

## DIFFERENTIAL GEOMETRY HW 5

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Show that the law of cosines in spherical geometry is

$$\cos c = \cos a \cos b + \sin a \sin b \cos \theta.$$

*Proof.* Consider the spherical triangle depicted below:

Form radii from each of the vertices of the triangle to the origin of the sphere, denoted by  $OA$ ,  $OB$  and  $OC$ . Since this triangle is on a unit sphere, the length of the arc  $a$  is given by the measure of the angle  $BOC$  in radians, the length of  $b$  is the measure of  $AOC$  and the length of  $c$  is the measure of  $AOB$ . Thus, we may as well think of  $BOC$  as angle  $a$  and similarly for these other two angles. Now,  $\theta$  is, in addition to the angle between  $a$  and  $b$ , the dihedral angle between the plane  $P_1$  defined by  $OB$  and  $OC$  and the plane  $P_2$  defined by  $OA$  and  $OC$ . Let  $N_i$  be the unit normal to  $P_i$  for  $i = 1, 2$ ; then  $\theta$  is the angle between  $N_1$  and  $N_2$ . Now  $N_1 = \overrightarrow{OB} \times \overrightarrow{OC}$  and  $N_2 = \overrightarrow{OA} \times \overrightarrow{OC}$ . Therefore, by definition of the inner product,

$$\begin{aligned} \langle N_1, N_2 \rangle &= |\overrightarrow{OB} \times \overrightarrow{OC}| |\overrightarrow{OA} \times \overrightarrow{OC}| \cos \theta \\ &= (|\overrightarrow{OB}| |\overrightarrow{OC}| \sin a) (|\overrightarrow{OA}| |\overrightarrow{OC}| \sin b) \cos \theta \\ &= \sin a \sin b \cos \theta. \end{aligned}$$

On the other hand,  $N_1 = \overrightarrow{OB} \times \overrightarrow{OC}$  and  $N_2 = \overrightarrow{OA} \times \overrightarrow{OC}$ , so

$$\begin{aligned} \langle N_1, N_2 \rangle &= \langle \overrightarrow{OB} \times \overrightarrow{OC}, \overrightarrow{OA} \times \overrightarrow{OC} \rangle \\ &= \langle \overrightarrow{OB}, \overrightarrow{OA} \rangle \langle \overrightarrow{OC}, \overrightarrow{OC} \rangle - \langle \overrightarrow{OB}, \overrightarrow{OC} \rangle \langle \overrightarrow{OC}, \overrightarrow{OA} \rangle \\ &= |\overrightarrow{OB}| |\overrightarrow{OA}| \cos c - \left( |\overrightarrow{OB}| |\overrightarrow{OC}| \cos a \right) \left( |\overrightarrow{OC}| |\overrightarrow{OA}| \cos b \right) \\ &= \cos c - \cos a \cos b. \end{aligned}$$

Equating these two, we see that

$$\cos c - \cos a \cos b = \sin a \sin b \cos \theta,$$

so

$$\cos c = \cos a \cos b + \sin a \sin b \cos \theta.$$

□

## 10

Let  $S \subset \mathbb{R}^3$  be a smooth surface homeomorphic to  $S^2$ . Suppose  $\Gamma \subset S$  is a simple closed geodesic, and let  $A$  and  $B$  be the two regions on  $S$  which have  $\Gamma$  as boundary. Let  $N : S \rightarrow S^2$  be the Gauss map. Prove that  $N(A)$  and  $N(B)$  have the same area on  $S^2$ .

*Proof.* Recall that, if  $R$  is a region on a surface, then  $\int_R \kappa d(\text{area})$  is equal to the signed area of  $N(R)$  on  $S^2$ . Note that, since  $\Gamma$  is a simple closed curve and  $S$  is homeomorphic to  $S^2$ ,  $A$  and  $B$  are both homeomorphic to a disk; hence,  $\chi(A) = \chi(B) = 1$ . Now, using the Gauss-Bonnet Theorem,

$$\begin{aligned} 2\pi &= 2\pi\chi(A) = \int_A \kappa d(\text{area}) + \int_{\partial A} \kappa_g(s) ds + \sum_{i=0}^k \theta_i \\ &= \text{Area } N(A) + \int_{\partial A} 0 ds + 0 \\ &= \text{Area } N(A). \end{aligned}$$

Similarly,

$$\begin{aligned} 2\pi &= 2\pi\chi(B) = \int_B \kappa d(\text{area}) + \int_{\partial B} \kappa_g(s) ds + \sum_{i=0}^k \theta_i \\ &= \text{Area } N(B) + \int_{\partial B} 0 ds + 0 \\ &= \text{Area } N(B). \end{aligned}$$

Thus, we see that  $N(A)$  and  $N(B)$  have the same area on  $S^2$ .

□

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Let  $S \subset \mathbb{R}^3$  be a surface homeomorphic to a cylinder and with Gaussian curvature  $K < 0$ . Show that  $S$  has at most one simple closed geodesic.

*Proof.* Suppose there are two simple closed geodesics on  $S$ ,  $\gamma_1$  and  $\gamma_2$ . Of course, there are 3 possibilities, either  $\gamma_1$  and  $\gamma_2$  don't intersect, they intersect in one point, or they intersect in more than one point. If they intersect in more than one point, then, looking at two adjacent intersection points (as viewed traveling along, say,  $\gamma_1$ ), the region bounded by  $\gamma_1$  and  $\gamma_2$  is homeomorphic to a disc, since  $\gamma_1$  and  $\gamma_2$  are simple closed curves. However, by problem 11 (discussed in class), this is impossible, since  $K < 0$  on  $S$ , so we see that  $\gamma_1$  and  $\gamma_2$  can intersect in at most 1 point.

On the other hand, suppose  $\gamma_1$  and  $\gamma_2$  don't intersect at all. Let  $S'$  be the region bounded by  $\gamma_1$  and  $\gamma_2$ . Then, since  $\gamma_1$  and  $\gamma_2$  are simple closed curves and  $S$  is homeomorphic to a cylinder,  $S'$  is also homeomorphic to a cylinder. Hence,  $\chi(S') = 0$ . Now, by the Gauss-Bonnet Theorem,

$$\begin{aligned} 0 = 2\pi\chi(S') &= \int_{S'} \kappa d(\text{area}) + \int_{\partial S'} \kappa_g(s) ds + \sum_{i=0}^k \theta_i \\ &= \int_{S'} \kappa d(\text{area}) + \int_{\partial S'} 0 ds + 0 \\ &= \int_{S'} \kappa d(\text{area}) \\ &< 0, \end{aligned}$$

which is an obvious impossibility.

Thus, we conclude that  $\gamma_1$  and  $\gamma_2$  must intersect in exactly one point. However, for this to be the case,  $\gamma_1$  and  $\gamma_2$  must be tangent at their point of intersection, which, by the uniqueness of geodesics, implies that  $\gamma_1$  and  $\gamma_2$  describe the same curve. Therefore, we conclude that  $S$  has at most one simple closed geodesic.  $\square$

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Let  $S \subset \mathbb{R}^3$  be a smooth closed surface of positive curvature, and thus homeomorphic to  $S^2$ . Show that if  $\Gamma_1$  and  $\Gamma_2$  are two simple closed geodesics on  $S$ , then they must intersect each other.

*Proof.* Suppose  $\Gamma_1$  and  $\Gamma_2$  don't intersect each other. Then, since  $\Gamma_1$  and  $\Gamma_2$  are simple closed curves and  $S$  is homeomorphic to  $S^2$ , the region  $S'$  bounded by  $\Gamma_1$  and  $\Gamma_2$  is homeomorphic to a cylinder. Therefore, since

$$\chi(S') = 0,$$

$$\begin{aligned} 0 = 2\pi\chi(S') &= \int_{S'} \kappa d(\text{area}) + \int_{\partial S'} \kappa_g ds + \sum_{i=0}^k \theta_i \\ &= \int_{S'} \kappa d(\text{area}) + \int_{\partial S'} 0 ds + 0 \\ &= \int_{S'} \kappa d(\text{area}) \\ &> 0, \end{aligned}$$

which is impossible. Therefore, we conclude that  $\Gamma_1$  and  $\Gamma_2$  must intersect each other.  $\square$

(B)

Summarize the proof of the Gauss-Bonnet Theorem.

**Theorem 13.1.** *Let  $U$  be an open set in  $\mathbb{R}^2$  and  $X : U \rightarrow S \subset \mathbb{R}^3$  a regular local parametrization of the surface  $S$ . Let  $R \subset X(U)$  be a region homeomorphic to a disk. Let  $\alpha : I \rightarrow \partial R$  be a piecewise smooth parametrization of  $\partial R$  by arc length, with vertices at  $\alpha(s_0), \alpha(s_1), \dots, \alpha(s_k)$  and with exterior angles  $\theta_0, \theta_1, \dots, \theta_k$  at these vertices. Then*

$$\int_R K d(\text{area}) + \int_{\partial R} \kappa_g(s) ds + \sum_{i=0}^k \theta_i = 2\pi.$$

*Proof.* (sketch) We may as well assume  $X$  gives an orthogonal parametrization of  $S$ . Let  $W(s) = \alpha'(s)$ , which is unit length and well-defined on the smooth pieces of  $\partial R$ . If  $V(s) = \frac{X_u}{|X_u|}$ , then  $\phi$  is the angle from  $V$  to  $W$ . Now, the rate of change of  $W$  with respect to the frame  $(X_u, X_v)$  is equal to the rate of change of  $V$  with respect to the same frame plus the rate of change of the angle between  $W$  and  $V$ . Since the algebraic value of a covariant derivative of a vector field tells us how that vector field changes with respect to  $(X_u, X_v)$ , this means that

$$\left[ \frac{DW}{ds} \right] = \left[ \frac{DV}{ds} \right] + \frac{d\phi}{ds}$$

along each smooth arc of  $\partial R$ . Now, since  $\frac{dV}{ds} = \sqrt{E} \frac{dX_u}{ds} + \frac{d\sqrt{E}}{ds} X_u$ , only the first term in this sum contributes to

$$\left[ \frac{DW}{ds} \right] = \left[ \frac{DV}{ds} \right] \frac{d\phi}{ds} = \left\langle \frac{dV}{ds}, N \times V \right\rangle + \frac{d\phi}{ds} = \left\langle \frac{dV}{ds}, \sqrt{G} X_v \right\rangle + \frac{d\phi}{ds},$$

so a simple computation shows that

$$\left[ \frac{DW}{ds} \right] = \frac{1}{2} (EG)^{-1/2} \left[ G_u \frac{dv}{ds} - E_v \frac{du}{ds} \right] + \frac{d\phi}{ds}.$$

Hence, along each smooth arc,

$$\kappa_g(s) = \left[ \frac{D\alpha'}{ds} \right] = \frac{1}{2}(EG)^{-1/2} \left[ G_u \frac{dv}{ds} - E_v \frac{du}{ds} \right] + \frac{d\phi}{ds}.$$

Furthermore,  $\int_{\partial R} \frac{d\phi}{ds} ds + \sum_{i=0}^k \theta_i$  gives the total change in angle of inclination with respect to the frame  $(X_u, X_v)$  along the simple closed curve  $\partial R$ ; that is,  $2\pi$ . Hence,

$$\begin{aligned} \int_{\partial R} \kappa_g(s) ds + \sum_{i=0}^k \theta_i &= \int_{\partial R} \frac{1}{2}(EG)^{-1/2} \left[ G_u \frac{dv}{ds} - E_v \frac{du}{ds} \right] ds + \int_{\partial R} \frac{d\phi}{ds} ds + \sum_{i=0}^k \theta_i \\ &= \int_{\partial R} \frac{1}{2}(EG)^{-1/2} \left[ G_u \frac{dv}{ds} - E_v \frac{du}{ds} \right] ds + 2\pi. \end{aligned}$$

Now, using Green's Theorem to convert the remaining integral to a surface integral, we see that

$$\begin{aligned} \int_{\partial R} \frac{1}{2}(EG)^{-1/2} \left[ G_u \frac{dv}{ds} - E_v \frac{du}{ds} \right] ds &= \int_{\partial R} \left[ \frac{1}{2}(EG)^{-1/2} G_u \frac{dv}{ds} - \frac{1}{2}(EG)^{-1/2} E_v \frac{du}{ds} \right] ds \\ (1) \qquad \qquad \qquad &= \int_R \left[ \left( \frac{1}{2}(EG)^{-1/2} G_u \right)_u + \left( \frac{1}{2}(EG)^{-1/2} E_v \right)_v \right] dudv. \end{aligned}$$

With a little manipulation, we can see that this is  $-\int_R K d(\text{area})$ . Using the Gauss equation

$$-EK = \Gamma_{12}^1 \Gamma_{11}^2 + \Gamma_{12,u}^2 + \Gamma_{12}^2 \Gamma_{12}^2 - \Gamma_{11}^2 \Gamma_{12}^2 - \Gamma_{11,v}^2 - \Gamma_{11}^2 \Gamma_{22}^2$$

and making the appropriate substitutions for the Christoffel symbols, we get an equation equivalent to

$$K = \frac{-1}{2}(EG)^{-1/2} \left[ (E_v(EG)^{-1/2})_v + (G_u(EG)^{-1/2})_u \right].$$

Since  $\sqrt{EG}$  is the volume multiplier (as per problem 4 below),

$$\begin{aligned} - \int_R K d(\text{area}) &= - \int_R K(EG)^{1/2} dudv = - \int_R - \left[ \left( \frac{1}{2}(EG)^{-1/2} G_u \right)_u + \left( \frac{1}{2}(EG)^{-1/2} E_v \right)_v \right] dudv \\ &= \int_R \left[ \left( \frac{1}{2}(EG)^{-1/2} G_u \right)_u + \left( \frac{1}{2}(EG)^{-1/2} E_v \right)_v \right] dudv, \end{aligned}$$

which is the integral in (1). Therefore, we see that

$$\int_{\partial R} \kappa_g(s) ds + \sum_{i=0}^k \theta_i = - \int_R K d(\text{area}) + 2\pi,$$

so

$$\int_R K d(\text{area}) + \int_{\partial R} \kappa_g(s) ds + \sum_{i=0}^k \theta_i = \int_R K d(\text{area}) - \int_R K d(\text{area}) + 2\pi = 2\pi.$$

□

4

**(a):** Let  $A$  and  $B$  be two vectors in the plane, with an angle  $\theta$  between them. Then the area of the parallelogram that they span is  $|A||B|\sin\theta$ .

*Proof.* If  $A$  and  $B$  define two sides of the parallelogram, then, as in the below drawing,  $\sin\theta = \frac{h}{|B|}$  where  $h$  is the altitude of the parallelogram.

Hence,  $h = |B|\sin\theta$ , so the area of the parallelogram is

$$|A|h = |A||B|\sin\theta.$$

□

**(b):** If  $A_1, \dots, A_n$  are vectors in  $\mathbb{R}^n$ , and

$$g_{ij} = \langle A_i, A_j \rangle,$$

show that

$$\sqrt{\det(g_{ij})} = \text{vol}(A_1, \dots, A_n).$$

*Proof.* Let  $A$  be the matrix with  $A_i$  as its  $i$ th row. Then  $A$  is a linear transformation that takes the standard oriented basis of  $\mathbb{R}^n$  to  $(A_1, \dots, A_n)$ . The volume of the solid spanned by  $(A_1, \dots, A_n)$  is given by  $\text{vol}(e_1, \dots, e_n) \cdot \det A = 1 \cdot \det A = \det A$ . Now, if  $A_i = (A_{i1}, \dots, A_{in})$ , then

$$AA^t = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix} \begin{pmatrix} A_{11} & \cdots & A_{n1} \\ \vdots & \ddots & \vdots \\ A_{1n} & \cdots & A_{nn} \end{pmatrix} = \begin{pmatrix} \langle A_1, A_1 \rangle & \cdots & \langle A_1, A_n \rangle \\ \vdots & \ddots & \vdots \\ \langle A_n, A_1 \rangle & \cdots & \langle A_n, A_n \rangle \end{pmatrix} = (g_{ij}).$$

Therefore,  $\det(g_{ij}) = \det A \cdot \det A^t = (\det A)^2 = (\text{vol}(A_1, \dots, A_n))^2$ , so we see that

$$\sqrt{\det(g_{ij})} = \text{vol}(A_1, \dots, A_n).$$

□

(c): Let  $x : U \rightarrow M$  be a coordinate neighborhood in a Riemannian manifold  $M$ , and  $R \subset x(U)$  an open set with compact closure. Show that

$$\text{vol } R = \int_{x^{-1}R} \sqrt{\det(g_{ij})} dx_1 \cdots dx_n,$$

where  $g_{ij} = \langle \partial/\partial x_i, \partial/\partial x_j \rangle$ .

*Proof.* This is the definition of the volume of  $R$ . To see why it makes sense, let  $g_{ij} = \langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle$ . Then, by part (b),  $\sqrt{\det(g_{ij})}$  gives the volume of the box  $\left( \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right)$ . Then pulling back to  $\mathbb{R}^n$  and adding over all these boxes gives

$$\text{vol } R = \int_{x^{-1}R} \sqrt{\det(g_{ij})} dx_1 \cdots dx_n.$$

□

(d): Double check that the right hand side is invariant under change of coordinates.

*Proof.* Suppose  $y : V \rightarrow M$  is another coordinate neighborhood containing the region  $R$ . Let  $h_{ij} = \langle \frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j} \rangle$ . Then

$$\text{vol} \left( \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n} \right) = \sqrt{\det(h_{ij})}$$

and so, by the argument in (c),

$$\text{vol } R = \int_{y^{-1}R} \sqrt{\det(h_{ij})} dy_1 \cdots dy_n.$$

On the other hand,

$$\sqrt{\det(g_{ij})} = \text{vol} \left( \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right) = J \text{vol} \left( \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n} \right) = J \sqrt{\det(h_{ij})},$$

where  $J = \det \left( \frac{\partial x_i}{\partial y_j} \right)$  is the jacobian of the change of coordinate function  $y^{-1} \circ x$ . Hence,

$$\int_{y^{-1}R} \sqrt{\det(h_{ij})} dy_1 \cdots dy_n = \int_{x^{-1}R} \frac{\sqrt{\det(g_{ij})}}{J} J dx_1 \cdots dx_n = \int_{x^{-1}R} \sqrt{\det(g_{ij})} dx_1 \cdots dx_n,$$

so we see that the computation of the volume of  $R$  is independent of the choice of coordinates. □

A function  $g : \mathbb{R} \rightarrow \mathbb{R}$  given by  $g(t) = yt + x$ ,  $t, x, y \in \mathbb{R}$ ,  $y > 0$ , is called a *proper affine function*. The subset of all such functions with respect to the usual composition law forms a Lie group  $G$ . As a differentiable manifold  $G$  is simply the upper half-plane  $\{(x, y) \in \mathbb{R}^2 : y > 0\}$  with the differentiable structure induced from  $\mathbb{R}^2$ . Prove that:

- (a): The left-invariant Riemannian metric of  $G$  which at the neutral element  $e = (0, 1)$  coincides with the Euclidean metric ( $g_{11} = g_{22} = 1$ ,  $g_{12} = 0$ ) is given by  $g_{11} = g_{22} = \frac{1}{y^2}$ ,  $g_{12} = 0$  (this is the metric of the non-euclidean geometry of Lobatchevski).

*Proof.*  $g_{11} = g_{22} = \frac{1}{y^2}$ ,  $g_{12} = 0$  certainly defines a metric  $\langle \cdot, \cdot \rangle_p$  on each point  $p \in G$ ; since  $g_{ij}$  is smooth for each  $i, j \in \{1, 2\}$ , this defines a Riemannian metric on  $G$ . Furthermore, at  $(0, 1)$ ,  $\frac{1}{y^2} = \frac{1}{1} = 1$ , so this metric coincides with the Euclidean metric at the neutral element of  $G$ . Thus, it only remains to show that this metric is left-invariant on  $G$ .

To that end, let  $g_0(t) = y_0t + x_0$  be an element of  $G$ . Then left translation by  $g_0$  is given by

$$L_{g_0}(g) = g_0 \circ g(t) = y_0(yt + x) + x_0 = y_0yt + (y_0x + x_0),$$

which is represented by the element  $(y_0x + x_0, y_0y) \in \mathbb{R}^2$ . Therefore,

$$\begin{aligned} dL_{g_0} &= \begin{pmatrix} \frac{\partial L_{g_0}}{\partial x} & \frac{\partial L_{g_0}}{\partial y} \\ \frac{\partial L_{g_0}}{\partial x} & \frac{\partial L_{g_0}}{\partial y} \end{pmatrix} \\ &= \begin{pmatrix} y_0 & 0 \\ 0 & y_0 \end{pmatrix}. \end{aligned}$$

If  $v = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} \in T_pG$  and  $w = c\frac{\partial}{\partial x} + d\frac{\partial}{\partial y} \in T_pG$ , then

$$\langle v, w \rangle_g = \left\langle a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}, c\frac{\partial}{\partial x} + d\frac{\partial}{\partial y} \right\rangle_{(x,y)} = acg_{11} + (ad+bc)g_{12} + bdg_{22} = \frac{ac + bd}{y^2}.$$

On the other hand,

$$\begin{aligned} \langle (dL_{g_0})_g v, (dL_{g_0})_g w \rangle_{L_{g_0}(g)} &= \langle y_0 v, y_0 w \rangle_{(y_0x+x_0, y_0y)} \\ &= y_0^2 \langle v, w \rangle_{(y_0x+x_0, y_0y)} \\ &= y_0^2 \frac{ac + bd}{(y_0y)^2} \\ &= \frac{ac + bd}{y^2} \\ &= \langle v, w \rangle_{g_0}, \end{aligned}$$

so the metric is left-invariant.  $\square$

**(b):** Putting  $(x, y) = z = x + iy$ ,  $i = \sqrt{-1}$ , the transformation  $z \mapsto z' = \frac{az+b}{cz+d}$ ,  $a, b, c, d \in \mathbb{R}$ ,  $ad - bc = 1$  is an isometry of  $G$ .

*Proof.* Let  $f$  denote the above Möbius transformation. Note that  $f$  is defined on all of  $\mathbb{C} = \mathbb{R}^2$  except when  $z = \frac{-d}{c} \notin G$ , so  $f$  is defined on all of  $G$ . Now, if  $z = x + iy$ ,

$$\begin{aligned} f(z) = f(x + iy) &= \frac{a(x + iy) + b}{c(x + iy) + d} \\ &= \frac{ax + b + iay}{cx + d + icy} \cdot \frac{cx + d - icy}{cx + d - icy} \\ &= \frac{acx^2 + (ad + bc)x + acy^2 + bd + i(acxy - acxy + (ad - bc)y)}{(cx + d)^2 + c^2y^2} \\ &= \frac{acx^2 + (ad + bc)x + acy^2 + bd}{(cx + d)^2 + c^2y^2} + i \frac{y}{(cx + d)^2 + c^2y^2}, \end{aligned}$$

which is in  $G$ , since  $\frac{y}{(cx+d)^2+c^2y^2} > 0$ . Therefore, we see that  $f : G \rightarrow G$ . Furthermore, we know from complex analysis that Möbius transformations are biholomorphic from their domain onto their image, so we see that  $f : G \rightarrow G$  is actually a diffeomorphism. Therefore, if we can show that  $f$  preserves the first fundamental form, that will suffice to show that  $f$  is an isometry.

Recall that the first fundamental form is given by

$$ds^2 = g_{11}dx dx + 2g_{12}dx dy + g_{22}dy dy = \frac{dx^2}{y^2} + \frac{dy^2}{y^2} = \frac{dx^2 + dy^2}{y^2}.$$

Now, letting  $z = x + iy$ , we know that  $dz = dx + idy$  and  $d\bar{z} = dx - idy$ . Hence,

$$dz d\bar{z} = (dx + idy)(dx - idy) = dx^2 + idxdy - idxdy - i^2 dy^2 = dx^2 + dy^2.$$

Also,  $z - \bar{z} = (x + iy) - (x - iy) = 2iy$ , so  $y = \frac{z - \bar{z}}{2i}$ . Therefore,

$$ds^2 = \frac{dx^2 + dy^2}{y^2} = \frac{dz d\bar{z}}{\left(\frac{z - \bar{z}}{2i}\right)^2} = \frac{-4dz d\bar{z}}{(z - \bar{z})^2}.$$

Since  $z$  and  $\bar{z}$  are the coordinates of  $G$ ,  $f(z)$  and  $f(\bar{z})$  are the coordinates on the image, which is also  $G$ . Hence, since the first fundamental form on the domain is given by  $ds^2 = \frac{-4dz d\bar{z}}{(z - \bar{z})^2}$ , the fundamental form on the range is given by

$$ds^2 = \frac{-4df(z)df(\bar{z})}{(f(z) - f(\bar{z}))^2} = \frac{-4f'(z)dz f'(\bar{z})d\bar{z}}{(f(z) - f(\bar{z}))^2}.$$

Now, using the quotient rule,

$$f'(z) = \frac{a(cz + d) - c(az + b)}{(cz + d)^2} = \frac{ad - bc}{(cz + d)^2} = \frac{1}{(cz + d)^2}$$

and

$$f'(\bar{z}) = \frac{a(c\bar{z} + d) - c(a\bar{z} + b)}{(c\bar{z} + d)^2} = \frac{ad - bc}{(c\bar{z} + d)^2} = \frac{1}{(c\bar{z} + d)^2}.$$

Hence,

$$f'(z)dz f'(\bar{z})d\bar{z} = \frac{dzd\bar{z}}{(cz + d)^2(c\bar{z} + d)^2}.$$

On the other hand,

$$\begin{aligned} f(z) - f(\bar{z}) &= \frac{az + b}{cz + d} - \frac{a\bar{z} + b}{c\bar{z} + d} = \frac{(az + b)(c\bar{z} + d) - (a\bar{z} + b)(cz + d)}{(cz + d)(c\bar{z} + d)} \\ &= \frac{(ad - bc)z + (bc - ad)\bar{z}}{(cz + d)(c\bar{z} + d)} \\ &= \frac{z - \bar{z}}{(cz + d)(c\bar{z} + d)}. \end{aligned}$$

Hence,

$$ds^2 = \frac{-4df(z)df(\bar{z})}{(f(z) - f(\bar{z}))^2} = \frac{-4\frac{dzd\bar{z}}{(cz+d)^2(c\bar{z}+d)^2}}{\left(\frac{z-\bar{z}}{(cz+d)(c\bar{z}+d)}\right)^2} = \frac{-4dzd\bar{z}}{(z - \bar{z})^2}.$$

Hence,  $f$  preserves the first fundamental form on  $G$ , so we see that  $f$  is an isometry.  $\square$