DIFFERENTIAL GEOMETRY HW 9

CLAY SHONKWILER

1

Let G be a Lie group with a bi-invariant metric $\langle , \rangle$. Let $X, Y, Z \in \mathfrak{X}(G)$ be unit left invariant vector fields on G.

(a): Show that $\nabla_X Y = \frac{1}{2}[X, Y]$.

Proof. As we saw in problem 3 from chapter 3, for any $U \in \mathfrak{X}(G)$, since the metric is bi-invariant, $\nabla_U U = 0$. Thus,

$$\begin{aligned} 0 = \nabla_{X+Y}(X+Y) &= \nabla_X X + \nabla_X Y + \nabla_Y X + \nabla_Y Y \\ &= \nabla_X Y + \nabla_Y X \\ &= \nabla_Y X - \nabla_X Y + 2\nabla_X Y \\ &= [Y, X] + 2\nabla_X Y. \end{aligned}$$

Therefore, $\nabla_X Y = -\frac{1}{2}[Y, X] = \frac{1}{2}[X, Y]$. □

(b): Conclude from (a) that $R(X, Y)Z = \frac{1}{4}[[X, Y], Z]$.

Proof. By defintion,

$$\begin{aligned} R(X, Y)Z &= \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z \\ &= \frac{1}{2}\nabla_Y [X, Z] - \frac{1}{2}\nabla_X [Y, Z] + \frac{1}{2}[[X, Y], Z] \\ &= \frac{1}{2}\left(\frac{1}{2}[Y, [X, Z]] + \frac{1}{2}[X, [Y, Z]] + [[X, Y], Z]\right) \\ &= \frac{1}{2}\left(\frac{1}{2}[Y, [X, Z]] + \frac{1}{2}[X, [Y, Z]] + [Z, [Y, X]]\right) \\ &= \frac{1}{4}[Z, [Y, X]] \end{aligned}$$

by the Jacobi identity. Thus,

$$R(X, Y)Z = \frac{1}{4}[Z, [Y, X]] = \frac{1}{4}[[X, Y], Z].$$

□

(c): Prove that, if X and Y are orthonormal, the sectional curvature $K(\sigma)$ of G with respect to the plane σ generated by X and Y is given by

$$K(\sigma) = \frac{1}{4}\|[X, Y]\|^2.$$

Therefore, *the sectional curvature $K(\sigma)$ of a Lie group with bi-invariant metric is non-negative and is zero if and only if σ is generated by vectors X, Y which commute, that is, such that $[X,Y]=0$.*

Proof. By definition,

$$K(\sigma) = \frac{\langle R(X,Y)X, Y\rangle}{|X\wedge Y|^2} = \frac{\left\langle \frac{1}{4}[[X,Y],X],Y\right\rangle}{\|X\|^2\|Y\|^2 - \langle X,Y\rangle^2}$$
$$= \frac{1}{4}\langle [[X,Y],X],Y\rangle$$

since the denominator is 1 because X and Y are orthonormal. Now, using the identity $\langle [U,X],V\rangle = -\langle U,[V,X]\rangle$ (since this is a bi-invariant metric), we see that the above gives that

$$K(\sigma) = -\frac{1}{4}\langle [X,Y],[X,Y]\rangle = \frac{1}{4}\|[X,Y]\|^2.$$

□

2

Let X be a Killing field on a Riemannian manifold M. Define a mapping $A_X : \mathfrak{X}(M) \to \mathfrak{X}(M)$ by $A_X(Z) = \nabla_Z X$, $Z \in \mathfrak{X}(M)$. Consider the function $f : M \to R$ given by $f(q) = \langle X, X\rangle_q$, $q \in M$. Let $p \in M$ be a critical point of f (that is, $df_p = 0$). Prove that for any $Z \in \mathfrak{X}(M)$, at p,

(a): $\langle A_X(Z), X\rangle(p) = 0$.

Proof. Note that

$$Z\langle X,X\rangle = \langle \nabla_Z X, X\rangle + \langle X, \nabla_Z X\rangle = 2\langle A_X(Z), X\rangle.$$

Hence,

$$\langle A_X(Z), X\rangle(p) = \frac{1}{2}Z\langle X,X\rangle(p) = \frac{1}{2}Zf(p).$$

Since p is a critical point of f, all derivatives of f at p are zero, so we conclude that

$$\langle A_X(Z), X\rangle(p) = 0.$$

□

(b): $\langle A_X(Z), A_X(Z)\rangle(p) = \frac{1}{2}Z_p(Z\langle X,X\rangle) + \langle R(X,Z)X, Z\rangle$.

Proof. Let $S = \frac{1}{2}ZZ\langle X,X\rangle + \langle R(X,Z)X,Z\rangle$. Then, since $Z\langle X,X\rangle = 2\langle \nabla_Z X, X\rangle$,

$$S = Z\langle \nabla_Z X, X\rangle + \langle R(X,Z)X,Z\rangle = Z(-\langle \nabla_X X, Z\rangle) + \langle R(X,Z)X,Z\rangle$$

by the Killing equation. In turn, this means

$$S = -\langle \nabla_Z\nabla_X X, Z\rangle - \langle \nabla_X X, \nabla_Z Z\rangle + \langle \nabla_Z\nabla_X X - \nabla_X\nabla_Z X + \nabla_{[X,Z]}X, Z\rangle$$
$$= \langle \nabla_{[X,Z]}X, Z\rangle - \langle \nabla_X X, \nabla_Z Z\rangle - \langle \nabla_X\nabla_Z X, Z\rangle.$$

Since X is a Killing field, $\langle \nabla_{[X,Z]} X, Z\rangle = -\langle \nabla_Z X, [X,Z]\rangle = \langle \nabla_Z X, \nabla_Z X - \nabla_X Z\rangle$. Hence,

$$\begin{aligned} S &= \langle \nabla_Z X, \nabla_Z X\rangle - \langle \nabla_Z X, \nabla_X Z\rangle - \langle \nabla_X X, \nabla_Z Z\rangle - \langle \nabla_X \nabla_Z X, Z\rangle \\ &= \langle \nabla_Z X, \nabla_Z X\rangle - \langle \nabla_Z X, \nabla_X Z\rangle - \langle \nabla_X X, \nabla_Z Z\rangle - (X\langle \nabla_Z X, Z\rangle - \langle \nabla_Z X, \nabla_X Z\rangle) \\ &= \langle \nabla_Z X, \nabla_Z X\rangle - \langle \nabla_X X, \nabla_Z Z\rangle \end{aligned}$$

since $\langle \nabla_Z X, Z\rangle = -\langle \nabla_Z X, Z\rangle$ by the Killing equation, we see that $\langle \nabla_Z X, Z\rangle \equiv 0$ by part (a) and so $X\langle \nabla_Z X, Z\rangle = 0$. Now, by the Killing equation,

$$\langle \nabla_X X, X\rangle(p) = -\langle \nabla_X X, X\rangle(p),$$

so $\nabla_X X = 0$. Therefore, putting all these equations together, we see that

$$S(p) = \langle \nabla_Z X, \nabla_Z X\rangle(p) = \langle A_X(Z), A_X(Z)\rangle(p),$$

as expected. □

3

Let M be a compact Riemannian manifold of even dimension whose sectional curvature is positive. Prove that every Killing field X on M has a singularity.

Proof. Let $f : M \to \mathbb{R}$ be given by $f(q) = \langle X, X\rangle(q)$ as in problem 2 above. Let $p \in M$ be a minimum of f. Suppose $X(p) \neq 0$. Define $A : T_pM \to T_pM$ by $A(y) = A_X Y = \nabla_Y X$ where Y is an extension of y. Then, as in problem 2, this is a linear map. Let $E \subset T_pM$ be orthogonal to $X(p)$. Then, for $y \in E$, 2(a) above demonstrates that

$$\langle A(y), X\rangle(p) = 0.$$

Hence, $A : E \to E$. Moreover, if Y, Z are vector fields,

$$\langle A(Y), Z\rangle(p) = \langle \nabla_Y X, Z\rangle(p) = -\langle \nabla_Z X, Y\rangle = -\langle Y, A(Z)\rangle(p)$$

by the Killing equation, so we see that A is skew-symmetric. Moreover, if Z is a vector field such that $Z(p) \neq 0$, then, by 2(b),

$$\|A(Z)\|_p^2 = \frac{1}{2}Z(Z\langle X, X\rangle)(p) + \langle R(X,Z)X, Z\rangle(p) = \frac{1}{2}ZZf(p) + \langle R(X,Z)X, Z\rangle(p).$$

Now, since the sectional curvature is positive, $\langle R(X,Z)X, Z\rangle(p) > 0$. Also, since p is a minimum of f, all second derivatives of f are postive at p; in particular, $ZZf(p) > 0$. Thus, we see that for all non-zero Z, $\|A(Z)\|^2 > 0$, so $A : E \to E$ is injective and, therefore, an isomorphism. Therefore, $\det A \neq 0$; since A is skew-symmetric,

$$\det A = (-1)^{\dim E} \det A^t,$$

so $\dim E = \dim M - 1$ is even, contradicting the fact that M is even-dimensional. From this contradiction, then, we conclude that, indeed, $X(p) =$

0. Since our choice of Killing field X was arbitrary, we conclude that every Killing field on M has a singularity. □

4

Let M be a Riemannian manifold with the following property: given any two points $p, q \in M$, the parallel transport form p to q does not depend on the curve that joins p and q. Prove that the curvature of M is identically zero, that is, for all $X, Y, Z \in \mathfrak{X}(M)$, $R(X, Y)Z = 0$.

Proof. Consider the surface $f : U \subset \mathbb{R}^2 \to M$ where

$$U = \{(s,t) \in \mathbb{R}^2 | -\epsilon < t < 1+\epsilon, -\epsilon < s < 1+\epsilon, \epsilon > 0\}$$

and $f(s,0) = f(0,0)$ for all s. Let $V_0 \in T_{f(0,0)}$ and define the vector field V along f by $V(s,0) = V(0,0) = V_0$ and, for $t \neq 0$, $V(s,t)$ is the parallel transport of V_0 along the curve $t \mapsto f(s,t)$. Then, by definition of what it means to be parallel, $\frac{D}{\partial t}V \equiv 0$ and so, by Lemma 4.1,

$$\frac{D}{\partial s}\frac{D}{\partial t}V = 0 = \frac{D}{\partial t}\frac{D}{\partial s}V + R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)V. \tag{1}$$

Now, consider the vector $V(s,1) \in T_{f(s,1)}M$. Then $V(s,1)$ can be obtained by parallel transporting $V(0,1)$ down to $f(0,0)$ and then up to $f(s,1)$. On the other hand, since, by hypothesis, parallel transport from $f(0,1)$ to $f(s,1)$ does not depend on the curve we transport along, $V(s,1)$ is also the parallel transport of $V(s,0)$ along the curve $s \mapsto f(s,1)$. Therefore, again by what it means to be parallel, $\frac{D}{\partial s}V(s,1) = 0$ for all s. Hence, $\frac{D}{\partial t}\frac{D}{\partial s}V(s,1) = 0$. All of the above was independent of our choices of f and V_0, so, for any $X, Y, Z \in \mathfrak{X}(M)$, we can choose appropriate f and V_0 such that equation (1) reduces to:

$$0 = R_{f(0,1)}\left(\frac{\partial f}{\partial t}(0,1), \frac{\partial f}{\partial s}(0,1)\right)V(0,1) = R(X,Y)Z.$$

Thus, we conclude that the curvature of M is identically 0. □

7

Prove the *2nd Bianchi Identity*:

$$\nabla R(X,Y,Z,W,T) + \nabla R(X,Y,W,T,Z) + \nabla R(X,Y,T,Z,W) = 0$$

for all $X, Y, Z, W, T \in \mathfrak{X}(M)$.

Proof. Since everything is a tensor, it suffices to demonstrate this at a point $p \in M$. Let $\{e_i\}$ be a geodesic frame at p. Then $\nabla_{e_i} e_j(p) = 0$, so

$$\nabla R(e_i, e_j, e_k, e_\ell, e_h) = e_h \langle R(e_i, e_j)e_k, e_\ell\rangle = e_h\langle R(e_k, e_\ell)e_i, e_j\rangle$$

$$= \langle \nabla_{e_h}\nabla_{e_\ell}\nabla_{e_k}e_i - \nabla_{e_h}\nabla_{e_k}\nabla_{e_\ell}e_i + \nabla_{e_h}\nabla_{[e_k,e_\ell]}e_i, e_j\rangle. \tag{2}$$

Now,

$$R(e_h, [e_k, e_\ell])e_i = \nabla_{[e_k,e_\ell]}\nabla_{e_h}e_i - \nabla_{e_h}\nabla_{[e_k,e_\ell]}e_i + \nabla_{[e_h,[e_k,e_\ell]]}e_i,$$

so

$$\nabla_{e_h}\nabla_{[e_k,e_\ell]}e_i = \nabla_{[e_k,e_\ell]}\nabla_{e_h}e_i + \nabla_{[e_h,[e_k,e_\ell]]}e_i - R(e_h,[e_k,e_\ell])e_i.$$

Now,

$$\nabla_{[e_h,[e_k,e_\ell]]}e_i + \nabla_{[e_k,[e_\ell,e_h]]}e_i + \nabla_{[e_\ell,[e_h,e_k]]}e_i = 0$$

by the Jacobi identity. Also,

$$\nabla_{e_h}\nabla_{e_\ell}\nabla_{e_k}e_i - \nabla_{e_\ell}\nabla_{e_h}\nabla_{e_k}e_i = R(e_\ell,e_h)\nabla_{e_k}e_i - \nabla_{[e_\ell,e_h]}\nabla_{e_k}e_i.$$

Hence, using similar calculations to that done in (2) and making the cancellations indicated above,

$$\begin{aligned}&\nabla R(e_i,e_j,e_k,e_\ell,e_h) + \nabla R(e_i,e_j,e_\ell,e_h,e_k) + \nabla R(e_i,e_j,e_h,e_k,e_\ell)\\ &= R(e_\ell,e_h,\nabla_{e_k}e_i,e_j) + R(e_h,e_k,\nabla_{e_\ell}e_i,e_j) + R(e_k,e_\ell,\nabla_{e_h}e_i,e_j)\\ &= 0\end{aligned}$$

since each of the terms on the second line vanishes at p. Since these tensors are linear, the result follows. □

8

(*Schur's Theorem*) Let M^n be a connected Riemannian manifold with $n \geq 3$. Suppose that M is *isotropic*, that is, for each $p \in M$, the sectional curvature $K(p,\sigma)$ does not depend on $\sigma \in T_pM$. Prove that M has constant sectional curvature, that is, $K(p,\sigma)$ also does not depend on p.

Proof. Define the tensor R' by

$$R'(X,Z,X,Y) = \langle W,X\rangle\langle Z,Y\rangle - \langle Z,X\rangle\langle W,Y\rangle.$$

If $K(\sigma,p) = K$ does not depend on σ, then Lemma 3.4 implies that $R = KR'$. Now, suppose $U \in \mathfrak{X}(M)$. Then

$$\nabla_U R = (UK)R' + K\nabla_U R'.$$

Now, if $W,Z,X,Y \in \mathfrak{X}(M)$,

$$\begin{aligned}K\nabla_U R'(W,Z,X,Y) &= K\big[U(R'(W,Z,X,Y)) - R'(\nabla_U W,Z,X,Y) - R'(W,\nabla_U Z,X,Y)\\ &\quad -R'(W,Z,\nabla_U X,Y) - R'(W,Z,X,\nabla_U Y)\big]\\ &= K\left[U\left(\langle W,X\rangle\langle Z,Y\rangle\right) - \langle\nabla_U W,X\rangle\langle Z,Y\rangle\right.\\ &\quad \left.-\langle W,X\rangle\langle\nabla_U Z,Y\rangle - \langle W,\nabla_U X\rangle\langle Z,Y\rangle - \langle W,X\rangle\langle Z,\nabla_U Y\rangle\right]\\ &= 0,\end{aligned}$$

so we see that $\nabla_U R = (UK)R'$. Therefore, by the Bianchi identity proved in problem 7 above,

$$\begin{aligned}0 &= \nabla R(W,Z,X,Y,U) + \nabla R(W,Z,Y,U,X) + \nabla R(W,Z,U,X,Y)\\ &= (UK)\left(\langle W,X\rangle\langle Z,Y\rangle - \langle Z,X\rangle\langle W,Y\rangle\right)\\ &\quad +(XK)\left(\langle W,Y\rangle\langle Z,U\rangle - \langle Z,Y\rangle\langle W,U\rangle\right)\\ &\quad +(UK)\left(\langle W,U\rangle\langle Z,X\rangle - \langle Z,U\rangle\langle W,X\rangle\right).\end{aligned} \tag{3}$$

Now, fix $p \in M$. Since $n \geq 3$, if we fix X at p, we can find Y and Z at p such that $\langle X, Y\rangle = \langle Y, Z\rangle = \langle Z, X\rangle = 0$ and $\langle Z, Z\rangle = 1$. If we let $U = Z$, then (3) yields, for all W,

$$0 = (XK)\langle W, Y\rangle - (YK)\langle W, X\rangle = \langle (XK)Y - (YK)X, W\rangle.$$

Since X and Y are linearly independent at p and W is arbitrary, we see that the coefficients of the first term in this inner product must be 0; in particular, $XK = 0$. Since our choice of $X \in T_pM$ was arbitrary, we see that $XK = 0$ for all $X \in T_pM$, so K must be constant. □

10

(Einstein manifolds) A Riemannian manifold M^n is called an *Einstein Manifold* if, for all $X, Y \in \mathfrak{X}(M)$, $\mathrm{Ric}(X, Y) = \lambda\langle X, Y\rangle$, where $\lambda : M \to \mathbb{R}$ is a real valued function. Prove that:

(a): If M^n is connected and Einstein, with $n \geq 3$, then λ is constant on M.

Proof. Consider a geodesic frame $\{e_i\}$, $i = 1, \ldots, n \geq 3$ at a point $p \in M$. The 2nd Bianchi identity from problem 7 becomes

$$\begin{aligned} 0 &= \nabla R(e_i, e_j, e_k, e_h, e_s) + \nabla R(e_i, e_j, e_h, e_s, e_k) + \nabla R(e_i, e_j, e_s, e_k, e_h) \\ &= e_s(R(e_i, e_j, e_k, e_h)) + e_k(R(e_i, e_j, e_h, e_s)) + e_h(R(e_i, e_j, e_s, e_k)) \\ &= e_s(R_{ijkh}) + e_k(R_{ijhs}) + e_h(R_{ijsk}). \end{aligned} \tag{4}$$

Now, $\langle e_j, e_h\rangle = g_{jh} = \delta_{jh}$; multiplying (4) by $\delta_{jh}\delta_{ik}$ gives, for the first term,

$$\begin{aligned} \sum_{i,j,k,h} \delta_{ik}\delta_{jh} e_s(R_{ijkh}) &= e_s\left(\sum_{i,j,k,h} \delta_{ik}\delta_{jh} R_{ijkh}\right) \\ &= e_s\left(\sum_{i,k} \delta_{i,k} R_{ijkj}\right) \\ &= e_s\left(\sum_{i,k} \delta_{i,k} R_{ik}\right) \\ &= e_s\left(\sum_{i,k} \delta_{i,k}(\lambda\langle e_i, e_k\rangle)\right) \\ &= e_s\left(\sum_{i,k} \delta_{i,k}(\lambda\delta_{ik})\right) \\ &= ne_s(\lambda). \end{aligned}$$

The second term in (4) becomes

$$\begin{aligned}\sum_{i,j,k,h} \delta_{ik}\delta_{jh}e_k(R_{ijhs}) &= \sum_{i,k} \delta_{ik}e_k \left(\sum_{j,h} -\delta_{jh}R_{ijsh}\right) \\ &= -\sum_{i,k} \delta_{ik}e_k(R_{ijsj}) \\ &= -\sum_{i,k} \delta_{ik}e_k(\lambda\delta_{is}) \\ &= -e_s(\lambda).\end{aligned}$$

Finally, the third term in (4) becomes

$$\begin{aligned}\sum_{i,j,k,h} \delta_{ik}\delta_{jh}e_h(R_{ijsk}) &= \sum_{j,h} \delta_{jh}e_h \left(\sum_{i,k} -\delta_{ik}R_{ijks}\right) \\ &= -\sum_{j,h} \delta_{jh}e_h(R_{ijis}) \\ &= -\sum_{j,h} \delta_{jh}e_h(\lambda\delta_{js}) \\ &= -e_s(\lambda).\end{aligned}$$

Hence, (4) implies that

$$0 = ne_s(\lambda) - e_s(\lambda) - e_s(\lambda) = (n-2)e_s(\lambda).$$

Since $n \geq 3$, $e_s(\lambda) = 0$ at p; since our choice of p was arbitrary, this implies that λ is constant on each component of M; since M is connected, λ is constant on M. □

(b): If M^3 is a connected Einstein manifold then M^3 has constant sectional curvature.

Proof. By problem 8 above, if we can show that $K(\sigma, p)$ is independent of σ, then it follows that K is constant on M. Now, let $p \in M$ and suppose v_1, v_2, v_3 are orthonormal in T_pM. Since K is bilinear, if we can show that $K(v_i, v_j) = \frac{\lambda}{2}$ for all $i \neq j$, then that will suffice to show that $K(\sigma) = \frac{\lambda}{2}$ for all planes σ in T_pM, which in turn, by the argument in problem 8, suffices to show that M has constant sectional curvature. Now,

$$\begin{aligned}\lambda = \lambda\langle v_i, v_i\rangle &= \mathrm{Ric}(v_i, v_i) \\ &= \frac{1}{2}\sum_j \langle R(v_i, v_j)v_i, v_j\rangle \\ &= \frac{1}{2}\sum_{j\neq i} K(v_i, v_j).\end{aligned}$$

Hence,

$$\lambda = K(v_1, v_2) + K(v_1, v_3)$$
$$\lambda = K(v_2, v_3) + K(v_2, v_1)$$
$$\lambda = K(v_3, v_1) + K(v_3, v_2).$$

Since $K(v_i, v_j) = K(v_j, v_i)$, this implies that $K(v_1, v_2) = K(v_1, v_3) = K(v_2, v_3) = \frac{\lambda}{2}$. Thus, we conclude that, indeed, M has constant sectional curvature. □

DRL 3E3A, University of Pennsylvania
E-mail address: shonkwil@math.upenn.edu