DIFFERENTIAL GEOMETRY HW 4

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3

Show that a catenoid and helicoid are locally isometric.

Proof. Let $X(u,v) = (a\cosh v\cos u, a\cosh v\sin u, av)$ be the parametrization of the catenoid and let $Y(z,w) = (w\cos z, w\sin z, az)$ be the parametrization of the helicoid, where $0 < u, z < 2\pi$ and $-\infty < v, w < \infty$. Now, let $z = u$ and let $w = a\sinh v$, which is valid since the map $(u,v) \mapsto (u, a\sinh v)$ is smooth and bijective on the region where u and v are defined. Then the change-of-coordinate matrix is given by

$$\begin{pmatrix} 1 & 0 \\ 0 & a\cosh v \end{pmatrix},$$

so the Jacobian of this coordinate change is given by the determinant of this matrix, $a\cosh v$, which is everywhere nonzero. In other words, this change of coordinates induces a local diffeomorphism between these surfaces. Hence, a new parametrization of the helicoid is

$$Y(u,v) = (a\sinh v\cos u, a\sinh v\sin u, au).$$

Now,

$$\begin{aligned} Y_u &= (-a\sinh v\sin u, a\sinh v\cos u, a) \\ Y_v &= (a\cosh v\cos u, a\cosh v\sin u, 0), \end{aligned}$$

so the terms of the first fundamental form of the reparametrized helicoid are given by

$$\begin{aligned} E &= \langle Y_u, Y_u\rangle = a^2\sinh^2 v + a^2 = a^2(\sinh^2 v + 1) = a^2\cosh^2 v \\ F &= \langle Y_u, Y_v\rangle = -a^2\sinh v\cosh v\sin u\cos u + a^2\sinh v\cosh v\sin u\cos u = 0 \\ G &= \langle Y_v, Y_v\rangle = a^2\cosh^2 v \end{aligned}$$

On the other hand,

$$\begin{aligned} X_u &= (-a\cosh v\sin u, a\cosh v\cos u, 0) \\ X_v &= (a\sinh v\cos u, a\sinh v\sin u, a), \end{aligned}$$

so the terms of the first fundamental form of the catenoid are given by

$$E = \langle X_u, X_u\rangle = a^2\cosh^2 v$$
$$F = \langle X_u, X_v\rangle = -a^2\sinh v\cosh v\sin u\cos u + a^2\sinh v\cosh v\sin u\cos u = 0$$
$$G = \langle X_v, X_v\rangle = a^2\sinh^2 v + a^2 = a^2(\sinh^2 v + 1) = a^2\cosh^2 v.$$

Therefore, the local diffeomorphism induced by the change of coordinates described above preserves the first fundamental form. Since the First Fundamental Form is just a repackaging of the metric on a surface, this means that this local diffeomorphism preserves the metric, so the helicoid and catenoid are locally isometric. □

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Show that

$$\langle X_{uu}, X_u\rangle = \frac{1}{2}E_u, \qquad \langle X_{uu}, X_v\rangle = F_u - \frac{1}{2}E_v$$
$$\langle X_{uv}, X_u\rangle = \frac{1}{2}E_v, \qquad \langle X_{uv}, X_v\rangle = \frac{1}{2}G_u$$
$$\langle X_{vv}, X_u\rangle = F_v - \frac{1}{2}G_u, \qquad \langle X_{vv}, X_v\rangle = \frac{1}{2}G_v.$$

Proof. Note that

$$E_u = \langle X_{uu}, X_u\rangle + \langle X_u, X_{uu}\rangle,$$

so $\langle X_{uu}, X_u\rangle = \frac{1}{2}E_u$. Also,

$$E_v = \langle X_{uv}, X_u\rangle + \langle X_u, X_{uv}\rangle,$$

so $\langle X_{uv}, X_u\rangle = \frac{1}{2}E_v$. Now,

$$G_u = \langle X_{vu}, X_v\rangle + \langle X_v, X_{vu}\rangle,$$

so $\langle X_{uv}, X_v\rangle = \frac{1}{2}G_u$. Also,

$$G_v = \langle X_{vv}, X_v\rangle + \langle X_v, X_{vv}\rangle$$

so $\langle X_{vv}, X_v\rangle = \frac{1}{2}G_v$. Now,

$$F_u = \langle X_{uu}, X_v\rangle + \langle X_u, X_{vu}\rangle$$

and

$$F_v = \langle X_{uv}, X_v\rangle + \langle X_u, X_{vv}\rangle,$$

so

$$F_v - \frac{1}{2}G_u = \langle X_{uv}, X_v\rangle + \langle X_u, X_{vv}\rangle - \langle X_{uv}, X_v\rangle = \langle X_{vv}, X_u\rangle$$

and

$$F_u - \frac{1}{2}E_v = \langle X_{uu}, X_v\rangle + \langle X_{uv}, X_u\rangle - \langle X_{uv}, X_u\rangle = \langle X_{uu}, X_v\rangle.$$

□

5

Compute the Christoffel symbols Γ^k_{ij} for a surface of revolution parametrized by

$$X(u,v) = (f(v)\cos u, f(v)\sin u, g(v))$$

with $f(v) \neq 0$.

Answer: If the surface is given by $X(u,v) = (f(v)\cos u, f(v)\sin u, g(v))$, then

$$\begin{aligned} X_u &= (-f(v)\sin u, f(v)\cos u, 0) \\ X_v &= (f'(v)\cos u, f'(v)\sin u, g'(v)). \end{aligned}$$

Hence

$$\begin{aligned} E &= \langle X_u, X_u \rangle = f(v)^2\sin^2 u + f(v)^2\cos^2 u = f(v)^2 \\ F &= \langle X_u, X_v \rangle = -f(v)f'(v)\sin u\cos u + f(v)f'(v)\sin u\cos u = 0 \\ G &= \langle X_v, X_v \rangle = f'(v)^2\cos^2 u + f'(v)^2\sin^2 u + g'(v)^2 = (f'(v))^2 + (g'(v))^2. \end{aligned}$$

Hence,

$$\begin{array}{ccc} E_u = 0 & F_u = 0 & G_u = 0 \\ E_v = 2f(v)f'(v) & F_v = 0 & G_v = 2f'(v)f''(v) + 2g'(v)g''(v) \end{array}$$

Using the defining equations of the Christoffel symbols:

$$\begin{aligned} 0 = \frac{1}{2}E_u &= \Gamma^1_{11}E + \Gamma^2_{11}F = \Gamma^1_{11}f(v)^2 \\ -f(v)f'(v) = F_u - \frac{1}{2}E_v &= \Gamma^1_{11}F + \Gamma^2_{11}G = \Gamma^2_{11}[(f'(v))^2 + (g'(v))^2] \\ f(v)f'(v) = \frac{1}{2}E_v &= \Gamma^1_{12}E + \Gamma^2_{12}F = \Gamma^1_{12}f(v)^2 \\ 0 = \frac{1}{2}G_u &= \Gamma^1_{12}F + \Gamma^2_{12}G = \Gamma^2_{12}[(f'(v))^2 + (g'(v))^2] \\ 0 = F_v - \frac{1}{2}G_u &= \Gamma^1_{22}E + \Gamma^2_{22}F = \Gamma^1_{22}f(v)^2 \\ f'(v)f''(v) + g'(v)g''(v) = \frac{1}{2}G_v &= \Gamma^1_{22}F + \Gamma^2_{22}G = \Gamma^2_{22}[(f'(v))^2 + (g'(v))^2] \end{aligned}$$

Hence,

$$\begin{array}{cc} \Gamma^1_{11} = 0 & \Gamma^2_{11} = \frac{-f(v)f'(v)}{(f'(v))^2+(g'(v))^2} \\ \Gamma^1_{12} = \frac{f'(v)}{f(v)} & \Gamma^2_{12} = 0 \\ \Gamma^1_{22} = 0 & \Gamma^2_{22} = \frac{f'(v)f''(v)+g'(v)g''(v)}{(f'(v))^2+(g'(v))^2} \end{array}$$

♣

6

(a): Consider the equation $A_1 = 0$, which comes from the equation $(X_{uu})_v - (X_{uv})_u = 0$ by setting the coefficients of X_u equal to 0. Show that this yields

$$\Gamma^1_{12,u} - \Gamma^1_{11,v} + \Gamma^2_{12}\Gamma^1_{12} - \Gamma^2_{11}\Gamma^1_{22} = FK.$$

If $F \neq 0$, this also shows that the Gaussian curvature depends only on the first fundamental form.

Proof. Using the values of $(X_{uu})_v$ and $(X_{uv})_u$ already computed, we know that the coefficients of X_u for them are $\Gamma^1_{11,v} + \Gamma^1_{11}\Gamma^1_{12} + \Gamma^2_{11}\Gamma^1_{22} + ea_{12}$ and $\Gamma^1_{12,u} + \Gamma^1_{12}\Gamma^1_{11} + \Gamma^2_{12}\Gamma^1_{12} + fa_{11}$, respectively. Hence, equating these coefficients, we see that

$$ea_{12} - fa_{11} = \Gamma^1_{12,u} + \Gamma^1_{12}\Gamma^1_{11} + \Gamma^2_{12}\Gamma^1_{12} - \Gamma^1_{11,v} - \Gamma^1_{11}\Gamma^1_{12} - \Gamma^2_{11}\Gamma^1_{22}.$$

On the other hand,

$$\begin{aligned} ea_{12} - fa_{11} &= \frac{e(gF - fG) - f(fF - eG)}{EG - F^2} \\ &= \frac{egF - f^2F}{EG - F^2} \\ &= F\frac{eg - f^2}{EG - F^2} \\ &= FK. \end{aligned}$$

Hence,

$$FK = \Gamma^1_{12,u} + \Gamma^1_{12}\Gamma^1_{11} + \Gamma^2_{12}\Gamma^1_{12} - \Gamma^1_{11,v} - \Gamma^1_{11}\Gamma^1_{12} - \Gamma^2_{11}\Gamma^1_{22}.$$

□

(b): Consider the equation $C_1 = 0$, which comes from the equation $(X_{uu})_v - (X_{uv})_u = 0$ by setting the coefficient of N equal to 0. Show that this yields

$$e_v - f_u = e\Gamma^1_{12} + f(\Gamma^2_{12} - \Gamma^1_{11}) - g\Gamma^2_{11}.$$

This is one of the two *Mainardi-Codazzi equations.*

Proof. Again using the values of $(X_{uu})_v$ and $(X_{uv})_u$ already computed, we know that the coefficients of N for them are $\Gamma^1_{11}f + \Gamma^2_{11}g + e_v$ and $\Gamma^1_{12}e + \Gamma^2_{12}f + f_u$, respectively. Hence, equating these coefficients yields

$$e_v - f_u = \Gamma^1_{12}e + \Gamma^2_{12}f - \Gamma^1_{11}f - \Gamma^2_{11}g = e\Gamma^1_{12} + f(\Gamma^2_{12} - \Gamma^1_{11}) - g\Gamma^2_{11}.$$

□

(c): Consider the equations $A_2 = 0$ and $B_2 = 0$, which come from the equation $(X_{vv})_u - (X_{uv})_v = 0$ by setting the coefficients of X_u and X_v, respectively, equal to 0. Show that both of these equations again give the Gauss Formula for the Gaussian curvature K.

Proof. Note that

$$\begin{aligned}(X_{vv})_u &= \Gamma^1_{22,u}X_u + \Gamma^1_{22}X_{uu} + \Gamma^2_{22,u}X_v + \Gamma^2_{22}X_{uv} + g_uN + gN_u \\ &= \left(\Gamma^1_{22,u} + \Gamma^1_{22}\Gamma^1_{11} + \Gamma^2_{22}\Gamma^1_{12} + ga_{11}\right)X_u \\ &\quad + \left(\Gamma^2_{22,u} + \Gamma^1_{22}\Gamma^2_{11} + \Gamma^2_{22}\Gamma^2_{12} + ga_{21}\right)X_v \\ &\quad + \left(\Gamma^1_{22}e + \Gamma^2_{22}f + g_u\right)N\end{aligned}$$

and

$$\begin{aligned}(X_{uv})_v &= \Gamma^1_{12,v}X_u + \Gamma^1_{12}X_{uv} + \Gamma^2_{12,v}X_v + \Gamma^2_{12}X_{vv} + f_vN + fN_v \\ &= \left(\Gamma^1_{12,v} + \Gamma^1_{12}\Gamma^1_{12} + \Gamma^2_{12}\Gamma^1_{22} + fa_{12}\right)X_u \\ &\quad + \left(\Gamma^2_{12,v} + \Gamma^1_{12}\Gamma^2_{12} + \Gamma^2_{12}\Gamma^2_{22} + fa_{22}\right)X_v \\ &\quad + \left(\Gamma^1_{12}f + \Gamma^2_{12}g + f_v\right)N\end{aligned}$$

Thus, by equating the coefficients on X_u,

$$ga_{11} - fa_{12} = \Gamma^1_{22,u} + \Gamma^1_{22}\Gamma^1_{11} + \Gamma^2_{22}\Gamma^1_{12} - \Gamma^1_{12,v} - \Gamma^1_{12}\Gamma^1_{12} - \Gamma^2_{12}\Gamma^1_{22}.$$

On the other hand,

$$\begin{aligned}ga_{11} - fa_{12} &= \frac{g(fF - eG) - f(gF - fG)}{EG - F^2} \\ &= \frac{f^2G - egG}{EG - F^2} \\ &= -G\frac{eg - f^2}{EG - F^2} \\ &= -GK.\end{aligned}$$

Hence,

$$-GK = \Gamma^1_{22,u} + \Gamma^1_{22}\Gamma^1_{11} + \Gamma^2_{22}\Gamma^1_{12} - \Gamma^1_{12,v} - \Gamma^1_{12}\Gamma^1_{12} - \Gamma^2_{12}\Gamma^1_{22}.$$

Equating the coefficients on X_v, we see that

$$ga_{21} - fa_{22} = \Gamma^2_{22,u} + \Gamma^1_{22}\Gamma^2_{11} + \Gamma^2_{22}\Gamma^2_{12} - \Gamma^2_{12,v} - \Gamma^1_{12}\Gamma^2_{12} - \Gamma^2_{12}\Gamma^2_{22}.$$

On the other hand,

$$\begin{aligned}ga_{21} - fa_{22} &= \frac{g(eF - fE) - f(fF - gE)}{EG - F^2} \\ &= \frac{egF - f^2F}{EG - F^2} \\ &= F\frac{eg - f^2}{EG - F^2} \\ &= FK.\end{aligned}$$

Hence,

$$FK = \Gamma_{22,u}^2 + \Gamma_{22}^1\Gamma_{11}^2 + \Gamma_{22}^2\Gamma_{12}^2 - \Gamma_{12,v}^2 - \Gamma_{12}^1\Gamma_{12}^2 - \Gamma_{12}^2\Gamma_{22}^2.$$

□

(d): Consider the equation $C_2 = 0$, which comes from the equation $(X_{vv})_u - (X_{uv})_v = 0$ by setting the coefficient of N equal to 0. Show that this yields

$$f_v - g_u = e\Gamma_{22}^1 + f(\Gamma_{22}^2 - \Gamma_{12}^1) - g\Gamma_{12}^2.$$

This is the second of the two Mainardi-Codazzi equations.

Proof. From our computation of $(X_{vv})_u$ and $(X_{uv})_v$ in (b) above, we know that setting the coefficients of N in each equal yields

$$\begin{aligned} f_v - g_u &= \Gamma_{22}^1 e + \Gamma_{22}^2 f - \Gamma_{12}^1 f - \Gamma_{12}^2 g \\ &= e\Gamma_{22}^1 + f(\Gamma_{22}^2 - \Gamma_{12}^1) - g\Gamma_{12}^2. \end{aligned}$$

□

(e): Do the same for the coefficients A_3, B_3 and C_3 of the third equation $(N_u)_v = (N_v)_u$. Show that the equations $A_3 = 0$ and $B_3 = 0$ yield again the two Mainardi-Codazzi equations, and that the equation $C_3 = 0$ is an identity.

Proof. Note that

$$\begin{aligned} (N_u)_v &= a_{11,v}X_u + a_{11}X_{uv} + a_{21,v}X_v + a_{21}X_{vv} \\ &= \left(a_{11,v} + a_{11}\Gamma_{12}^1 + a_{21}\Gamma_{22}^1\right)X_u \\ &\quad + \left(a_{21,v} + a_{11}\Gamma_{12}^2 + a_{21}\Gamma_{22}^2\right)X_v \\ &\quad + (a_{11}f + a_{21}g)N \end{aligned}$$

and

$$\begin{aligned} (N_v)_u &= a_{12,u}X_u + a_{12}X_{uu} + a_{22,u}X_v + a_{22}X_{uv} \\ &= \left(a_{12,u} + a_{12}\Gamma_{11}^1 + a_{22}\Gamma_{12}^1\right)X_u \\ &\quad + \left(a_{22,u} + a_{12}\Gamma_{11}^2 + a_{22}\Gamma_{12}^2\right)X_v \\ &\quad + (a_{12}e + a_{22}f)N \end{aligned}$$

Equating the coefficients of X_u yields

$$\begin{aligned} a_{11,v} - a_{12,u} &= a_{12}\Gamma_{11}^1 + a_{22}\Gamma_{12}^1 - a_{11}\Gamma_{12}^1 - a_{21}\Gamma_{22}^1 \\ &= \frac{\Gamma_{11}^1(gF - fG) + \Gamma_{12}^1(fF - gE) - \Gamma_{12}^1(fF - eG) - \Gamma_{22}^1(eF - fE)}{EG - F^2} \\ &= \frac{\Gamma_{11}^1(gF - fG) + \Gamma_{12}^1(eG - gE) + \Gamma_{22}^1(fE - eF)}{EG - F^2}. \end{aligned}$$

On the other hand,

$$a_{11,v} - a_{12,u} = \frac{(EG-F^2)(f_vF + fF_v - e_vG - eG_v) - (fF-eG)(E_vG + EG_v - 2FF_v)}{(EG-F^2)^2} - \frac{(EG-F^2)(g_uF + gF_u - f_uG - fG_u) - (gF-fG)(E_uG + EG_u - 2FF_u)}{(EG-F^2)^2}$$

□

8

Is there a regular surface $X : U \to \mathbb{R}^3$ with $E \equiv 1$, $F \equiv 0$, $G = \cos^2 u$, $e = \cos^2 u$, $f \equiv 0$, $g \equiv 1$?

(a): Show that such a surface would have Gaussian curvature $K \equiv 1$.

Proof. Recall that

$$\begin{aligned} K &= \frac{eg - f^2}{EG - F^2} \\ &= \frac{\cos^2 u}{\cos^2 u} \\ &= 1. \end{aligned}$$

□

(b): Compute the Christoffel symbols for such a surface.

Answer: Using the defining equations of the Christoffel symbols,

$$\begin{aligned} 0 = \frac{1}{2}E_u &= \Gamma^1_{11}E + \Gamma^2_{11}F = \Gamma^1_{11} \\ 0 = F_u - \frac{1}{2}E_v &= \Gamma^1_{11}F + \Gamma^2_{11}G = \Gamma^2_{11} \\ 0 = \frac{1}{2}E_v &= \Gamma^1_{12}E + \Gamma^2_{12}F = \Gamma^1_{12} \\ -\sin u \cos u = \frac{1}{2}G_u &= \Gamma^1_{12}F + \Gamma^2_{12}G = \Gamma^2_{12}\cos^2 u \\ \sin u \cos u = F_v - \frac{1}{2}G_u &= \Gamma^1_{22}E + \Gamma^2_{22}F = \Gamma^1_{22} \\ 0 = \frac{1}{2}G_v &= \Gamma^1_{22}F + \Gamma^2_{22}G = \Gamma^2_{22}\cos^2 u \end{aligned}$$

Hence,

$$\begin{array}{ll} \Gamma^1_{11} = 0 & \Gamma^2_{11} = 0 \\ \Gamma^1_{12} = 0 & \Gamma^2_{12} = \frac{-\sin u}{\cos u} = -\tan u \\ \Gamma^1_{22} = \sin u \cos u & \Gamma^2_{22} = 0 \end{array}$$

♣

(c): Show that such a surface would satisfy the Gauss formula.

Proof. The Gauss Formula is

$$\begin{aligned}-1 = -EK &= \Gamma^1_{12}\Gamma^2_{11} + \Gamma^2_{12,u} + \Gamma^2_{12}\Gamma^2_{12} - \Gamma^1_{11}\Gamma^2_{12} - \Gamma^2_{11,v} - \Gamma^2_{11}\Gamma^2_{22} \\ &= -\sec^2 u + \tan^2 u,\end{aligned}$$

which is an identity, since $\tan^2 u + 1 = \sec^2 u$. Thus, the surface satisfies the Gauss formula. ☐

(d): Show that such a surface would satisfy the first Mainardi-Codazzi equation, but violate the second one. Conclude that no such surface exists.

Proof. The first Mainardi-Codazzi equation is

$$\begin{aligned}0 = e_v - f_u &= e\Gamma^1_{12} + f(\Gamma^2_{12} - \Gamma^1_{11}) - g\Gamma^2_{11} \\ &= e(0) + 0(\Gamma^2_{12} - \Gamma^1_{11}) - g(0) \\ &= 0.\end{aligned}$$

The second Mainardi-Codazzi equation, if it held, would be

$$\begin{aligned}0 = f_v - g_u &= e\Gamma^1_{22} + f(\Gamma^2_{22} - \Gamma^1_{12}) - g\Gamma^2_{12} \\ &= \cos^3 u \sin u + \tan u.\end{aligned}$$

If this holds, then, dividing by $\tan u$, we would have that $\cos^4 u + 1 = 0$, which is clearly impossible since $\cos^4 u \geq 0$, so $\cos^4 u + 1 \geq 1$.

Therefore, since the second Mainardi-Codazzi equation does not hold, no surface with E, F, G and e, f, g as given exists. ☐

9

In this problem, we will see how the Mainardi-Codazzi equations simplify when the coordinate neighborhood contains no umbilical points and the coordinate curves are lines of curvature ($F = 0$ and $f = 0$).

(a): Show that in such a case, the Mainardi-Codazzi equations may be written as

$$e_v = e\Gamma^1_{12} - g\Gamma^2_{11} \text{ and } g_u = g\Gamma^2_{12} - e\Gamma^1_{22}.$$

Proof. The first Mainardi-Codazzi equation is, in this situation,

$$e_v = e_v - f_u = e\Gamma^1_{12} + f(\Gamma^2_{12} - \Gamma^1_{11}) - g\Gamma^2_{11} = e\Gamma^1_{12} - g\Gamma^2_{11}.$$

The second Mainardi-Codazzi equation is

$$-g_u = f_v - g_u = e\Gamma^1_{22} + f(\Gamma^2_{22} - \Gamma^1_{12}) - g\Gamma^2_{12} = e\Gamma^1_{22} - g\Gamma^2_{12},$$

so $g_u = g\Gamma^2_{12} - e\Gamma^1_{22}$. ☐

(b): Show also that

$$\begin{aligned}\Gamma^2_{11} &= -E_v/2G & \Gamma^1_{12} &= E_v/2E \\ \Gamma^1_{22} &= -G_u/2E & \Gamma^2_{12} &= G_u/2G.\end{aligned}$$

Proof. Using the defining equations of the Christoffel equations,

$$\begin{aligned}\frac{1}{2}E_u &= \Gamma^1_{11}E + \Gamma^2_{11}F = \Gamma^1_{11}E\\ -\frac{1}{2}E_v = F_u - \frac{1}{2}E_v &= \Gamma^1_{11}F + \Gamma^2_{11}G = \Gamma^2_{11}G\\ \frac{1}{2}E_v &= \Gamma^1_{12}E + \Gamma^2_{12}F = \Gamma^1_{12}E\\ \frac{1}{2}G_u &= \Gamma^1_{12}F + \Gamma^2_{12}G = \Gamma^2_{12}G\\ -\frac{1}{2}G_u = F_v - \frac{1}{2}G_u &= \Gamma^1_{22}E + \Gamma^2_{22}F = \Gamma^1_{22}E\\ \frac{1}{2}G_v &= \Gamma^1_{22}F + \Gamma^2_{22}G = \Gamma^2_{22}G\end{aligned}$$

Hence,

$$\begin{array}{ll}\Gamma^1_{11} = \frac{E_u}{2E} & \Gamma^2_{11} = \frac{-E_v}{2G}\\ \Gamma^1_{12} = \frac{E_v}{2E} & \Gamma^2_{12} = \frac{G_u}{2G}\\ \Gamma^1_{22} = \frac{-G_u}{2E} & \Gamma^2_{22} = \frac{G_v}{2G}\end{array}$$

□

(c): Conclude that the Mainardi-Codazzi equations take the following form:

$$e_v = \frac{1}{2}E_v(e/E + g/G) \text{ and } g_u = \frac{1}{2}G_u(e/E + g/G).$$

Proof. The first Mainardi-Codazzi equation is

$$\begin{aligned}e_v &= e\Gamma^1_{12} - g\Gamma^2_{11}\\ &= e\frac{E_v}{2E} + g\frac{E_v}{2G}\\ &= \frac{1}{2}E_v\left(\frac{e}{E} + \frac{g}{G}\right).\end{aligned}$$

The second Mainardi-Codazzi equation is

$$\begin{aligned}g_u &= g\Gamma^2_{12} - e\Gamma^1_{22}\\ &= g\frac{G_u}{2G} + e\frac{G_u}{2E}\\ &= \frac{1}{2}G_u\left(\frac{g}{G} + \frac{e}{E}\right).\end{aligned}$$

□

3

Let V and W be parallel vector fields along a curve $\alpha : I \to S$. Show that the inner product $\langle V, W\rangle$ is constant along α. Conclude that the lengths $|V|$ and $|W|$ are also constant along α.

Proof. Let $V(t) = a(t)X_u + b(t)X_v$ and let $W(t) = c(t)X_u + d(t)X_v$. Note that

$$\begin{aligned}\frac{d}{dt}\langle V, W\rangle &= \langle \frac{dV}{dt}, W\rangle + \langle V, \frac{dW}{dt}\rangle \\ &= \langle \frac{DV}{dt} + (au'e + av'f + bu'f + bv'g)N, W\rangle + \langle V, \frac{DW}{dt} + (cu'e + cv'f + du'f + dv'g)N\rangle \\ &= \langle (au'e + av'f + bu'f + bv'g)N, W\rangle + \langle (cu'e + cv'f + du'f + dv'g)N\rangle \\ &= 0\end{aligned}$$

since N is normal to both V and W. Thus, $\langle V, W\rangle$ is constant along α. Since our choices of V and W were arbitrary, we see that $|V|^2 = \langle V, V\rangle$ and $|W|^2 = \langle W, W\rangle$ are constant along α as well. □

4

Let $\alpha : I \to S^2$ parametrize a great circle at constant speed. Show that the velocity field α' is parallel along α.

Proof. If necessary, rotate S^2 so that the great circle in question is the equator; let $\beta : I \to S^2$ be the composition of α with the rotation. Since the rotation doesn't affect the velocity of α, we see that α' is parallel along α if and only if β' is parallel along β.

Now, we can parametrize S^2 by $X(u, v) = (\sin v \cos u, \sin v \sin u, \cos v)$ for $0 \leq u \leq 2\pi$ and $0 \leq v \leq \pi$. Then

$$\beta(t) = X(t, \pi/2) = (\cos t, \sin t, 0),$$

so

$$\beta'(t) = (-\sin t, \cos t, 0).$$

On the other hand,

$$\begin{aligned}X_u &= (-\sin v \sin u, \sin v \cos u, 0) \\ X_v &= (\cos v \cos u, \sin v \sin u, -\sin v).\end{aligned}$$

Along the β, $X_u = (-\sin t, \cos t, 0)$ and $X_v = (0, 0, 1)$. Thus, $\beta' = X_u$ and so, by the definition of the covariant derivative,

$$\frac{D\beta'}{dt} = \Gamma^1_{11} X_u + \Gamma^2_{11} X_v.$$

Now, recall that, on the sphere, $E = 1$, $F = 0$, $G = \sin^2 v$, so, from the defining equations of the Christoffel symbols,

$$\begin{aligned}0 = \frac{1}{2}E_u &= \Gamma^1_{11}E + \Gamma^2_{11}F = \Gamma^1_{11} \\ 0 = F_u - \frac{1}{2}E_v &= \Gamma^1_{11}F + \Gamma^2_{11}G = \Gamma^2_{11}\sin^2 v,\end{aligned}$$

so $\Gamma^1_{11} = \Gamma^2_{11} = 0$, meaning that $\frac{D\beta'}{dt} = 0$. □

9

Show that the geodesic curvature of the curve $\alpha : I \to S$ at the point $\alpha(s)$ is the same as the ordinary curvature at that point of the plane curve obtained by projecting α orthogonally onto the tangent plane $T_{\alpha(s)}S$.

Proof. We'll find the projected curvature at the point $s = s_0$. If X_u and X_v are the standard basis for the tangent plane, re-parametrize, if necessary, such that $X_u(\alpha(s_0))$, $X_v(\alpha(s_0))$ forms an orthonormal basis of $T_{\alpha(s_0)}S$. Now, define

$$\beta(s) := \langle \alpha(s), X_u \rangle X_u + \langle \alpha(s), X_v \rangle X_v.$$

This is the projection of α onto the tangent plane. Now, since X_u and X_v are fixed,

$$\beta'(s) = \langle \alpha'(s), X_u \rangle X_u + \langle \alpha'(s), X_v \rangle X_v$$

and

$$\beta''(s) = \langle \alpha''(s), X_u \rangle X_u + \langle \alpha''(s), X_v \rangle X_v.$$

Thus,

$$\begin{aligned} \beta'(s) \times \beta''(s) &= (\langle \alpha', X_u \rangle X_u + \langle \alpha', X_v \rangle X_v) \times (\langle \alpha'', X_u \rangle X_u + \langle \alpha'', X_v \rangle X_v) \\ &= \langle \alpha', X_u \rangle \langle \alpha'', X_u \rangle (X_u \times X_u) + \langle \alpha', X_v \rangle \langle \alpha'', X_u \rangle (X_v \times X_u) \\ &\quad + \langle \alpha', X_u \rangle \langle \alpha'', X_v \rangle (X_u \times X_v) + \langle \alpha', X_v \rangle \langle \alpha'', X_v \rangle (X_v \times X_v) \\ &= \big(\langle \alpha', X_u \rangle \langle \alpha'', X_v \rangle - \langle \alpha', X_v \rangle \langle \alpha'', X_u \rangle \big) N \end{aligned}$$

since $N = \frac{X_u \times X_v}{|X_u \times X_v|}$. Thus, the signed projected curvature at the point we're interested in is given by

$$\kappa_p = \frac{\langle \beta' \times \beta'', N \rangle}{|\beta'|^3} = \frac{\langle \alpha', X_u \rangle \langle \alpha'', X_v \rangle - \langle \alpha'', X_u \rangle \langle \alpha', X_v \rangle}{|\beta'|^3}.$$

Since $\beta' = \alpha'$ at this particular point and α is parametrized by arc length, the denominator here is 1. Thus,

$$\kappa_p = \langle \alpha', X_u \rangle \langle \alpha'', X_v \rangle - \langle \alpha'', X_u \rangle \langle \alpha', X_v \rangle.$$

On the other hand,

$$\begin{aligned} \kappa_g = \langle \alpha'', M \rangle &= \langle \alpha'', N(\alpha(s)) \times T(s) \rangle \\ &= \left\langle \alpha'', \frac{X_u \times X_v}{|X_u \times X_v|} \times \alpha' \right\rangle \\ &= \big\langle \alpha'', \langle X_u, \alpha' \rangle X_v - \langle X_v, \alpha' \rangle X_u \big\rangle \\ &= \langle X_u, \alpha' \rangle \langle X_v, \alpha'' \rangle - \langle X_v, \alpha' \rangle \langle X_u, \alpha'' \rangle \\ &= \kappa_p. \end{aligned}$$

□

10

Let $\alpha : I \to S$ be a smooth curve on the regular surface S in $\mathbb{R}^3$. Let $\kappa(s)$ be the ordinary curvature of the curve α in $\mathbb{R}^3$, let $\kappa_g(s)$ be its geodesic curvature on the surface S, and let $k_n(s)$ be the normal curvature of the surface S at the point $\alpha(s)$ in the direction $\alpha'(s)$. Show that

$$\kappa(s)^2 = \kappa_g(s)^2 + k_n(s)^2.$$

Check this equation when α is a small circle on a round sphere.

Proof. By the result we just proved in problem 9 above, we know that $\kappa_g = \kappa_p$. Since the curvature κ is the length of the vector α'', this means that κ_g is the length of the orthogonal projection of α'' onto $T_{\alpha(s)}S$. On the other hand, $k_n(s)$ is defined to be the length of the projection of α'' onto the normal to the surface, $N(\alpha(s))$. Since $N(\alpha(s))$ is perpendicular to $T_{\alpha(s)}S$, this gives an orthogonal decomposition of α''. Thus, simply using the Pythagorean Theorem,

$$\kappa(s)^2 = \kappa_g(s)^2 + k_n(s)^2.$$

□

12

Check that $u =$ constant and $v = v(t)$ is a solution of the geodesic equations for some choice of $v(t)$.

Answer: Equation (1) holds trivially, since $u' = u'' = 0$. Also, since $u' = 0$, the second equation reduces to

$$v'' + \frac{f'f'' + g'g''}{(f')^2 + (g')^2}(v')^2 = 0.$$

Now, consider the curve

$$\alpha(t) = X(u, v(t)) = (f(v(t))\cos u, f(v(t))\sin u, g(v(t))).$$

Since α is a curve in 3-space, we can re-parametrize it by arc length. Thus, we can choose $v(t)$ such that α *is* parametrized by arc length. That being the case,

$$\alpha'(t) = (f'v'\cos u, f'v'\sin u, g'v');$$

since α is parametrized by arc length,

$$1 = |\alpha'(t)|^2 = (f'v')^2\cos^2 u + (f'v')^2\sin^2 u + (g'v')^2 = (f'v')^2 + (g'v')^2.$$

Differentiating with respect to t, this implies that

$$\begin{aligned} 0 &= 2f'v'(f''(v')^2 + f'v'') + 2g'v'(g''(v')^2 + g'v'') \\ &= 2\left[f'f''(v')^3 + (f')^2v'v'' + g'g''(v')^3 + (g')^2v'v''\right] \\ &= 2\left[(f'f'' + g'g'')(v')^3 + ((f')^2 + (g')^2)v'v''\right]. \end{aligned}$$

Since v is certainly non-constant and f and g cannot both be constant, $((f')^2+(g')^2)v' \neq 0$. Hence, dividing both sides by $2((f')^2+(g')^2)v'$ yields

$$0 = \frac{f'f''+g'g''}{(f')^2+(g')^2}(v')^2 + v'',$$

which is equation (2). Therefore, we see that this choice of $v(t)$ satisfies the geodesic equations.

♣

13

Show that the curve $X(u(t), v(t))$ on our surface of revolution is travelled at constant speed if and only if

$$u''f^2u' + v''(f'^2+g'^2)v' + (f'f''+g'g'')v'^3 + ff'u'^2v' = 0. \tag{3}$$

Proof. Consider $X(u(t), v(t))$. Then

$$X' = (f'v'\cos u - f\sin u u', f'v'\sin u + f\cos u u', g'v')$$

and

$$\begin{aligned} |X'|^2 &= (f'v')^2\cos^2 u - 2ff'v'u'\sin u\cos u + (fu')^2\sin^2 u + (f'v')^2\sin^2 u \\ &\quad +2ff'v'u'\sin u\cos u + (fu')^2\cos^2 u + (g'v')^2 \\ &= (f'v')^2 + (fu')^2 + (g'v')^2 \end{aligned}$$

Then $X(u(t), v(t))$ is travelled at constant speed if and only if the derivative of this expression is constant. That is, the curve is travelled at constant speed if and only if

$$\begin{aligned} 0 &= 2f'v'(f''(v')^2 + f'v'') + 2fu'(f'v'u' + fu'') + 2g'v'(g''(v')^2 + g'v'') \\ &= 2u''f^2u' + 2v''((f')^2+(g')^2)v' + 2(f'f''+g'g'')(v')^3 + 2ff'(u')^2v'. \end{aligned}$$

Dividing by 2 gives the desired equation. ☐

14

Show that equations (1) and (2) together imply equation (3).

Proof. From the sum $u'f^2(1) + v'((f')^2+(g')^2)(2)$ we have the linear combination

$$\begin{aligned} 0 &= u'f^2\left(u'' + 2\frac{f'}{f}u'v'\right) + v'((f')^2+(g')^2)\left(v'' + \frac{-ff'}{(f')^2+(g')^2}(u')^2 + \frac{f'f''+g'g''}{(f')^2+(g')^2}(v')^2\right) \\ &= u''f^2u' + 2v'ff'(u')^2 + v'v''((f')^2+(g')^2) - v'ff'(u')^2 + (f'f''+g'g'')(v')^3 \\ &= u''f^2u' + v''((f')^2+(g')^2)v' + (f'f''+g'g'')(v')^3 + ff'(u')^2v', \end{aligned}$$

which is equation (3). ☐

15

Show that if $f'(v_0) = 0$, then the circle $u = ct$ and $v = v_0$ satisfies equations (1) and (2), and hence is a geodesic.

Proof. With u and v as given and since $f' = 0$, equation (1) reduces to

$$0 = u'' + 2\frac{f'}{f}u'v' = 0 + 2\frac{0}{f}c(0) = 0.$$

Similarly, equation (2) reduces to

$$0 = v'' + \frac{-ff'}{(f')^2 + (g')^2}(u')^2 + \frac{f'f'' + g'g''}{(f')^2 + (g')^2}(v')^2 = 0 + \frac{0}{(g')^2}c^2 + \frac{0 + g'g''}{(g')^2}(0) = 0.$$

□

16

Show that if $v' \neq 0$, then equations (1) and (3) together imply equation (2). So to get a geodesic, just satisfy equations (1) and make sure you travel at constant speed.

Proof. Since f and g cannot both be constant (else we have only a circle, not a surface), $(f')^2 + (g')^2 \neq 0$. Hence, since $v' \neq 0$, we can safely divide by $v'((f')^2 + (g')^2)$. Now, we know from problem 14 above that $u'f^2(1) + v'((f')^2 + (g')^2)(2) = (3)$, so, dividing both sides by $v'((f')^2 + (g')^2)$,

$$(2) = \frac{u'f^2}{v'((f')^2 + (g')^2)}(1) - \frac{1}{v'((f')^2 + (g')^2)}(3).$$

Therefore, (1) and (3) imply (2). □

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