DIFFERENTIAL GEOMETRY HW 2

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4.

Give $Diff(S^1)$ any reasonable topology, such as the compact-open topology, the C^r topology or the C^∞ topology. The orthogonal group $O(2)$ of rigid motions of a circle has two components, the rotations $SO(2)$ and the flips. Find an explicit strong deformation retraction of $DIff(S^1)$ to $O(2)$.

Answer: We may as well consider elements of $Diff(S^1)$ as functions of ϕ, the angle of elevation above the x-axis. Now, if $f \in Diff(S^1)$, define $H : Diff(S^1) \times [0,1] \to Diff(S^1)$ by

$$H(f,t)(\phi) = (1-t)f(\phi) + tA_f(\phi)$$

where $A_f \in O(2)$ is chosen such that $A_f(0) = f(0)$ and, if f is orientation-preserving, so is A_f (i.e. A_f is a rotation) and if f is orientation-reversing, so is A_f (i.e. A_f is a flip). Note that if $f \in Diff(S^1)$,

$$H(f,0)(\phi) = f(\phi)$$
$$H(f,1)(\phi) = A_f(\phi)$$

for all ϕ and, if $B \in O(s)$, $A_B = B$, so $H(B,t)(\phi) = B(\phi)$ for all ϕ. Since $H(f,t) \in Diff(S^1)$ for all t, we need only show that H is continuous to demonstrate that H is a strong deformation retraction of $Diff(S^1)$ onto $O(2)$.

We give $Diff(S^1)$ the compact-open topology; since addition and multiplication are continuous, we need only demonstrate that the choice of A_f is continuous in this topology. We can denote $A_f = A(f)$ where $A : Diff(S^1) \to O(2)$ is as defined above. Let

$$V_{C,U} = \{f \in O(2) | C \text{ compact, } U \text{ open and } f(C) \subset U\}$$

be a basic open set in $O(2)$ where C is a closed interval $[\phi_1, \psi_1]$ and U is an open interval (ϕ_2, ψ_2). Then an orientation-preserving element in $V_{C,U}$ (i.e. a rotation) must rotate $\phi = 0$ to at least $\phi_2 - \phi_1$ and at most $\psi_2 - \psi_1$; that is, for all rotations $B \in V_{C,U}$, $B(0) \in (\phi_2 - \phi_1, \psi_2 - \psi_1)$. Hence, if $f \in A^{-1}(V_{C,U})$ is orientation-preserving, then $f(0) \in (\phi_2 - \phi_1, \psi_2 - \psi_1)$. In other words, if O denotes the set of orientation-preserving elements of $Diff(S^1)$, then

$$A^{-1}(V_{C,U}) \cap O = W_{\{0\},(\phi_2-\phi_1,\psi_2-\psi_1)} \cap O,$$

where

$$W_{X,Y} = \{f \in Diff(S^1) | f(X) \subset Y\}.$$

Since $\{0\}$ is compact and $(\phi_2 - \phi_1, \psi_2 - \psi_1)$ is open in S^1, this set is open in $Diff(S^1)$ provided we can show that O is open in $Diff(S^1)$ (see below).

On the other hand, an orientation-reversing element of $V_{C,U}$ must flip $\phi = 0$ to at least $\psi_2 - \phi_1$ and at most $\phi_2 - \psi_1$; that is, for all flips $B \in V_{C,U}$, $B(0) \in (\psi_2 - \phi_1, \phi_2 - \psi_1)$. Hence, if $f \in A^{-1}(V_{C,U})$ is orientation-reversing, then $f(0) \in (\psi_2 - \phi_1, \phi_2 - \psi_1)$. In other words, if $\overline{O}$ denotes the set of orientation-reversing elements of $Diff(S^1)$, then

$$A^{-1}(V_{C,U}) \cap \overline{O} = W_{\{0\},(\psi_2-\phi_1,\phi_2-\psi_1)} \cap \overline{O}.$$

Again, this set is open in $Diff(S^1)$ provided $\overline{O}$ is open. Putting the above facts together, we see that

$$A^{-1}(V_{C,U}) = (W_{\{0\},(\phi_2-\phi_1,\psi_2-\psi_1)} \cap O) \cup (W_{\{0\},(\psi_2-\phi_1,\phi_2-\psi_1)} \cap \overline{O}),$$

which is open in $Diff(S^1)$ if O and $\overline{O}$ are. Before we prove that O and $\overline{O}$ are open, note that this will suffice to complete the proof. Since the open sets in S^1 are generated by the open intervals, when considering open sets $V_{C,U}$ in $O(2)$, it suffices to consider the case where U is an interval. On the other hand, if K is any compact set in S^1, let C_K be the interval determined by the "largest" and "smallest" elements of K (that is, C_K is a [possibly degenerate] interval of shortest length containing K; note that there may be more than one such: in such a case, simply choose one). Then, since we're only considering rigid motions of S^1, $V_{K,U} = V_{C_K,U}$. Therefore, we can restrict our attention to the case where K is an interval (again, possibly a degenerate interval consisting of a single point).

Now, we turn to demonstrating that O and $\overline{O}$ are both open in $Diff(S^1)$. Since $O \cap \overline{O} = \emptyset$ and $O \cup \overline{O} = Diff(S^1)$, this will mean that O and $\overline{O}$ are precisely the connected components of $Diff(S^1)$. Let $\phi_1 < \phi_2 < \phi_3$. For each $f \in Diff(S^1)$, choose neighborhoods $U_{f,i}$ about each of the ϕ_i such that the $U_{f,i}$ are disjoint and the $f(U_{f,i})$ are disjoint. Note that, since f is a diffeomorphism, $f(U_{f,i})$ is open for all i. Hence

$$W_f := W_{\{\phi_1\},f(U_{f,1})} \cap W_{\{\phi_2\},f(U_{f,2})} \cap W_{\{\phi_3\},f(U_{f,3})}$$

is open. Furthermore, $f \in W_f$. Also, if f is orientation-preserving, then W_f contains only orientation-preserving maps, and if f is orientation-reversing, then W_f contains only orientation-reversing maps. That is, W_f is an open neighborhood containing f and entirely contained in either O or $\overline{O}$, depending on whether f is orientation-preserving or -reversing. Since our choice of $f \in Diff(S^1)$ was arbitrary, this implies that both O and $\overline{O}$ are open, completing the proof.

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5.

Prove that the subgroup $\mathcal{D}$ is contractible.

Proof. Let $f \in \mathcal{D}$. Then, since f fixes an interval about $\phi = 0$, f is an orientation-preserving diffeomorphism. Furthermore, $f(0) = 0$, so, using the notation from problem 4 above, $H(f, 1) = Id$, the identity map. Since our choice of $f \in \mathcal{D}$ was arbitrary, we see that $H|_{\mathcal{D}}$ homotopes all of $\mathcal{D}$ to the identity. Thus, $\mathcal{D}$ is contractible. $\square$

7

(1) Compute $I_1(h_{t,d})$ explicitly for all $h_{t,d} \in B^2$.
Answer: Recall that

$$h_{t,d}(\phi) = \begin{cases} \frac{\pi}{2} + (1+d)\left(\phi - \frac{\pi}{2}\right) - t & \text{for } \phi \in D_2 \\ \frac{3\pi}{2} + (1-d)\left(\phi - \frac{3\pi}{2}\right) + t & \text{for } \phi \in D_4 \end{cases}$$

Also,

$$\begin{aligned} I_1(h) &= \int_0^{2\pi} gh^{-1}(\phi) N(\phi) d\phi \\ &= \int_0^{2\pi} mN(\phi) d\phi + \int_{h(D_2 \cup D_4)} (M-m) N(\phi) d\phi \\ &= \int_{h(D_2 \cup D_4)} (M-m) N(\phi) d\phi, \end{aligned}$$

so

$$\begin{aligned} I_1(h_{t,d}) &= \int_{h_{t,d}(D_2 \cup D_4)} (M-m) N(\phi) d\phi \\ &= (M-m) \int_{h_{t,d}(7\pi/16)}^{h_{t,d}(9\pi/16)} (\cos\phi, \sin\phi) d\phi + (M-m) \int_{h_{t,d}(23\pi/16)}^{h_{t,d}(25\pi/16)} (\cos\phi, \sin\phi) d\phi \\ &= (M-m) \left[(\sin\phi, -\cos\phi)\right]_{7\pi/16 - d\pi/16 - t}^{9\pi/16 + d\pi/16 - t} + (M-m) \left[(\sin\phi, -\cos\phi)\right]_{23\pi/16 + d\pi/16 + t}^{25\pi/16 - d\pi/16 + t} \\ &= (M-m)(\sin(9\pi/16 + d\pi/16 - t) - \sin(7\pi/16 - d\pi/16 - t) \\ &\quad + \sin(25\pi/16 - d\pi/16 + t) - \sin(23\pi/16 + d\pi/16 + t), \\ &\quad - \cos(9\pi/16 + d\pi/16 - t) + \cos(7\pi/16 - d\pi/16 - t) \\ &\quad - \cos(25\pi/16 - d\pi/16 + t) + \cos(23\pi/16 + d\pi/16 + t)) \end{aligned}$$

See attached Maple sheet for a (slight) simplification of this expression.

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(2) Show that the arrows along the outside of the "proof in one picture" are a fair portrayal of the values of $I_1(h_{t,d}) \in \Sigma^1$.

Answer: Using the above,

$$\begin{aligned}
I_1(h_{3\pi/16,0}) &\approx (M-m)(0.434, 0)\\
I_1(h_{3\pi/16,1/2}) &\approx (M-m)(0.431, 0.320)\\
I_1(h_{0,1/2}) &\approx (M-m)(0, 0.385)\\
I_1(h_{-3\pi/16,1/2}) &\approx (M-m)(-0.431, 0.320)\\
I_1(h_{-3\pi/16,0}) &\approx (M-m)(-0.434, 0)\\
I_1(h_{-3\pi/16,-1/2}) &\approx (M-m)(-0.431, -0.320)\\
I_1(h_{0,-1/2}) &\approx (M-m)(0, -0.385)\\
I_1(h_{3\pi/16,-1/2}) &\approx (M-m)(0.431, -0.320)
\end{aligned}$$

So the arrows on the "proof in one picture" really are realistic.

♣

(3) Show that I_1 embeds B^2 into the plane $\mathbb{R}^2$ and takes the center of B^2 to the origin. Conclude that $I_1(\Sigma^1)$ loops once around the origin.

Proof. I can't figure out how to simplify $I_1(H_{t,d})$ enough to make any headway showing that it's injective (which would be enough to show that it's an embedding, given that I_1 is a continuous map from a compact set into $\mathbb{R}^2$). Note that when $(t, d) = (0, 0)$,

$$\begin{aligned}
I_1(h_{0,0}) &= (M-m)(\sin 9\pi/16 - \sin 7\pi/16 + \sin 25\pi/16 - \sin 23\pi/16,\\
&\quad -\cos 9\pi/16 + \cos 7\pi/16 - \cos 25\pi/16 + \cos 23\pi/16)\\
&= (0, 0).
\end{aligned}$$

Assuming I_1 embeds B^2 into the plane and since we know I_1 maps $(0, 0)$ to the origin, the boundary Σ^1 of B^2 wraps exactly once about the origin. □

(4) Find explicitly the points on $I_1(\Sigma^1)$ which are closest to the origin.

Answer: We restrict I_1 to each of the four sides of Σ^1 and look for critical points (see attached Maple sheets for computations). The critical points are at

$$(t, d) = (\pm 3\pi/16, 0), (0, \pm 1/2).$$

Checking each critical point and the corners of Σ^1, we see that the distance to the origin is minimized at $(t, d) = (0, \pm 1/2)$, with the distance given by $\approx 0.38(M - m)$.

♣

(5) Show how to choose ϵ small enough so that $I(\Sigma^1)$ also loops once around the origin.

Answer: Note that so long as $\|I(h_{t,d}) - I_1(h_{t,d})\| < 0.38(M-m)$, then $I(\Sigma^1)$ must loop once around the origin. Note that f is positive and bounded above on S^1; let B be an upper bound on f which is at least 1. Now, let $\epsilon = \frac{M-m}{25B}$. Recall that the measure of A is less than 4ϵ. Then

$$\begin{aligned}
\|I(h_{t,d}) - I_1(h_{t,d})\| &= \left\| \int_0^{2\pi} f h_{t,d}^{-1}(\phi) N(\phi) d\phi - \int_0^{2\pi} g h_{t,d}^{-1}(\phi) N(\phi) d\phi \right\| \\
&= \left\| \int_0^{2\pi} (f h_{t,d}^{-1} - g h_{t,d}^{-1})(\phi) N(\phi) d\phi \right\| \\
&= \left\| \int_{h_{t,d}(E_1 \cup D_2 \cup E_3 \cup D_4)} (f h_{t,d}^{-1}(\phi) - g h_{t,d}^{-1}(\phi)) N(\phi) d\phi \right. \\
&\quad \left. + \int_{h_{t,d}(A)} (f h_{t,d}^{-1}(\phi) - g h_{t,d}^{-1}(\phi)) N(\phi) d\phi \right\| \\
&\leq \left\| \int_{h_{t,d}(E_1 \cup D_2 \cup E_3 \cup D_4)} (f h_{t,d}^{-1}(\phi) - g h_{t,d}^{-1}(\phi)) N(\phi) d\phi \right\| \\
&\quad + \left\| \int_{h_{t,d}(A)} (f h_{t,d}^{-1}(\phi) - g h_{t,d}^{-1}(\phi)) N(\phi) d\phi \right\| \\
&\leq \left\| \int_{h_{t,d}(E_1 \cup D_2 \cup E_3 \cup D_4)} \epsilon N(\phi) d\phi \right\| + \left\| \int_{h_{t,d}(A)} f h_{t,d}^{-1}(\phi) N(\phi) d\phi \right\|
\end{aligned}$$

since both f and g are positive. Now, since f is bounded by B, this implies that

$$\begin{aligned}
\|I(h_{t,d}) - I_1(h_{t,d})\| &\leq 2\pi\epsilon + B \left\| \int_{h_{t,d}(A)} N(\phi) d\phi \right\| \\
&\leq 2\pi\epsilon + B\text{measure}(A)
\end{aligned}$$

Since $h_{t,d}$ stretches by a factor of at most $3/2$, measure$(A) \leq 3\epsilon/2$, so

$$\begin{aligned}
\|I(h_{t,d}) - I_1(h_{t,d})\| \leq 2\pi\epsilon + 3B\epsilon/2 &= \frac{2\pi(M-m)}{25B} + \frac{3(M-m)}{50} \\
&\leq (M-m)\left(\frac{2\pi}{25} + \frac{3}{50}\right) \\
&\approx 0.311(M-m) \\
&< 0.38(M-m),
\end{aligned}$$

since $B \geq 1$. Hence, $I(\Sigma^1)$ does indeed wrap once around the origin.

♣

(6) Conclude that there must be a root of the equation

$$I(h) = \int_0^{2\pi} f h^{-1}(\phi) N(\phi) d\phi = 0$$

somewhere inside that 2-cell B^2.

Proof. As we showed in (5) above, $I(h_{t,d})$ has winding number about the origin of 1 where $(t,d) \in \Sigma^1$. On the other hand, for $|(h_{t,d})| = 0$, $I(t,d) = (0,0)$ is a single point and so certainly has winding number 0 about the origin. Now, since B^2 is contractible, we can continuously deform it to the point $(0,0)$; composing with I yields another continuous deformation. Suppose, for the sake of contradiction, that $I(h_{t,d})$ is never 0. Then the image under I of the contraction of Σ^1 can never cross the origin, so its winding number about the origin can never change. But this implies that $1 = 0$, which is obviously bad. Thus, we conclude that $I(h) = 0$ does indeed have a solution; in particular, it has a solution contained in $\mathcal{D}$. $\square$

B

Let C be a smooth simple closed curve in 3-space. Let $r > 0$ be a real number, and let $N(C,r)$ denote the set of points in 3-space whose distance from C is at most r. If r is sufficiently small, then these disks will not intersect one another, and hence $N(C,r)$ will be homeomorphic to the product of C and any one of these disks, and will look like a curvy, perhaps knotted, solid donut. Assume r is this small, and prove that the volume of $N(C,r)$ equals the length of C times the area, πr^2, of any one of the disks.

Proof. First, suppose C has non-vanishing curvature everywhere. Then the Frenet frame is defined everywhere on C. We may as well assume C is parametrized by arc length. Let L be the length of C. Consider a vertical cylinder Cyl with radius r and height L; that is,

$$Cyl := \{(x,y,z) \in \mathbb{R}^3 | x^2 + y^2 \leq r, z \in [0,L]\}.$$

Now, using the notation that $C(s) = (C_1(s), C_2(s), C_3(s))$, the normal vector $N(s) = (N_1(s), N_2(s), N_3(s))$ and the binormal vector $B(s) = (B_1(s), B_2(s), B_3(s))$, define the function $f : Cyl \to N(C,r)$ by

$$f(x,y,z) = C(z) + xN(z) + yB(z);$$

then $f(x,y,z) = (f_1(x,y,z), f_2(x,y,z), f_3(x,y,z))$ where

$$\begin{aligned} f_1(x,y,z) &= C_1(z) + xN_1(z) + yB_1(z) \\ f_2(x,y,z) &= C_2(z) + xN_2(z) + yB_2(z) \\ f_3(x,y,z) &= C_3(z) + xN_3(z) + yB_3(z). \end{aligned}$$

Thus, the Jacobian of f is given by

$$J(f) = \begin{pmatrix} N_1(z) & B_1(z) & C_1'(z) + xN_1'(z) + yB_1'(z) \\ N_2(z) & B_2(z) & C_2'(z) + xN_2'(z) + yB_2'(z) \\ N_3(z) & B_3(z) & C_3'(z) + xN_3'(z) + yB_3'(z) \end{pmatrix}$$

Now, since z is equal to arc length, the first column of this matrix is just $N(z)$, the second is $B(z)$ and the third is

$$C'(z) + xN'(z) + yB'(z) = T(z) + x(-\kappa(z)T(z) + \tau(z)B(z)) + y(-\tau(z)N(z))$$

by the Frenet equations. Hence,

$$\begin{aligned}\det J(f) &= \langle N(z) \times B(z), T(z) - \kappa(z)xT(z) + \tau(z)xB(z) - \tau(z)yN(z)\rangle \\ &= \langle T(z), T(z) - \kappa(z)xT(z) + \tau(z)xB(z) - \tau(z)yN(z)\rangle \\ &= 1 - x\kappa(z)\end{aligned}$$

Now, the volume of $N(C, r)$ is given by

$$\begin{aligned}\int\int\int_{Cyl} \det J(f) &= \int_0^L \int_0^r \int_0^{2\pi} (1 - \rho\kappa(z)\cos\theta)\rho d\theta d\rho dz \\ &= \int_0^L \int_0^r 2\pi\rho d\rho dz \\ &= \int_0^L \pi r^2 dz \\ &= L\pi r^2,\end{aligned}$$

as expected.

Now, if C has zero curvature somewhere, we'll split up the computation of the volume. Over those regions where the curvature is non-zero, we'll integrate the Jacobian as above. Over those regions with zero curvature, C is cylindrical, so the volume of each such region is simply given by $L_i\pi r^2$ where L_i is the length of each region C_i with zero curvature. Then, simply adding the volumes for each of these regions yields a total volume of $L\pi r^2$. □

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