PDE HOMEWORK

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4.

On a bounded open set $\mathcal{D} \subset \mathbb{R}^n$ let the vector $u(x) = (u_1(x), \ldots, u_k(x))$ be a smooth solution of

$$-\Delta u + M(x)u = 0 \quad \text{with} \quad u = 0 \text{ on } \partial\mathcal{D},$$

where $M(x)$ is a smooth positive definite matrix-valued function. Show that $u \equiv 0$.

Proof. Supposing that u is such a solution, the fact that $Lu := -\Delta u + M(x)u = 0$ implies that $\langle Lu, u\rangle = 0$. Therefore,

$$\begin{aligned} 0 &= \int_{\mathcal{D}} \langle Lu, u\rangle dx \\ &= \int_{\mathcal{D}} \left[\langle -\Delta u, u\rangle + \langle M(x)u, u\rangle\right] dx \\ &= \int_{\mathcal{D}} \left[\sum_{i=1}^{k} -u_i \Delta u_i + \langle M(x)u, u\rangle\right] dx \\ &= \int_{\mathcal{D}} \left[\sum_{i=1}^{k} \|\nabla u_i\|^2 + \langle M(x)u, u\rangle\right] dx. \end{aligned}$$

Since M is positive definite, this expression can only be zero if $u \equiv 0$. □

5.

On a compact manifold without boundary, use the methods of sub- and super-solutions to show that if $|f(x,u)| \leq 1$, then the equation

$$\Delta u = f(x,u) + \sin u$$

has infinitely many solutions.

Also, if $c > 1$, show that the equation $\Delta u = c + \sin u$ has *no* solutions.

Proof. Consider the function $u_k(x) = \frac{(4k+1)\pi}{2}$. Then

$$\Delta u_k = 0 \leq f(x, u_k) + 1 = f(x, u_k) + \sin u_k,$$

since $|f(x,u)| \leq 1$. Hence, for all k, u_k is a supersolution of the given equation. Similarly, if $v_k = \frac{(4k+3)\pi}{2}$, then

$$\delta v_k = 0 \geq f(x, v_k) - 1 = f(x, v_k) + \sin v_k,$$

so the v_k are all subsolutions of the given equation. Therefore, there is at least one solution u of $\Delta u = f(x,u) + \sin u$ in each of the intervals

$$\frac{(4k+1)\pi}{2} \leq u(x) \leq \frac{(4k+3)\pi}{2};$$

since there are infinitely many such intervals, there are infinitely many solutions of this equation.

Now, consider $\Delta u = c + \sin u$ where $c > 1$. If u is a solution to this equation then, since the manifold is compact, u must have a maximum somewhere. At the maximum, $\Delta u \leq 0$; however, since $c > 1$,

$$\Delta u = c + \sin u > 0.$$

From this contradiction, then, we conclude that the equation has no solutions. □

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