(z)}{z-w}dz.$$

Now, since $f \circ \gamma$ is a differentiable curve,

$$\begin{aligned}\int_{f\circ\gamma} \frac{g'(z)}{z-w}dz &= \int_0^1 \frac{g'(f(\gamma(t)))f'(\gamma(t))\gamma'(t)dt}{f(\gamma(t))-w} \\ &= \int_0^1 \frac{g'(f(\gamma(t)))f'(\gamma(t))\gamma'(t)dt}{f\circ g(f(\gamma(t)))-w} \\ &= \int_{g\circ f\circ\gamma} \frac{dz}{f(z)-w} \\ &= \int_\gamma \frac{dz}{f(z)-w},\end{aligned}$$

since $g \circ f = f \circ g = 1$. Therefore, we see that, indeed,

$$g'(w) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{f(z)-w}.$$

□

(200.2): Deduce that $g(w) = \sum_{n=1}^\infty c_n w^n$ with

$$c_n = \frac{1}{2n\pi i}\oint_C \frac{dz}{[f(z)]^n}.$$

Proof. Since we know that $g'(w) = \frac{1}{2\pi i}\int_C \frac{dz}{f(z)-w}$, we will first find a power series for g'. To that end, note that

$$\frac{1}{f(z)-w} = \frac{1}{f(z)}\frac{1}{1-\frac{w}{f(z)}} = \frac{1}{f(z)}\sum_{n=0}^\infty \left(\frac{w}{f(z)}\right)^n = \sum_{n=0}^\infty \frac{w^n}{[f(z)]^{n+1}}, \tag{1}$$

which converges uniformly so long as $|f(z)| = r$ if $|w| < r$. Therefore,

$$g'(w) = \frac{1}{2\pi i}\int_C \frac{dz}{f(z)-w} = \frac{1}{2\pi i}\int_C \left[\sum_{n=0}^\infty \frac{w^n}{[f(z)]^{n+1}}\right]dz = \sum_{n=0}^\infty \left[\frac{1}{2\pi i}\int_C \frac{dz}{[f(z)]^{n+1}}\right]w^n.$$

Therefore, integrating both sides with respect to w, we see that

$$g(w) = \sum_{n=0}^{\infty} \left[\frac{1}{2\pi i} \int_C \frac{dz}{[f(z)]^{n+1}} \right] \frac{1}{n+1} w^{n+1},$$

which is to say that $g(w) = \sum_{n=1}^{\infty} c_n w^n$ where

$$c_n = \frac{1}{2n\pi i} \int_C \frac{dz}{[f(z)]^n}$$

(the constant term disappears because $g(0) = 0$). □

(200.3): With $h(z) := z/f(z)$, prove that

$$c_n = \frac{1}{n!} \frac{\partial^{(n-1)}(h^n)}{\partial z^{(n-1)}}(0).$$

Proof. Note, first of all, that $f'(0) \neq 0$ (else f could not be invertible), so f has a simple zero at 0, meaning $\frac{1}{f}$ has a simple pole at 0. Now, z has a simple zero at 0 as well, and so

$$\text{ord}_0(h) = \text{ord}_0\left(z \cdot \frac{1}{f}\right) = \text{ord}_0(z) + \text{ord}_0\left(\frac{1}{f}\right) = 1 + (-1) = 0,$$

so $h(z) = \frac{z}{f(z)}$ is holomorphic at 0. Since this is the only point at which $f(z) = 0$, we see that h and, hence, h^n, is holomorphic on $\overline{D(0,r)}$. Therefore, for any $n > 0$, we know, by Cauchy's Theorem, that

$$[h(w)]^n = \frac{1}{2\pi i} \int_C \frac{(h(z))^n dz}{z - w}.$$

Hence, by a slight modification of (1),

$$\begin{aligned}
[h(w)]^n &= \frac{1}{2\pi i} \int_C \left[\sum_{k=0}^{\infty} \frac{w^k [h(z)]^n}{z^{k+1}} dz \right] \\
&= \sum_{k=0}^{\infty} \left[\frac{1}{2\pi i} \int_C \frac{[h(z)]^n}{z^{k+1}} dz \right] w^k \\
&= \sum_{k=0}^{\infty} \left[\frac{1}{2\pi i} \int_C \frac{\frac{z^n}{[f(z)]^n}}{z^{k+1}} dz \right] w^k \\
&= \sum_{k=0}^{\infty} \left[\frac{1}{2\pi i} \int_C \frac{z^{n-k-1}}{[f(z)]^n} dz \right] w^k.
\end{aligned}$$

Hence, focusing on the $(n-1)$st term in this series,

$$\frac{1}{2\pi i} \int_C \frac{dz}{[f(z)]^n} = \frac{1}{(n-1)!} \frac{\partial^{(n-1)} h^n}{\partial z^{(n-1)}}(0)$$

by the uniqueness of the Taylor series. Now, notice that this is almost the nth term in the series for g; we need only divide both sides by n to see that

$$c_n = \frac{1}{2n\pi i}\int_C \frac{dz}{[f(z)]^n} = \frac{1}{n!}\frac{\partial^{(n-1)} h^n}{\partial z^{(n-1)}}(0).$$

□

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Show that there is *no* holomorphic function $f \in \mathcal{H}(\mathbb{C}\backslash\{1,-1\})$ with

$$f'(z) = \frac{1}{z^2-1}.$$

Determine whether the same results hold for $\mathbb{C}\backslash\overline{D(0,1)} = \{z \in \mathbb{C} | 1 < |z|\}$.

Proof. Suppose there exists $f \in \mathcal{H}(\mathbb{C}\backslash\{-1,1\})$ such that $f'(z) = 1/(z^2-1)$. Then, for any closed curve γ contained in $\mathbb{C}\backslash\{-1,1\}$,

$$\int_\gamma \frac{1}{z^2-1}dz = 0.$$

Now, consider the curve $\gamma : [0,1] \to \mathbb{C}\backslash\{-1,1\}$ given by

$$t \mapsto 1 + e^{2\pi i t}.$$

Then

$$\int_\gamma \frac{dz}{z^2-1} = \int_\gamma \frac{\frac{1}{z+1}}{z-1}dz.$$

Since $\frac{1}{z+1}$ is holomorphic on $\overline{D(1,1)}$, we know, by Cauchy's theorem, that

$$\int_\gamma \frac{\frac{1}{z+1}}{z-1}dz = 2\pi i \frac{1}{1+1} = \pi i \neq 0.$$

Therefore, we see that $\frac{1}{z^2-1}$ does not have a primitive on all of $\mathbb{C}\backslash\{-1,1\}$ and, hence, there is no such f.

On the other hand, consider the region $\Omega = \mathbb{C}\backslash\overline{D(0,1)}$. Let γ be a closed curve in Ω. Then, by the result proved in Problem Set 6 #6, γ is a closed curve in $\mathbb{C}\backslash\{0\}$ homotopic to $\eta(t) = e^{2\pi i n t}$ where $n = n(\gamma, 0)$. Since the image of the homotopy used in that problem was entirely contained in $\mathbb{C}\backslash\overline{D(0,1)}$, we can modify it slightly to see that γ is homotopic to the curve $\eta_r(t) = re^{2\pi i n t}$, which is contained in $\mathbb{C}\backslash\overline{D(0,1)}$. Let $g = \frac{1}{z^2-1}$. Now, by the residue theorem,

$$\int_\gamma g(z)dz = \int_{\eta_r} g(z)dz = \text{res}(g)_{-1} n(\eta_r, -1) + \text{res}(g)_1 n(\eta_r, 1). \tag{2}$$

Since 1 and -1 lie in the same connected component of $\mathbb{C}\backslash\eta_r$ as 0, we see that $n(\eta_r, -1) = n(\eta_r, 1) = n(\eta_r, 0) = n$. Since the residue of g at a point

a is just equal to $\frac{1}{2\pi i}\int_{\gamma_a} g$ where $\gamma_a(t) = a + \rho e^{2\pi i t}$ for some small ρ, we see that

$$\text{res}(g)_{-1} = \frac{1}{2\pi i}\int_{\gamma_{-1}} \frac{dz}{z^2-1} = \frac{1}{2\pi i}\int_{\gamma_{-1}} \frac{\frac{1}{z-1}}{z+1}dz = \frac{1}{-1-1} = \frac{-1}{2}$$

by Cauchy's Theorem. Similarly,

$$\text{res}(g)_1 = \frac{1}{2\pi i}\int_{\gamma_1} \frac{dz}{z^2-1} = \frac{1}{2\pi i}\int_{\gamma_1} \frac{\frac{1}{z+1}}{z-1}dz = \frac{1}{1+1} = \frac{1}{2},$$

again by Cauchy's Theorem. Therefore, putting all the pieces of (2) together, we see that

$$\int_\gamma g(z)dz = \text{res}(g)_{-1}n(\eta_r, -1) + \text{res}(g)_1 n(\eta_r, 1) = \frac{-1}{2}n + \frac{1}{2}n = 0.$$

Since our choice of γ was arbitrary, we see that $\int_\gamma g(z)dz = 0$ for any closed curve γ in Ω. This, in turn means that g has a primitive on Ω, so there is indeed an $f \in \mathcal{H}(\mathbb{C}\backslash\overline{D(0,1)})$ such that

$$f'(z) = \frac{1}{z^2-1}.$$

□

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This exercise provides a quantitative version of the statement that if $f'(0) \neq 0$ then f is injective in a neighborhood of 0. Specifically, consider a function $f \in \mathcal{H}[D(0,1)]$ with a MacLaurin series

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

such that

$$|c_1| > \sum_{n=2}^{\infty} n|c_n|.$$

Prove that f is injective on the open unit disc $D(0,1)$.

Proof. Suppose $w \in D(0,1)$ such that $f'(w) = 0$. Then

$$0 = f'(w) = \sum_{n=1}^{\infty} nc_n w^{n-1};$$

hence, $c_1 = -\sum_{n=2}^{\infty} nc_n w^{n-1}$. However,

$$\left| -\sum_{n=2}^{\infty} nc_n w^{n-1} \right| \leq \sum_{n=2}^{\infty} |nc_n w^{n-1}| \leq \sum_{n=2}^{\infty} n|c_n| < |c_1|,$$

so this is impossible. Hence, we see that $f'(z) \neq 0$ for all $z \in D(0,1)$. Hence, for each $w \in D(0,1)$, there exists a neighborhood U_w and a function g_w such that $f|_{U_w}$ is injective and $g_w \circ f|_{U_w} = 1$. Now, whenever $U_w \cap U_{w'} \neq \emptyset$, this in turn implies that $g_w|_{U_w \cap U_{w'}} = g_{w'}|_{U_w \cap U_{w'}}$, so we can analytically continue g_0 to a well-defined function g on all of $D(0,1)$. Now, at any point $z \in D(0,1)$,

$$g \circ f(z) = g_z \circ f|_{U_z}(z) = 1,$$

so f has a global inverse on all of $D(0,1)$. Therefore, it must be the case that f is injective on all of $D(0,1)$. □

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Verify that the method of residues applies to the calculation of

$$\int_{-\infty}^{\infty} \exp(-x^2)dx = \sqrt{\pi}.$$

For instance, integrate $z \mapsto \exp(\pi i z^2) \cdot \tan(\pi z)$ along the parallelogram with vertices at $\pm(r+ir)$ and $1 \pm (r+ir)$.

Proof. First off, let $f(z) = \cos(\pi z)$. Then $f'(z) = -\pi \sin(\pi z)$, so

$$\frac{f'}{f} = \frac{-\pi \sin(\pi z)}{\cos(\pi z)} = -\pi \tan(\pi z).$$

Now, $\frac{f'}{f}$ has a simple pole wherever f has a zero; in this region, the only such point is $z = 1/2$. Hence, by the argument principle, we know that

$$\operatorname{res}(f'/f, 1/2) = \operatorname{ord}_{1/2}(\cos(\pi z)) = 1$$

since $-\pi \sin(\pi 1/2) = -\pi \neq 0$. Hence,

$$\operatorname{res}(\tan(\pi z), 1/2) = \frac{-1}{\pi} \operatorname{res}(f'/f, 1/2) = \frac{-1}{\pi}.$$

In turn, this implies that

$$\operatorname{res}(e^{i\pi z^2} \tan(\pi z), 1/2) = \frac{-1}{\pi},$$

since the power series for $e^{i\pi z^2}$ is has the first few terms $1 + \frac{i\pi}{1!} z - \frac{\pi^2}{2!} z^2 - \ldots$. Thus, if R denotes the indicated region,

$$\int_R e^{i\pi z^2} \tan \pi z dz = \frac{-1}{\pi}.$$

Now, let L_1 denote the horizontal piece of R given by $t \mapsto t+ir$ for t between r and $r+1$ and let L_2 denote the piece given by $t \mapsto 1-t-ir$. Then

$$\begin{aligned}
\int_{L_1} + \int_{L_2} e^{i\pi z^2} \tan(\pi z) dz &= \int_{r+1}^{r} e^{i\pi(t+ir)^2} \tan(\pi(t+ir))dt + \int_{r}^{r+1} e^{i\pi(1-t-ir)^2} \tan(\pi(1-z-ir))dt \\
&= -\int_{r}^{r+1} e^{i\pi(t^2-r^2)} e^{-2\pi rt} \tan(\pi(t+ir))dt \\
&\quad + \int_{r}^{r+1} e^{\pi i} e^{\pi i(-2t+t^2-r^2)} e^{2\pi r-2\pi rt} \tan(\pi(-t-ir))dt \\
&= -\int_{r}^{r+1} e^{i\pi(t^2-r^2)} e^{-2\pi rt} \tan(\pi(t+ir))dt \\
&\quad - \int_{r}^{r+1} e^{\pi i} e^{\pi i(-2t+t^2-r^2)} e^{2\pi r-2\pi rt} \tan(\pi(t+ir))dt \\
&= \int_{r}^{r+1} e^{\pi i(-2t+2t^2-2r^2)} e^{2\pi r-4\pi rt} \tan(\pi(t+ir))dt.
\end{aligned}$$

Now, note that $e^{\pi i(-2t+2t^2-2r^2)}$ and $\tan(\pi(t+ir))$ are bounded and $2\pi r - 4\pi rt \to -\infty$ as $r \to \infty$, so $e^{2\pi r-4\pi rt} \to 0$ as r gets large. This, then, implies that the above integral goes to zero as r gets large.

Therefore, the only contribution to the integral $\int_R e^{i\pi z^2} \tan(\pi z)dz$ must come from the diagonal pieces. Now, we can parametrize the diagonal piece D_1 passing through the origin by $t \mapsto t+it$ and the other diagonal D_2 by $t \mapsto 1+t+it$. Then

$$\begin{aligned}
\int_{D_1} + \int_{D_2} e^{i\pi z^2} \tan(\pi z)dz &= \int_{r}^{-r} e^{i\pi(t+it)^2} \tan(\pi(t+it))dt + \int_{-r}^{r} e^{\pi i(1+t+it)^2} \tan(\pi(1+t+it))dt \\
&= -\int_{-r}^{r} e^{i\pi(t+it)^2} \tan(\pi(t+it))dt + \int_{-r}^{r} e^{i\pi(1+2t+2it+(t+it)^2)} \tan(\pi(t+it))dt \\
&= \int_{-r}^{r} \left(-1 - e^{i\pi(2t+2it)}\right) e^{\pi i(t+it)^2} \tan(\pi(t+it))dt \\
&= \int_{-r}^{r} \left(-1 - e^{i\pi(2t+2it)}\right) e^{-2\pi t^2} \tan(\pi(t+it))dt.
\end{aligned}$$

Now,

$$\begin{aligned}
\left(-1 - e^{i\pi(2t+2it)}\right) \tan(\pi(t+it)) &= \left(-1 - e^{i\pi(2t+2it)}\right) \frac{1}{i} \frac{e^{i\pi(t+it)} - e^{-i\pi(t+it)}}{e^{i\pi(t+it)} + e^{-i\pi(t+it)}} \\
&= \frac{1}{i} \left(-1 - e^{i\pi(2t+2it)}\right) \frac{e^{i\pi(t+it)} - e^{-i\pi(t+it)}}{e^{-i\pi(t+it)}(e^{2\pi i(t+it)} + 1)} \\
&= \frac{1}{i} \frac{-e^{i\pi(t+it)} + e^{-i\pi(t+it)}}{e^{-i\pi(t+it)}} \\
&= \frac{1}{i} \left[-e^{2i\pi(t+it)} + 1\right].
\end{aligned}$$

Hence,

$$\int_{D_1} + \int_{D_2} = -\frac{1}{i}\int_{-r}^{r} e^{2\pi i(t+it)} e^{-2\pi t^2}dt + \frac{1}{i}\int_{-r}^{r} e^{-2\pi t^2}dt.$$

We would like to conclude that, as $r \to \infty$, the right hand side goes to $\frac{1}{\pi\sqrt{\pi}}\int_{-r\sqrt{\pi}}^{r\sqrt{\pi}} e^{-x^2}dx$ and, hence, that

$$\frac{-1}{\pi} = \operatorname{res}(e^{i\pi z^2}\tan(\pi z), 1/2) = \frac{1}{2\pi i}\int_R e^{i\pi z^2}\tan(\pi z)dz = \frac{-1}{\pi\sqrt{\pi}}\int_{-r\sqrt{\pi}}^{r\sqrt{\pi}} e^{-x^2}dx,$$

meaning that that $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$. □

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Let R_N be the square with vertices at $(N+1/2)(\pm 1 \pm i)$. Prove that there exists a real constant $C > 0$ such that $|\cot(\pi z)| \leq C$ for every positive integer N and every $z \in R_N$.

Proof. On the horizontal sides of R_N, $z = x \pm i(N+1/2)$, and so

$$\begin{aligned}
|\cot(\pi z)| &= \left|\frac{\cos(\pi z)}{\sin(\pi z)}\right| \\
&= \left| i\frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}}\right| \\
&= \left|\frac{e^{i\pi(x\pm i(N+1/2))} + e^{-i\pi(x\pm i(N+1/2))}}{e^{i\pi(x\pm i(N+1/2))} - e^{-i\pi(x\pm i(N+1/2))}}\right| \\
&= \left|\frac{e^{i\pi x}e^{\mp\pi(N+1/2)} + e^{-i\pi x}e^{\pm\pi(N+1/2)}}{e^{i\pi x}e^{\mp\pi(N+1/2)} - e^{-i\pi x}e^{\pm\pi(N+1/2)}}\right| \\
&\leq \frac{|e^{i\pi x}||e^{\mp\pi(N+1/2)}| + |e^{-i\pi x}||e^{\pm\pi(N+1/2)}|}{|e^{i\pi x}e^{\mp\pi(N+1/2)} - e^{-i\pi x}e^{\pm\pi(N+1/2)}|} \\
&= \frac{|e^{\mp\pi(N+1/2)}| + |e^{\pm i(N+1/2)}|}{|e^{i\pi x}e^{\mp\pi(N+1/2)} - e^{-i\pi x}e^{\pm\pi(N+1/2)}|} \\
&\leq \frac{|e^{\mp\pi(N+1/2)}| + |e^{\pm\pi(N+1/2)}|}{||e^{i\pi x}e^{\mp\pi(N+1/2)}| - |e^{-i\pi x}e^{\pm\pi(N+1/2)}||} \\
&= \frac{e^{\mp\pi(N+1/2)} + e^{\pm\pi(N+1/2)}}{|e^{\mp\pi(N+1/2)} - e^{\pm\pi(N+1/2)}|} \\
&= \frac{e^{\pi(N+1/2)} + e^{-\pi(N+1/2)}}{e^{\pi(N+1/2)} - e^{-\pi(N+1/2)}}.
\end{aligned}$$

Now, if $n > m > 0$, then

$$\frac{e^n + e^{-n}}{e^n + e^{-n}} - \frac{e^m + e^{-m}}{e^m - e^{-m}} = \frac{2e^{m-n} - 2e^{n-m}}{e^{n+m} - e^{m-n} - e^{n-m} + e^{-n-m}}.$$

Since $m < n$, $m - n$ is negative; as is $-n - m$; hence, we see that the numerator of the right hand side is negative while the denominator is positive. Thus, $\frac{e^x+e^{-x}}{e^x-e^{-x}}$ is a decreasing function for $x > 0$. Hence, applying this knowledge to the cotangent inequality above, we see that

$$|\cot(\pi z)| \leq \frac{e^{\pi(N+1/2)} + e^{-\pi(N+1/2)}}{e^{\pi(N+1/2)} - e^{-\pi(N+1/2)}} \leq \frac{e^{3\pi/2} + e^{-3\pi/2}}{e^{3\pi/2} - e^{-3\pi/2}} = C_1.$$

On the other hand, on the vertical sides of R_N, $z = \pm(N + 1/2) + iy$, so for these z,

$$\begin{aligned} |\cot(\pi z)| &= \left| \frac{e^{i\pi(\pm(N+1/2)+iy)} + e^{-i\pi(\pm(N+1/2)+iy)}}{e^{i\pi(\pm(N+1/2)+iy)} - e^{-i\pi(\pm(N+1/2)+iy)}} \right| \\ &= \left| \frac{e^{\pm i\pi(N+1/2)}e^{-\pi y} + e^{\mp i\pi(N+1/2)}e^{\pi y}}{e^{\pm i\pi(N+1/2)}e^{-\pi y} - e^{\mp i\pi(N+1/2)}e^{\pi y}} \right| \\ &= \left| \frac{\pm ie^{\pm i\pi N}e^{-\pi y} + \mp ie^{\mp i\pi N}e^{\pi y}}{\pm ie^{\pm i\pi N}e^{-\pi y} - \mp ie^{\mp i\pi N}e^{\pi y}} \right| \\ &= \left| \frac{e^{\pm i\pi N}e^{-\pi y} - e^{\mp i\pi N}e^{\pi y}}{e^{\pm i\pi N}e^{-\pi y} + e^{\mp i\pi N}e^{\pi y}} \right| \\ &= \left| \frac{e^{-\pi y} - e^{\pi y}}{e^{-\pi y} + e^{\pi y}} \right| \\ &= |\tan(i\pi y)| \\ &= 1. \end{aligned}$$

Hence, if $C = \max\{1, C_1\}$, we see that $|\cot(\pi z)| \leq C$ for all $z \in R_N$. □

(218.2): Prove that

$$\sum_{n\in\mathbb{Z}} f(n) = -\sum_{p\in P_f} \text{res}[\pi\cot(\pi z)f(z), p].$$

Proof. By the residue theorem, with R_N as above,

(3)
$$\frac{1}{2\pi i}\int_{R_N} \pi\cot(\pi z)f(z)dz = \sum_{k=-N}^{N} \text{res}(\pi\cot(\pi z)f(z), k) + \sum_{p\in P_f} \text{res}(\pi\cot(\pi z)f(z), p)$$

so long as R_N is large enough to contain all of the poles of f. Since $\cot(\pi z) = \frac{1}{\pi}\frac{\pi\cos(\pi z)}{\sin(\pi z)}$ and the derivative of $\sin(\pi z)$ is $\pi\cos(\pi z)$, we see that $\cot(\pi z)$ has a simple pole at n for each $n \in \mathbb{Z}$ and that $\text{res}(\pi\cot(\pi z), n) = \text{ord}_p(\sin(\pi z)) = 1$. Therefore,

$$\text{res}(\pi\cot(\pi z)f(z), n) = f(n)$$

for each such n. Therefore,

$$\sum_{k=-N}^{N} \text{res}(\pi\cot(\pi z)f(z),k) = \sum_{k=-N}^{N} f(k). \tag{4}$$

Now,

$$\left|\int_{R_N} \pi\cot(\pi z)f(z)dz\right| \le \pi\int_{R_N} |\cot(\pi z)||f(z)||dz| \le \pi CM\int_{R_N} \frac{1}{|z|^m}|dz| \le \frac{\pi M(4N+4)}{N^m}$$

which goes to 0 for N sufficiently large. Hence, applying this fact and (4) to (3), we see that,

$$\lim_{N\to\infty}\sum_{k=-N}^{N} f(k) + \sum_{p\in P_f} \text{res}(\pi\cot(\pi z)f(z),p) = 0,$$

so

$$\sum_{n\in\mathbb{Z}} f(n) = -\sum_{p\in P_f} \text{res}[\pi\cot(\pi z)f(z),p].$$

Now, we apply this fact to the sum $\sum_{n=1}^{\infty}\frac{1}{n^2}$. Note first that

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{2}\sum_{n=-\infty}^{\infty}\frac{1}{n^2}$$

since $(-n)^2 = n^2$. Now, we've already seen that $\cot(\pi z)$ has a simple pole at zero (and, in fact, all integers), so the Laurent expansion of $\cot(\pi z)$ is $\frac{c_{-1}}{z} + c_0 + c_1 z + \dots$ for some c_i, $i = -1,\dots,$. Now, since $\cot(\pi z) = \frac{\cos(\pi z)}{\sin(\pi z)}$, using the power series for sin and cos, we know that

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(\pi z)^{2n} = \left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(\pi z)^{2n+1}\right)\left(\frac{c_{-1}}{z} + c_0 + c_1 z + c_2 z + \dots\right). \tag{5}$$

Hence, $c_{-1} = \frac{1}{\pi}$, $c_0 = 0$, $c_1 = \frac{-\pi}{2} - \frac{\pi}{6} = \frac{-\pi}{3}$. Therefore,

$$\pi\cot(\pi z)f(z) = \frac{\pi\cot(\pi z)}{z^2} = \frac{\pi\left(\frac{1}{\pi z} - \frac{\pi z}{3} + \dots\right)}{z^2} = \frac{1}{z^3} - \frac{\pi^2}{3z} + \dots$$

so $\text{res}(\pi\cot(\pi z)f(z),0) = \frac{-\pi^2}{3}$. Hence,

$$\sum_{n=-\infty}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{3},$$

meaning that

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

To compute $\sum_{n=1}^{\infty} \frac{1}{n^4}$, we continue what we started in (5) above. Certainly $c_2 = 0$ since $c_0 = 0$. Now,

$$\frac{\pi^4}{4!} = \frac{-c_1\pi^3}{3!} + \frac{c_{-1}\pi^5}{5!} + c_3\pi = \frac{\pi^4}{18} + \frac{\pi^4}{120} + c_3\pi,$$

so

$$c_3 = \frac{\pi^3}{24} - \frac{\pi^3}{18} - \frac{\pi^3}{120} = \frac{-\pi^3}{45}.$$

Hence,

$$\frac{\pi\cot(\pi z)}{z^4} = \frac{\pi\left(\frac{1}{\pi z} - \frac{\pi z}{3} - \frac{\pi^3 z^3}{45} + \ldots\right)}{z^4} = \frac{1}{z^5} - \frac{\pi^2}{3z^3} - \frac{\pi^4}{45z},$$

so

$$\operatorname{res}\left(\frac{\pi\cot(\pi z)}{z^4}, 0\right) = -\frac{\pi^4}{45}.$$

Hence,

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{2}\sum_{n=-\infty}^{\infty} \frac{1}{n^4} = \frac{1}{2}\frac{\pi^4}{45} = \frac{\pi^4}{90}.$$

□

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