DIFFERENTIAL GEOMETRY HW 1

CLAY SHONKWILER

2.

Let $\alpha : I \to \mathbb{R}^3$ be parametrized by arc length. Thus, the tangent vector $\alpha'(s)$ has unit length. Show that the norm $|\alpha''(s)|$ of the second derivative measures the rate of change of the angle which neighboring tangents make with the tangent at s.

Proof. Suppose the tangent $\alpha'(s)$ at s is $(1, 0, 0)$ and the normal vector $\alpha''(s)$ points in the direction $(0, 1, 0)$ (rotate if necessary). Then, if ds is an infinitesimal,

$$\alpha'(s + ds) = \alpha'(s) + \alpha''(s)ds = (1, |\alpha''(s)|ds, 0).$$

Now, if we let $\theta(h)$ be the angle between $\alpha'(s)$ and $\alpha'(s + h)$, then $d\theta$ is the angle between $\alpha'(s)$ and $\alpha'(s + ds)$, which is given by

$$d\theta = \tan^{-1}\left(\frac{|\alpha''(s)|ds}{1}\right) = \tan^{-1}(|\alpha''(s)|ds),$$

which, since ds is infinitesimal, is essentially $|\alpha''(s)|ds$, so $\frac{d\theta}{ds} = \alpha''(s)$. □

4.

Show that the unit vector $N(s)$ is normal to the curve, in the sense that $N(s) \cdot T(s) = 0$, where $T(s)$ is the unit tangent vector to the curve.

Proof. Basically, we play around a bit with the definition of the unit tangent vector, noting that $T(s) = \alpha'(s)$, so

$$1 = |T(s)| = |\alpha'(s)| = \sqrt{(\alpha_1'(s))^2 + (\alpha_2'(s))^2 + (\alpha_3'(s))^2}.$$

where $\alpha(s) = (\alpha_1(s), \alpha_2(s), \alpha_3(s))$. Hence,

$$1 = \alpha_1'(s) + \alpha_2'(s) + \alpha_3'(s).$$

Differentiating both sides with respect to s, we see that

$$0 = 2\alpha_1'(s)\alpha_1''(s) + 2\alpha_2'(s)\alpha_2''(s) + 2\alpha_3'(s)\alpha_3''(s).$$

Now, note that the right side is simply $2\alpha'(s) \cdot \alpha''(s)$, so we see that

$$0 = \alpha''(s) \cdot \alpha'(s) = |\alpha''(s)|\frac{\alpha''(s)}{|\alpha''(s)|} \cdot \alpha'(s) = |\alpha''(s)|N(s) \cdot T(s),$$

so we conclude that $N(s) \cdot T(s) = 0$, meaning that $N(s)$ is normal to the curve. □

5.

Show that $B'(s)$ is parallel to $N(s)$.

Proof. We show this by a straightforward computation. Note that, since $B(s) = T(s) \times N(s)$,

$$B'(s) = T'(s) \times N(s) + T(s) \times N'(s).$$

Now, $T(s) = \alpha'(s)$, so $T''(s) = \alpha''(s) = |\alpha''(s)|N(s)$, so

$$B'(s) = (|\alpha''(s)|N(s)) \times N(s) + T(s) \times N'(s) = T(s) \times N'(s).$$

Now, to show that $B'(s)$ is parallel to $N(s)$, we want to show that $B'(s) \times N(s) = 0$. Using the above identity for $B'(s)$, we see that

$$B'(s) \times N(s) = (T(s) \times N'(s)) \times N(s) = (T(s) \cdot N(s))N'(s) - (N'(s) \cdot N(s))T(s) = -(N'(s) \cdot N(s))T(s).$$

Hence, the problem reduces to showing that $N'(s) \cdot N(s) = 0$. Now, if $\alpha(s) = (\alpha_1(s), \alpha_2(s), \alpha_3(s))$, then $\alpha''(s) = (\alpha_1''(s), \alpha_2''(s), \alpha_3''(s))$, so

$$\begin{aligned}
|\alpha''(s)|' &= \left(\sqrt{\alpha_1''(s)^2 + \alpha_2''(s)^2 + \alpha_3''(s)^2}\right)' \\
&= \frac{2(\alpha_1''(s)\alpha_1'''(s) + \alpha_2''(s)\alpha_3'''(s) + \alpha_3''(s)\alpha_3'''(s))}{2\sqrt{\alpha_1''(s)^2 + \alpha_2''(s)^2 + \alpha_3''(s)^2}} \\
&= \frac{\alpha''(s) \cdot \alpha'''(s)}{|\alpha''(s)|}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
N'(s) = \left(\frac{\alpha''(s)}{|\alpha''(s)|}\right)' &= \frac{\alpha'''(s)}{|\alpha''(s)} - \frac{1}{|\alpha''(s)|^2}\left(\frac{\alpha''(s) \cdot \alpha'''(s)}{|\alpha''(s)|}\right)\alpha''(s) \\
&= \frac{\alpha'''(s)}{|\alpha''(s)|} - \frac{\alpha''(s) \cdot \alpha'''(s)}{|\alpha''(s)|^3}\alpha''(s).
\end{aligned}$$

Hence,

$$\begin{aligned}
N'(s) \cdot N(s) &= \left(\frac{\alpha'''(s)}{|\alpha''(s)|} - \frac{\alpha''(s) \cdot \alpha'''(s)}{|\alpha''(s)|^3}\alpha''(s)\right) \cdot \frac{\alpha''(s)}{|\alpha''(s)|} \\
&= \frac{\alpha'''(s) \cdot \alpha''(s)}{|\alpha''(s)|^2} - \frac{\alpha''(s) \cdot \alpha'''(s)}{|\alpha''(s)|^4}\alpha''(s) \cdot \alpha''(s) \\
&= \frac{\alpha'''(s) \cdot \alpha''(s)}{|\alpha''(s)|^2} - \frac{\alpha''(s) \cdot \alpha'''(s)}{|\alpha''(s)|^2} \\
&= 0.
\end{aligned}$$

Therefore, we can conclude that $B'(s)$ is parallel to $N(s)$. □

6.

Find the curvature and torsion of the helix

$$\alpha(t) = (a\cos t, a\sin t, bt).$$

Answer: We know how to compute curvature and torsion given a curve parametrized by arc length, so we want to re-parametrize this curve by arc length and then use the definition of curvature and torsion. Let $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$. Now, arc length is given by

$$\begin{aligned} s(t) = \int_0^t |\alpha'(t)|dt &= \int_0^t \sqrt{\alpha_1'(t)^2 + \alpha_2'(t)^2 + \alpha_3'(t)}dt \\ &= \int_0^t \sqrt{a^2(\sin^2 t + \cos^2 t) + b^2}dt \\ &= \int_0^t \sqrt{a^2 + b^2}dt \\ &= \sqrt{a^2 + b^2}t. \end{aligned}$$

Hence, $t(s) = \frac{s}{\sqrt{a^2+b^2}}$, so

$$\beta(s) = \alpha(t(s)) = \left(a\cos\frac{s}{\sqrt{a^2+b^2}}, a\sin\frac{s}{\sqrt{a^2+b^2}}, \frac{bs}{\sqrt{a^2+b^2}}\right)$$

is a reparametrization of the curve by arc length. Differentiating, we see that

$$\alpha'(s) = \left(\frac{-a\sin\frac{s}{\sqrt{a^2+b^2}}}{\sqrt{a^2+b^2}}, \frac{a\cos\frac{s}{\sqrt{a^2+b^2}}}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}\right).$$

Therefore,

$$\alpha''(s) = \left(\frac{-a\cos\frac{s}{\sqrt{a^2+b^2}}}{a^2+b^2}, \frac{-a\sin\frac{s}{\sqrt{a^2+b^2}}}{a^2+b^2}, 0\right)$$

and so the curvature of the helix is given by

$$\kappa(s) = |\alpha''(s)| = \sqrt{\frac{a^2\cos^2\left(\frac{s}{\sqrt{a^2+b^2}}\right)}{(a^2+b^2)^2} + \frac{a^2\sin^2\left(\frac{s}{\sqrt{a^2+b^2}}\right)}{(a^2+b^2)^2}} = \sqrt{\frac{a^2}{a^2+b^2}} = \frac{|a|}{a^2+b^2}.$$

Now, $B'(s) = -\tau(s)N(s)$, so we need to compute $B'(s)$ and $N(s)$. To that end, if $a > 0$,

$$\begin{aligned} N(s) &= \frac{\alpha''(s)}{|\alpha''(s)|} \\ &= \frac{a^2+b^2}{|a|}\left(\frac{-a\cos\frac{s}{\sqrt{a^2+b^2}}}{a^2+b^2}, \frac{-a\sin\frac{s}{\sqrt{a^2+b^2}}}{a^2+b^2}, 0\right) \\ &= -\left(\cos\frac{s}{\sqrt{a^2+b^2}}, \sin\frac{s}{\sqrt{a^2+b^2}}, 0\right). \end{aligned}$$

Now, we showed in problem 5 above that $B'(s) = T(s) \times N'(s) = \alpha'(s) \times N'(s)$. Now,

$$N'(s) = -\left(\frac{-\sin \frac{s}{\sqrt{a^2+b^2}}}{\sqrt{a^2+b^2}}, \frac{\cos \frac{s}{\sqrt{a^2+b^2}}}{\sqrt{a^2+b^2}}, 0 \right) = \frac{1}{\sqrt{a^2+b^2}} \left(\sin \frac{s}{\sqrt{a^2+b^2}}, -\cos \frac{s}{\sqrt{a^2+b^2}}, 0 \right).$$

Therefore,

$$B'(s) = \alpha'(s) \times N'(s) = \frac{1}{a^2+b^2} \left(b \cos \frac{s}{\sqrt{a^2+b^2}}, b \sin \frac{s}{\sqrt{a^2+b^2}}, 0 \right).$$

Hence, we see that $B'(s) = -\frac{b}{a^2+b^2} N(s)$, so $\tau(s) = \frac{b}{a^2+b^2}$ when $a > 0$. When $a < 0$, we introduce a sign change to $N(s)$ and nowhere else, so $\tau(s) = \frac{-b}{a^2+b^2}$. Of course, when $a = 0$, $\tau(s) = 0$, since $\alpha(s)$ is a straight line in this case.

♣

7.

Let $\alpha : I \to \mathbb{R}^3$ be parametrized by arc length and have nowhere vanishing curvature $\kappa(s)$. Show that

$$\begin{array}{llll} T'(s) & = & \kappa(s)N(s) & \\ N'(s) & = -\kappa(s)T(s) & & +\tau(s)B(s) \\ B'(s) & = & -\tau(s)N(s) & \end{array}$$

Proof. We know that $B'(s) = -\tau(s)N(s)$ by definition of τ. Furthermore, since $T(s) = \alpha'(s)$,

$$T'(s) = \alpha''(s) = |\alpha''(s)| \frac{\alpha''(s)}{|\alpha''(s)|} = \kappa(s)N(s).$$

Finally, to compute $N'(s)$, we use an identity to re-express $N(s)$, then compute the derivative using the above identities. First, since $B(s) = T(s) \times N(s)$ by definition,

$$B(s) \times T(s) = (T(s) \times N(s)) \times T(s) = (T(s) \cdot T(s))N(s) - (N(s) \cdot T(s))T(s) = N(s),$$

since T is a unit vector and T and N are orthogonal. Hence, since the derivative of the cross product satisfies the Leibniz identity,

$$N'(s) = B'(s) \times T(s) + B(s) \times N'(s)$$

Using the identities we just proved,

$$N'(s) = -\tau(s)N(s) \times T(s) + B(s) \times (\kappa(s)N(s)).$$

Using the equality $B(s) = T(s) \times N(s)$, this reduces to

$$\begin{aligned} N'(s) &= \tau(s)B(s) + \kappa(s)(T(s) \times N(s)) \times N(s) \\ &= \tau(s)B(s) + \kappa(s)\left[(T(s) \cdot N(s))N(s) - (N(s) \cdot N(s))T(s)\right] \\ &= \tau(s)B(s) - \kappa(s)T(s), \end{aligned}$$

as desired. □

8.

The curvature of a smooth curve in the plane can be given a well-defined sign, just like the torsion of a curve in 3-space. Explain why this is so.

Answer: Intuitively, the idea is that a curve in the plane is always either going in a straight line or bending to the left of the tangent line or bending to the right of the tangent line. We can call bending "to the left" positive curvature (to accord with our notion of counter-clockwise rotation being positive) and bending "to the right" negative curvature. More rigorously, when the curve is bending to the left, the normal is to the left of the tangent, so the binormal B points up. When the curve is bending to the right, the normal is the the right of the tangent, so the binormal B points down. Since the binormal is a unit vector, it is always either $\pm(0,0,1)$. Hence, we can define the sign on curvature to be the sign on the binormal B as a multiple of $(0,0,1)$. In other words, if κ denotes curvature in 3-space and $\bar{\kappa}$ signed curvature in the plane, we define

$$\bar{\kappa} := |\kappa| \left(\langle B, (0,0,1)\rangle\right).$$

♣

9.

Given a smooth function $\kappa(s)$ defined for all s in the interval I, show that the arc length parametrized plane curve having $\kappa(s)$ as curvature is given by

$$\alpha(s) = \left(\int \cos\theta(s)ds + a, \int \sin\theta(s)ds + b\right)$$

where

$$\theta(s) = \int \kappa(s)ds + \theta_0.$$

Show that this solution is unique up to translation by (a,b) and rotation by θ_0.

Proof. First, we need to show that $\alpha(s)$ is parametrized by arc length. To show this, it suffices to show that $|\alpha'(s)| = 1$ for all s. Now,

$$\alpha'(s) = (\cos\theta(s), \sin\theta(s)),$$

so

$$|\alpha'(s)|^2 = \cos^2\theta(s) + \sin^2\theta(s) = 1,$$

so α is indeed parametrized by arc length.

Now, since $\frac{d}{ds}(\theta(s)) = \kappa(s)$,

$$\alpha''(s) = (-\sin\theta(s)\kappa(s), \cos\theta(s)\kappa(s)) = \kappa(s)(-\sin\theta(s), \cos\theta(s)).$$

Therefore, the absolute value of the curvature of α is given by

$$|\alpha''(s)| = |\kappa(s)|\sqrt{\sin^2\theta(s) + \cos^2\theta(s)} = |\kappa(s)|.$$

Furthermore, the binormal

$$\begin{aligned} B = \alpha'(s) \times \frac{\alpha''(s)}{|\alpha''(s)|} &= \frac{1}{|\alpha''(s)|}(\alpha'(s) \times \alpha''(s)) \\ &= \frac{1}{|\alpha''(s)|}(0, 0, \kappa(s)(\cos^2\theta + \sin^2\theta)) \\ &= \frac{\kappa(s)}{|\kappa(s)|}(0, 0, 1), \end{aligned}$$

so $\langle B, (0,0,1)\rangle = \frac{\kappa(s)}{|\kappa(s)|}$ has the same sign of $\kappa(s)$. Since we defined the sign of the curvature to be $\langle B, (0,0,1)\rangle$ in problem 8, we see that the signed curvature is given by $\kappa(s)$.

Now, since $\tau(s) = 0$ for all $s \in I$, the Frenet equations reduce to

$$\begin{aligned} T'(s) &= \kappa(s)N(s) \\ N'(s) &= -\kappa(s)T(s). \end{aligned}$$

Since $\alpha(s)$ has tangent and normal vectors satisfying this linear first-order system of ODEs, the uniqueness of solutions of linear first-order systems of ODEs guarantees that $\alpha(s)$ is the unique curve (up to choice of initial conditions a, b, θ_0) with curvature $\kappa(s)$. □

10.

Prove that if we start off with an orthonormal frame $T(s_0)$, $N(s_0)$, $B(s_0)$, then the solution will be an orthonormal frame for all $s \in I$.

Proof.

$$\begin{aligned} &f_1(s) = \langle T(s), T(s)\rangle, f_2(s) = \langle N(s), N(s)\rangle, \qquad f_3(s) = \langle B(s), B(s)\rangle \\ &f_4(s) = \langle T(s), N(s)\rangle f_5(s) = \langle T(s), B(s)\rangle \qquad f_6(s) = \langle N(s), B(s)\rangle \end{aligned}$$

When $s = s_0$, the above give $f_1(s_0) = f_2(s_0) = f_3(s_0) = 1$ and $f_4(s_0) = f_5(s_0) = f_6(s_0) = 0$.

These 6 also satisfy a system of 1st order linear ODEs:

$$f_1'(s) = 2\langle T'(s), T(s)\rangle = 2\kappa(s)\langle T(s), N(s)\rangle = 2\kappa(s)f_4(s)$$

$$f_2'(s) = 2\langle N'(s), N(s)\rangle = 2\left[-\kappa(s)\langle T(s), N(s)\rangle + \tau(s)\langle B(s), N(s)\rangle\right] = 2\left[-\kappa(s)f_4(s) + \tau(s)f_6(s)\right]$$

$$f_3'(s) = 2\langle B'(s), B(s)\rangle = -2\tau(s)\langle N(s), B(s)\rangle = -2\tau(s)f_6(s)$$

$$\begin{aligned} f_4'(s) &= \langle T'(s), N(s)\rangle + \langle T(s), N'(s)\rangle \\ &= \kappa(s)\langle N(s), N(s)\rangle - \kappa(s)\langle T(s), T(s)\rangle + \tau(s)\langle T(s), B(s)\rangle \\ &= \kappa(s)f_2(s) - \kappa(s)f_1(s) + \tau(s)f_5(s) \end{aligned}$$

$$f_5'(s) = \langle T'(s), B(s)\rangle + \langle T(s), B'(s)\rangle = \kappa(s)\langle N(s), B(s)\rangle - \tau(s)\langle T(s), N(s)\rangle = \kappa(s)f_6(s) - \tau(s)f_4(s)$$

$$\begin{aligned} f_6'(s) &= \langle N'(s), B(s)\rangle + \langle N(s), B'(s)\rangle \\ &= -\kappa(s)\langle T(s), B(s)\rangle + \tau(s)\langle B(s), B(s)\rangle - \tau(s)\langle N(s), N(s)\rangle \\ &= -\kappa(s)f_5(s) + \tau(s)f_3(s) - \tau(s)f_2(s) \end{aligned}$$

(using the Frenet equations). Note that the constant functions $f_1 \equiv f_2 \equiv f_3 \equiv 1$, $f_4 \equiv f_5 \equiv f_6 \equiv 0$ are solutions to this system of linear first order ODEs and satisfy the initial conditions. Therefore, by the uniqueness of solutions of linear systems of ODEs, the constant functions are the only solutions to this system and that, therefore, T, N, B are orthonormal for all $s \in I$. □

11.

Let $r = r(\theta)$, $a \leq \theta \leq b$, describe a plane curve in polar coordinates.

(a): Show that the arc length of this curve is given by

$$\int_a^b \left[r^2 + (r')^2\right]^{1/2} d\theta.$$

Proof. Let $\alpha(\theta) = (\alpha_1(\theta), \alpha_2(\theta))$ be the equation of the curve in rectangular coordinates. Then The length of the curve is given by

$$\int_a^b |\alpha'(\theta)| d\theta = \int_a^b \sqrt{\alpha_1'(\theta)^2 + \alpha_2'(\theta)^2} d\theta.$$

Now, $\alpha_1(\theta) = r(\theta)\cos\theta$ and $\alpha_2(\theta) = r(\theta)\sin\theta$, so

$$\begin{aligned} \alpha_1' &= r'\cos\theta - r\sin\theta \\ \alpha_2' &= r'\sin\theta + r\cos\theta. \end{aligned}$$

In turn, this implies that

$$\begin{aligned} (\alpha_1')^2 &= (r')^2\cos^2\theta - 2rr'\sin\theta\cos\theta + r^2\sin^2\theta \\ (\alpha_2')^2 &= (r')^2\sin^2\theta + 2rr'\sin\theta\cos\theta + r^2\cos^2\theta, \end{aligned}$$

so $(\alpha_1')^2 + (\alpha_2')^2 = (r')^2(\sin^2\theta + \cos^2\theta) + r^2(\sin^2\theta + \cos^2\theta) = (r')^2 + r^2$. Hence, the arc length is given by

$$\int_a^b \sqrt{(r')^2 + r^2} d\theta.$$

□

(b): Show that the curvature is given by

$$\kappa(\theta) = \left[2(r')^2 - rr'' + r^2\right] / \left[(r')^2 + r^2\right]^{3/2}.$$

Proof. Note that we showed in part (a) above that $|\alpha'| = \sqrt{(r')^2 + r^2}$, so $|\alpha'|^3 = \left[(r')^2 + r^2\right]^{3/2}$. Now, as we show in 12(a) below,

$$|\kappa(\theta)| = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3} = \frac{|\alpha' \times \alpha''|}{\left[(r')^2 + r^2\right]^{3/2}}.$$

Therefore, up to sign, it suffices to show that $|\alpha' \times \alpha''| = 2(r')^2 - rr' + r^2$. Now,

$$\alpha'(\theta) = (\alpha_1', \alpha_2') = (r' \cos\theta - r \sin\theta, r' \sin\theta + r\cos\theta)$$
$$\alpha''(\theta) = (\alpha_1'', \alpha_2'') = (r'' \cos\theta - 2r' \sin\theta - r\cos\theta, r'' \sin\theta + 2r' \cos\theta - r\sin\theta),$$

so

$$\begin{aligned}
\alpha' \times \alpha'' &= (0, 0, (r'\cos\theta - r\sin\theta)(r''\sin\theta + 2r'\cos\theta - r\sin\theta) \\
&\quad -(r'\sin\theta + r\cos\theta)(r''\cos\theta - 2r'\sin\theta - r\cos\theta)) \\
&= (0, 0, r'r''\sin\theta\cos\theta - rr''\sin^2\theta + 2(r')^2\cos^2\theta - 2rr'\sin\theta\cos\theta - rr'\sin\theta\cos\theta + r^2\sin \\
&\quad -r'r''\sin\theta\cos\theta - rr'\cos^2\theta + 2(r')^2\sin^2\theta + 2rr'\sin\theta\cos\theta + rr'\sin\theta\cos\theta + r^2\cos^2\theta) \\
&= (0, 0, 2(r')^2 - rr'' + r^2).
\end{aligned}$$

Therefore,

$$|\kappa(\theta)| = \frac{|(0, 0, 2(r')^2 - rr'' + r^2)|}{[(r')^2 + r^2]^{3/2}} = \frac{|2(r')^2 - rr'' + r^2|}{[(r')^2 + r^2]^{3/2}}.$$

Finally, let $\beta(s)$ be the arc-length parametrization of α. Then $B = \beta' \times \frac{\beta''}{|\beta''|}$ is the binormal vector. Since the pair (α', α'') has a positive orientation with respect to the pair (β', β''), $\alpha' \times \alpha''$ is in the same direction as B, so $2(r')^2 - rr'' + r^2$ has the same sign as $\langle B, (0, 0, 1)\rangle$, which we defined in problem 8 as the sign on κ. Hence,

$$\kappa(\theta) = \frac{2(r')^2 - rr'' + r^2}{[(r')^2 + r^2]^{3/2}}.$$

□

12.

Let $\alpha : I \to \mathbb{R}^3$ be a regular curve, not necessarily parametrized by arc length.

(a): Show that the curvature of α is given by

$$\kappa(t) = |\alpha' \times \alpha''| / |\alpha'|^3.$$

Proof. Let α' denote $\frac{d\alpha}{dt}$. Then

$$\alpha' = \frac{d\alpha}{ds}\frac{ds}{dt}$$
$$\alpha'' = \frac{d^2\alpha}{ds^2}\frac{ds}{dt}\frac{ds}{dt} + \frac{d\alpha}{ds}\frac{d^2 s}{dt^2} = \frac{d^2\alpha}{ds^2}\left(\frac{ds}{dt}\right)^2 + \frac{d\alpha}{ds}\frac{d^2 s}{dt^2}.$$

On the other hand, $\frac{\alpha'}{|\alpha'|}$ is the unit tangent vector, namely $\frac{d\alpha}{ds}$, so, using the equality in the first line, we see that $|\alpha'| = \frac{ds}{dt}$. Hence,

$\alpha' = \frac{d\alpha}{ds}|\alpha'|$ and $\alpha'' = \frac{d^2\alpha}{ds^2}|\alpha'|^2 + \frac{d\alpha}{ds}\frac{d^2s}{dt^2}$. Therefore,

$$\begin{aligned}\alpha' \times \alpha'' &= \left(\frac{d\alpha}{ds}|\alpha'|\right) \times \left(\frac{d^2\alpha}{ds^2}|\alpha'|^2 + \frac{d\alpha}{ds}\frac{d^2s}{dt^2}\right)\\ &= |\alpha'|^3\left(\frac{d\alpha}{ds} \times \frac{d^2\alpha}{ds^2}\right)\\ &= |\alpha'|^3\kappa(t)(T(t) \times N(t)),\end{aligned}$$

where the second equality is due to the fact that $\frac{d\alpha}{ds} \times \frac{d\alpha}{ds} = 0$ and the fourth equality is by the definition of N and κ. Therefore,

$$|\alpha' \times \alpha''| = \big||\alpha'|^3\kappa(t)(T(t) \times N(t))\big| = |\alpha'|^3\kappa(t)$$

since $T \times N = B$ is a unit vector. Therefore, since $\alpha' \neq 0$,

$$\kappa(t) = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3}.$$

□

(b): If the curvature is nonzero, so that the torsion is well-defined, show that the torsion is given by

$$\tau(t) = \frac{\langle(\alpha' \times \alpha''), \alpha'''\rangle}{|\alpha' \times \alpha''|}.$$

Proof. Using the same notation as in part (a), note that

$$\alpha''' = \frac{d^3\alpha}{ds^3}\left(\frac{ds}{dt}\right)^3 + 3\frac{d^2\alpha}{ds^2}\frac{ds}{dt}\frac{d^2s}{dt^2} + \frac{d\alpha}{ds}\frac{d^3s}{dt^3},$$

so

$$\begin{aligned}\langle(\alpha' \times \alpha''), \alpha'''\rangle &= \left\langle |\alpha'|^3\left(\frac{d\alpha}{ds} \times \frac{d^2\alpha}{ds^2}\right), \frac{d^3\alpha}{ds^3}|\alpha'|^3 + 3\frac{d^2\alpha}{ds^2}|\alpha'|\frac{d^2s}{dt^2} + \frac{d\alpha}{ds}\frac{d^3s}{dt^3}\right\rangle\\ &= |\alpha'|^3\left\langle \frac{d\alpha}{ds} \times \frac{d^2\alpha}{ds^2}, \frac{d^3\alpha}{ds^3}|\alpha'|^3\right\rangle\end{aligned}$$

since $\langle \frac{d\alpha}{ds}\frac{d^2\alpha}{ds^2}, \frac{d^2\alpha}{ds^2}\rangle = 0$ and $\langle \frac{d\alpha}{ds}\frac{d^2\alpha}{ds^2}, \frac{d\alpha}{ds}\rangle = 0$. Thus, since $\frac{d\alpha}{ds} = T$ and $\frac{d^2\alpha}{ds^2} = \kappa N$, we have

$$\begin{aligned}\langle(\alpha' \times \alpha''), \alpha'''\rangle &= |\alpha'|^6\left\langle T \times \kappa N, \frac{d}{ds}(\kappa N)\right\rangle\\ &= \kappa|\alpha'|^6\langle T \times N, \frac{d\kappa}{ds}\frac{ds}{dt}N + \kappa\frac{dN}{ds}\rangle\\ &= \kappa|\alpha'|^6\langle B, \kappa N'(s)\rangle\\ &= \kappa^2|\alpha'|^6\langle B, -\kappa T + \tau B\rangle\end{aligned}$$

since $\langle T \times N, N\rangle = 0$. Therefore,

$$\begin{aligned}\langle(\alpha' \times \alpha''), \alpha'''\rangle &= \left(\frac{|\alpha' \times \alpha''|}{|\alpha'|^3}\right)^2 |\alpha'|^6 \langle B, \tau B\rangle \\ &= |\alpha' \times \alpha''|^2 |\alpha'| \tau.\end{aligned}$$

Thus,

$$\tau = \frac{\langle(\alpha' \times \alpha''), \alpha'''\rangle}{|\alpha' \times \alpha''|^2}.$$

□

13.

(a): Show that the total curvature of a regular closed curve in the plane is $2n\pi$ for some integer n.

Proof. Let $\alpha : [0, b] \to \mathbb{R}^2$ be a regular closed curve in the plane. Then $\alpha'(b) = \alpha'(a)$. On the other hand, if $\kappa(s)$ is the curvature of the curve, then, by our work in problem 9, $\alpha(s) = \left(\int \cos\theta(s)ds + a, \int \sin\theta(s)ds + b\right)$ where $\theta(s) = \int \kappa(s)ds + \theta_0$ for some a, b, θ_0. Hence, $\alpha'(s) = (\cos\theta(s), \sin\theta(s))$ for all s. In particular,

$$(\cos\theta(b), \sin\theta(b)) = \alpha'(b) = \alpha'(a) = (\cos\theta(a), \sin\theta(a)).$$

Therefore, $\theta(b) - \theta(a) = 2n\pi$ for some integer n. By the fundamental theorem of calculus,

$$2n\pi = \theta(b) - \theta(a) = \int_a^b \kappa(s)ds,$$

since $\theta(s) = \int \kappa(s)ds + \theta_0$. Thus, the total curvature of the curve is $2n\pi$. □

DRL 3E3A, University of Pennsylvania
E-mail address: shonkwil@math.upenn.edu