DIFFERENTIAL GEOMETRY HW 7

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1

Show that within a local coordinate system $(x_1, \ldots, x_n)$ on M with coordinate vector fields $X_1 = \partial/\partial x_1, \ldots, X_n = \partial/\partial x_n$, if we pick n^3 smooth real-valued functions Γ_{ij}^k at random, and define

$$\nabla_V W = \sum \left[v^i X_i(w^k) + v^i w^j \Gamma_{ij}^k \right] X_k,$$

then ∇ will satisfy the definition of an affine connection within the range of that coordinate system.

Proof. First, we show linearity in the first vector field; let f_1, f_2 be smooth functions on the range of the coordinate system and let V_1, V_2, W be smooth vector fields on the range of the coordinate system. Then

$$\begin{aligned}
\nabla_{f_1V_1+f_2V_2} W &= \sum \left[(f_1v_1^i + f_2v_2^i)X_i(w^k) + (f_1v_1^i + f_2v_2^i)w^j\Gamma_{ij}^k \right] X_k \\
&= \sum \left[f_1v_1^iX_i(w^k) + f_1v_1^iw^j\Gamma_{ij}^k \right] X_k + \sum \left[f_2v_2^iX_i(w^k) + f_2v_2^iw^j\Gamma_{ij}^k \right] X_k \\
&= f_1 \sum \left[v_1^iX_i(w^k) + v_1^iw^j\Gamma_{ij}^k \right] X_k + f_2 \sum \left[v_2^iX_i(w^k) + v_2^iw^j\Gamma_{ij}^k \right] X_k \\
&= f_1\nabla_{V_1}W + f_2\nabla_{V_2}W.
\end{aligned}$$

Next, we show additivity in the second coordinate; let V, W_1, W_2 be smooth vector fields on the range of the coordinate system. Then

$$\begin{aligned}
\nabla_V(W_1 + W_2) &= \sum \left[v^iX_i(w_1^k + w_2^k) + v^i(w_1^j + w_2^j)\Gamma_{ij}^k \right] X_k \\
&= \sum \left[v^i \left(X_i(w_1^k) + X_i(w_2^k) \right) + v^i(w_1^j + w_2^j)\Gamma_{ij}^k \right] X_k \\
&= \sum \left[v^iX_i(w_1^k) + v^iw_1^j\Gamma_{ij}^k \right] X_k + \sum \left[v^iX_i(w_2^k) + v^iw_2^j\Gamma_{ij}^k \right] X_k \\
&= \nabla_V W_1 + \nabla_V W_2.
\end{aligned}$$

Finally, we show that the Leibniz rule is satisfied; let V, W be smooth vector fields on the range of the coordinate system and let g be a smooth function on the range of the coordinate system. Then

$$\begin{aligned}\nabla_V(gW) &= \sum \left[v^i X_i(gw^k) + v^i g w^j \Gamma_{ij}^k\right] X_k \\ &= \sum \left[v^i \left(X_i(g)w^k + gX_i(w^k)\right) + v^i g w^j \Gamma_{ij}^k\right] X_k \\ &= \sum v^i X_i(g) w^k X_k + \sum \left[v^i g X_i(w^k) + v^i g w^j \Gamma_{ij}^k\right] X_k \\ &= \sum V(g) w^k X_k + g \sum \left[v^i X_i(w^k) + v^i w^j \Gamma_{ij}^k\right] X_k \\ &= V(g)W + g\nabla_V W.\end{aligned}$$

Having shown linearity in the first coordinate, additivity in the second coordinate and satisfaction of the Leibniz rule, we conclude that ∇ is an affine connection on the range of the coordinate system. □

3

Show that an affine connection ∇ is compatible with a Riemannian metric on M if and only if, for any vector fields V and W along a smooth curve $c : I \to M$, we have

$$\frac{d}{dt}\langle V, W\rangle = \left\langle \frac{DV}{dt}, W\right\rangle + \left\langle V, \frac{DW}{dt}\right\rangle.$$

Proof. Suppose that, for any vector fields V and W along a smooth curve $c : I \to M$, we have

$$\frac{d}{dt}\langle V, W\rangle = \left\langle \frac{DV}{dt}, W\right\rangle + \left\langle V, \frac{DW}{dt}\right\rangle.$$

Let $c : I \to M$ be a smooth curve and let V and W be parallel vector fields along c. Then

$$\frac{d}{dt}\langle V, W\rangle = \left\langle \frac{DV}{dt}, W\right\rangle + \left\langle V, \frac{DW}{dt}\right\rangle = \langle 0, W\rangle + \langle V, 0\rangle = 0,$$

so we see that $\langle V, W\rangle$ is constant along c, meaning that ∇ is compatible with the Riemannian metric on M.

On the other hand, suppose ∇ is compatible with the Riemannian metric on M. Let $c : I \to M$ be a smooth curve and let V and W be vector fields along c. Let $U_1, \ldots, U_n$ be an orthonormal basis of $T_{c(0)}M$ and let $U_i(t)$ be the parallel transport of U_i along c. Then, since ∇ is compatible with the Riemannian metric on M,

$$\langle U_i(t), U_j(t)\rangle = \langle U_i, U_j\rangle = \delta_{ij},$$

so $U_1(t), \ldots, U_n(t)$ forms an orthonormal basis of $T_{c(t)}M$ for all $t \in I$. Hence, at each $c(t)$,

$$V = \sum_{i=1}^n f_i(t) U_i(t)$$

and

$$W = \sum_{j=1}^{n} g_j(t) U_j(t)$$

for smooth $f_1, \dots, f_n, g_1, \dots, g_n$. Now,

$$\begin{aligned}\langle V, W\rangle = \left\langle \sum_{i=1}^{n} f_i(t) U_i(t), \sum_{j=1}^{n} g_j(t) U_j(t) \right\rangle &= \sum_{i,j} f_i(t) g_j(t) \langle U_i(t), U_j(t)\rangle \\ &= \sum_{i,j} f_i(t) g_j(t) \delta_{ij} \\ &= \sum_{i=1}^{n} f_i(t) g_i(t),\end{aligned}$$

so

$$\frac{d}{dt}\langle V, W\rangle = \frac{d}{dt} \sum f_i(t) g_i(t) = \sum_{i=1}^{n} \left[f_i'(t) g_i(t) + f_i(t) g_i'(t) \right]. \tag{1}$$

On the other hand,

$$\frac{DV}{dt} = \frac{D}{dt}\left[\sum_{i=1}^{n} f_i(t) U_i(t)\right] = \sum_{i=1}^{n} \frac{D}{dt}\left[f_i(t) U_i(t)\right] = \sum_{i=1}^{n} \left[f_i'(t) U_i(t) + f_i(t) \frac{DU_i(t)}{dt} \right] = \sum_{i=1}^{n} f_i'(t) U_i(t)$$

and

$$\frac{DW}{dt} = \frac{D}{dt}\left[\sum_{j=1}^{n} g_j(t) U_j(t)\right] = \sum_{j=1}^{n} \frac{D}{dt}\left[g_j(t) U_j(t)\right] = \sum_{j=1}^{n} \left[g_j'(t) U_j(t) + g_j(t) \frac{DU_j(t)}{dt} \right] = \sum_{j=1}^{n} g_j'(t) U_j(t).$$

Hence

$$\begin{aligned}\left\langle \frac{DV}{dt}, W \right\rangle + \left\langle V, \frac{DW}{dt} \right\rangle &= \left\langle \sum_{i=1}^{n} f_i'(t) U_i(t), \sum_{j=1}^{n} g_j(t) U_j(t) \right\rangle + \left\langle \sum_{i=1}^{n} f_i(t) U_i(t), \sum_{j=1}^{n} g_j'(t) U_j(t) \right\rangle \\ &= \sum_{i=1}^{n} f_i'(t) g_i(t) + \sum_{i=1}^{n} f_i(t) g_i'(t) \qquad (2) \\ &= \sum_{i=1}^{n} \left[f_i'(t) g_i(t) + f_i(t) g_i'(t) \right].\end{aligned}$$

Equating (**??**) and (**??**), we see that

$$\frac{d}{dt}\langle V, W\rangle = \left\langle \frac{DV}{dt}, W \right\rangle + \left\langle V, \frac{DW}{dt} \right\rangle.$$

□

4

Conclude that an affine connection ∇ is compatible with a Riemannian metric on M if and only if

$$U\langle V,W\rangle = \langle \nabla_U V, W\rangle + \langle V, \nabla_U W\rangle .$$

Proof. Suppose

$$U\langle V,W\rangle = \langle \nabla_U V, W\rangle + \langle V, \nabla_U W\rangle.$$

Now, let V, W be vector fields on M, let $c : I \to M$ be a smooth curve and let $U = \frac{dc}{dt}$. Then

$$\begin{aligned}\frac{d}{dt}\langle V,W\rangle = U\langle V,W\rangle &= \langle \nabla_U V, W\rangle + \langle V, \nabla_U W\rangle \\ &= \langle \nabla_{dc/dt} V, W\rangle + \langle V, \nabla_{dc/dt} W\rangle \\ &= \left\langle \frac{DV}{dt}, W\right\rangle + \left\langle V, \frac{DW}{dt}\right\rangle .\end{aligned}$$

Therefore, by problem 3 above, ∇ is compatible with the Riemannian metric on M.

On the other hand, suppose ∇ is compatible with the Riemannian metric on M. Let U, V, W be vector fields on M, let $p \in M$ and let $c : I \to M$ be a smooth curve such that $c(0) = p$ and $c'(0) = U(p)$. Then $\nabla_U V(p) = \frac{DV}{dt}(p)$ and $\nabla_U W(p) = \frac{DW}{dt}(p)$. Hence, using our result from problem 3,

$$U\langle V,W\rangle_p = \frac{d}{dt}\langle V,W\rangle_p = \left\langle \frac{DV}{dt}, W\right\rangle_p + \left\langle V, \frac{DW}{dt}\right\rangle_p = \langle \nabla_U V, W\rangle_p + \langle V, \nabla_U W\rangle_p .$$

Since our choice of $p \in M$ was arbitrary, we see that

$$U\langle V,W\rangle = \langle \nabla_U V, W\rangle + \langle V, \nabla_U W\rangle$$

on all of M. □

5

Looking back over this proof, show that if U, V and W are three vector fields on M which do not necessarily commute with one another, then

$$2\langle \nabla_U V, W\rangle = U\langle V,W\rangle + V\langle W,U\rangle - W\langle U,V\rangle - \langle U,[V,W]\rangle + \langle V,[W,U]\rangle + \langle W,[U,V]\rangle .$$

Proof. Let U, V and W be vector fields on M. By problem 4 above,

$$\begin{aligned}U\langle V,W\rangle &= \langle \nabla_u V, W\rangle + \langle V, \nabla_U W\rangle \\ V\langle W,U\rangle &= \langle \nabla_V W, U\rangle + \langle W, \nabla_V U\rangle \\ W\langle U,V\rangle &= \langle \nabla_W U, V\rangle + \langle U, \nabla_W V\rangle.\end{aligned}$$

Hence,

$$\begin{aligned} U\langle V,W\rangle + V\langle W,U\rangle - W\langle U,V\rangle &= (\langle \nabla_u V, W\rangle + \langle V, \nabla_U W\rangle) + (\langle \nabla_V W, U\rangle + \langle W, \nabla_V U\rangle) \\ &\quad -(\langle \nabla_W U, V\rangle + \langle U, \nabla_W V\rangle) \\ &= \langle W, \nabla_U V + \nabla_V U\rangle + \langle U, [V,W]\rangle - \langle V, [W,U]\rangle \\ &= \langle W, 2\nabla_U V - [U,V]\rangle + \langle U, [V,W]\rangle - \langle V, [W,U]\rangle \\ &= 2\langle \nabla_U V, W\rangle - \langle W, [U,V]\rangle + \langle U, [V,W]\rangle - \langle V, [W,U],\rangle \end{aligned}$$

since $[U,V] = \nabla_U V - \nabla_V U$. Therefore, solving for $2\langle \nabla_U V, W\rangle$, we see that

$$2\left\langle \nabla_U V, W\right\rangle = U\left\langle V,W\right\rangle + V\left\langle W,U\right\rangle - W\left\langle U,V\right\rangle - \langle U,[V,W]\rangle + \langle V,[W,U]\rangle + \langle W,[U,V]\rangle .$$

□

6

Define a map $F : S^3 \times S^3 \to SO(4)$ by

$$F(x,y)(z) = xzy^{-1},$$

where x and y are unit quaternions, z is any quaternion, and quaternion multiplication is used on the right hand side. Show that F is a double covering.

Proof. In fact, we can show more than this. We can show that F is a Lie group homomorphism which is surjective with kernel $\{\pm(1,1)\}$, meaning that it induces a diffeomorphism $S^3 \times S^3/\{(x,y) \equiv (-x,-y)\}$. To that end, note that, if $(q,r),(p,s) \in S^3 \times S^3$, then

$$\begin{aligned} F((x,y)(z,w)) = F(xz,yw) &= [v \mapsto xzv(yw)^{-1}] \\ &= [v \mapsto xzvw^{-1}y^{-1}] \\ &= [v \mapsto xvy^{-1}] \circ [v \mapsto zvw^{-1}] \\ &= F(x,y) \circ F(z,w), \end{aligned}$$

so F is a group homomorphism.

Now, If $x, y \in S^3$ such that $x = a + bi + cj + dk$ and $y = e + fi + gj + hk$, then $y^{-1} = \bar{y}$, so:

$$\begin{aligned}
x(1)y^{-1} &= (a+bi+cj+dk)(e-fi-gj-hk) \\
&= (ae+bf+cg+dh)+i(be-af-ch+dg) \\
&\quad +j(ce-ag-df+bh)+k(de-ah-bg+cf) \\
x(i)y^{-1} &= (a+bi+cj+dk)i(e-fi-gj-hk) \\
&= (a+bi+cj+dk)(f+ei+hj-gk) \\
&= (af-be-ch+dg)+i(ae+bf-cg-dh) \\
&\quad +j(ah+cf+de+bg)+k(df-ag+bh-ce) \\
x(j)y^{-1} &= (a+bi+cj+dk)j(e-fi-gj-hk) \\
&= (a+bi+cj+dk)(g-hi+ej+fk) \\
&= (ag+bh-ce-df)+i(bf-ah+cf-de) \\
&\quad +j(ae+cg-dh-bf)+k(af+dg+be+ch) \\
x(k)y^{-1} &= (a+bi+cj+dk)k(e-fi-gj-hk) \\
&= (a+bi+cj+dk)(h+gi-fj+ek) \\
&= (ah-bg+cf-de)+i(ag+bh+ce+df) \\
&\quad +j(ch-af-be+dg)+k(ae+dh-bf-cg).
\end{aligned}$$

Hence, $F(x, y)$ is given by the matrix

$$\begin{pmatrix}
ae+bf+cg+dh & af-be-ch+dg & ag+bh-ce-df & ah-bg+cf-de \\
be-af-ch+dg & ae+bf-cg-dh & bf-ah+cf-de & ag+bh+ce+df \\
ce-ag-df+bh & ah+cf+de+bg & ae+cg-dh-bf & ch-af-be+dg \\
de-ah-bg+cf & df-ag+bh-ce & af+dg+be+ch & ae+dh-bf-cg
\end{pmatrix}.$$

We can extend this to a map from $S^3 \times S^3$ to $\mathbb{R}^{16}$. The coordinate functions of this map are clearly continuous (in fact, smooth); since restricting the range doesn't affect continuity, we see that F is continuous.

Now, if (x, y) as above (in terms of $a, b, \ldots$) and $(x, y) \in \ker F$, then, looking at the diagonal entries of $F(x, y)$ and recalling that $x\bar{x} = y\bar{y} = 1$, we see that

$$\begin{aligned}
ae+bf+cg+dh &= 1 \\
ae+bf-cg-dh &= 1 \\
ae+cg-dh-bf &= 1 \\
ae+dh-bf-cg &= 1 \\
a^2+b^2+c^2+d^2 &= 1 \\
e^2+f^2+g^2+h^2 &= 1.
\end{aligned}$$

Thus, it must be the case that $ae = 1$ and $b = c = d = f = g = h = 0$, so we see that

$$(x, y) = \pm(1, 1);$$

that is, $\ker F = \{\pm(1, 1)\}$. Hence,

$$SO(4) \approx (S^3 \times S^3)/\ker F = (S^3 \times S^3)/\{(x, y) \sim -(x, y)\}.$$

Clearly, this implies that F is a double covering. $\square$

7

Show that the orbits of the vector fields U, V and W are great circles on S^3, and that in each case the collection of orbits forms a Hopf fibration of S^3 by parallel great circles. Draw such a fibration.

Proof. First, suppose that $a + bi \in S^3$; then $U(a + bi) = (a + bi)i = -b + ai$. Hence, for any $x \in S^3$ in the x_0x_1-plane, $U(x)$ is also in the x_0x_1-plane. Specifically, the orbit of U through 1 lies in the x_0x_1-plane. Since the intersection of the x_0x_1-plane with S^3 is a great circle, we see that the orbit of U through 1 is a great circle. Similarly, $V(a + bj) = -b + aj$ and $W(a + bk) = -b + ak$, so the orbits of V and W through 1 are also great circles.

Now, for any $x \in S^3$, $L_x(1) = x \cdot 1 = x$. Since U, V and W are left-invariant, $(L_x)_*$ maps the orbits of U, V and W through 1 to the orbits of U, V and W, respectively, through x. Since L_x is an isometry (this is an easy computation, which I've done, but which is a huge pain to type up, so I don't reproduce the calculation here) and isometries take geodesics to geodesics, we see that the orbits of U, V and W through x are also great circles. Since our choice of $x \in S^3$ was arbitrary, we see that all the orbits of U, V and W are great circles on S^3.

Now, suppose $x_1i + x_2j + x_3k \in S^3$. Then

$$U(x_1i + x_2j + x_3k) = (x_1i + x_2j + x_3k)i = -x_1 + x_3j - x_2k,$$

so the orbit of U passing through $x_1i + x_2j + x_3k$ moves out of the purely imaginary quaternions, meaning the orbit does not lie in the purely imaginary quaternions. A similar argument shows that none of the orbits of V or W lie entirely in the purely imaginary quaternions. On the other hand, the purely imaginary quaternions form a surface on S^3, while the orbits of U, V and W are great circles on S^3, so each orbit intersects the purely imaginary quaternions in a single point. Since

$$\{x_0 + x_1i + x_2j + x_3k \in S^3 : x_0 = 0\} = S^2,$$

we can define the map $f_U : S^3 \to S^2$ where $f_U(x)$ is the unique purely imaginary quaternion lying on the orbit of U passing through x. A similar definition gives maps $f_V : S^3 \to S^2$ and $f_W : S^3 \to S^2$. If $p \in S^2$, we can represent p by a purely imaginary unit quaternion $x_1i + x_2j + x_3k$. Now, $f_Y^{-1}(x_1i + x_2j + x_3k)$ is the orbit of Y on S^3 passing through $x_1i + x_2j + x_3k$ for $Y = U, V, W$. Since these orbits are great circles, we see that the fibers of $f_Y : S^3 \to S^2$ are S^1 for $Y = U, V, W$. In other words, f_U, f_V, f_W are S^1 fibrations of S^2; that is, Hopf fibrations.

Here's a crude picture of a Hopf fibration:

□

8

(a): Justify the use of "symmetry" above.

Proof. We've already seen that $\nabla_U V = W$. Now,

$$\begin{aligned}
\nabla_V W &= \nabla_V \left(-x_3 \frac{\partial}{\partial x_0} + x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} + x_0 \frac{\partial}{\partial x_3} \right) \\
&= -V(x_3) \frac{\partial}{\partial x_0} + V(x_2) \frac{\partial}{\partial x_1} - V(x_1) \frac{\partial}{\partial x_2} + V(x_0) \frac{\partial}{\partial x_3} \\
&= -x_1 \frac{\partial}{\partial x_0} + x_0 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3} \\
&= U
\end{aligned}$$

and

$$\begin{aligned}
\nabla_W U &= \nabla_W \left(-x_1 \frac{\partial}{\partial x_0} + x_0 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3} \right) \\
&= -W(x_1) \frac{\partial}{\partial x_0} + W(x_0) \frac{\partial}{\partial x_1} + W(x_3) \frac{\partial}{\partial x_2} - W(x_2) \frac{\partial}{\partial x_3} \\
&= -x_2 \frac{\partial}{\partial x_0} - x_3 \frac{\partial}{\partial x_1} + x_0 \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3} \\
&= V.
\end{aligned}$$

□

(b): Show that

$$\nabla_V U = -W, \quad \nabla_W V = -U, \quad \nabla_U W = -V.$$

Proof.

$$\begin{aligned}\nabla_V U &= \nabla_V\left(-x_1\frac{\partial}{\partial x_0}+x_0\frac{\partial}{\partial x_1}+x_3\frac{\partial}{\partial x_2}-x_2\frac{\partial}{\partial x_3}\right)\\ &= -V(x_1)\frac{\partial}{\partial x_0}+V(x_0)\frac{\partial}{\partial x_1}+V(x_3)\frac{\partial}{\partial x_2}-V(x_2)\frac{\partial}{\partial x_3}\\ &= x_3\frac{\partial}{\partial x_0}-x_2\frac{\partial}{\partial x_1}+x_1\frac{\partial}{\partial x_2}-x_0\frac{\partial}{\partial x_3}\\ &= -W,\end{aligned}$$

$$\begin{aligned}\nabla_W V &= \nabla_W\left(-x_2\frac{\partial}{\partial x_0}-x_3\frac{\partial}{\partial x_1}+x_0\frac{\partial}{\partial x_2}+x_1\frac{\partial}{\partial x_3}\right)\\ &= -W(x_2)\frac{\partial}{\partial x_0}-W(x_3)\frac{\partial}{\partial x_1}+W(x_0)\frac{\partial}{\partial x_2}+W(x_1)\frac{\partial}{\partial x_3}\\ &= x_1\frac{\partial}{\partial x_0}-x_0\frac{\partial}{\partial x_1}-x_3\frac{\partial}{\partial x_2}+x_2\frac{\partial}{\partial x_3}\\ &= -U\end{aligned}$$

and

$$\begin{aligned}\nabla_U W &= \nabla_U\left(-x_3\frac{\partial}{\partial x_0}+x_2\frac{\partial}{\partial x_1}-x_1\frac{\partial}{\partial x_2}+x_0\frac{\partial}{\partial x_3}\right)\\ &= -U(x_3)\frac{\partial}{\partial x_0}+U(x_2)\frac{\partial}{\partial x_1}-U(x_1)\frac{\partial}{\partial x_2}+U(x_0)\frac{\partial}{\partial x_3}\\ &= x_2\frac{\partial}{\partial x_0}+x_3\frac{\partial}{\partial x_1}-x_0\frac{\partial}{\partial x_2}-x_1\frac{\partial}{\partial x_3}\\ &= -V.\end{aligned}$$

□

2

Let X and Y be differentiable vector fields on a Riemannian manifold M. Let $p \in M$ and let $c : I \to M$ be an integral curve of X through p; i.e. $c(t_0) = p$ and $\frac{dc}{dt} = X(c(t))$. Prove that the Riemannian connection of M is

$$(\nabla_X Y)(p) = \frac{d}{dt}\left(P^{-1}_{c,t_0,t}(Y(c(T)))\right)\bigg|_{t=t_0},$$

where $P_{c,t_0,t} : T_{c(t_0)}M \to T_{c(t)}M$ is the parallel transport along c, from t_0 to t (this shows how the connection can be reobtained from the concept of parallelism).

Proof. Let $X_1, \dots, X_n$ be an orthonormal basis for T_pM; then we can write $Y(p) = y^i X_i$. Let $P_i(t) = P_{c,t_0,t}(X_i)$. Then, for all $t \in I$,

$$Y(c(t)) = y^i P_i(t).$$

Then, for all $t \in I$,

$$P_{c,t_0,t}^{-1}(Y(c(t))) = P_{c,t_0,t}^{-1}(y^i P_i(t)) = y^i X_i.$$

Hence,

$$\frac{d}{dt}\left[P_{c,t_0,t}^{-1}(Y(c(T)))\right]\bigg|_{t=t_0} = \frac{d}{dt}\left(y^i X_i\right)\bigg|_{t=t_0} = \frac{dy^i}{dt}\bigg|_{t=t_0} X_i. \tag{3}$$

On the other hand,

$$\nabla_X Y = \frac{DY}{dt} = \left[\frac{dy^i}{dt} + \Gamma_{ij}^k y^j \frac{dx_i}{dt}\right] P_k(t),$$

so

$$(\nabla_X Y)(p) = \frac{dy^k}{dt}\bigg|_{t=t_0} P_k(t_0) = \frac{dy^k}{dt}\bigg|_{t=t_0} X_k.$$

This is equal to the right hand side of (**??**), so we conclude that

$$(\nabla_X Y)(p) = \frac{d}{dt}\left(P_{c,t_0,t}^{-1}(Y(c(T)))\right)\bigg|_{t=t_0}.$$

□

6

Let M be a Riemannian manifold and let p be a point of M. Consider a constant curve $f : I \to M$ given by $f(t) = p$, for all $t \in I$. Let V be a vector field along f (that is, V is a differentiable mapping of I into T_pM). Show that $\frac{DV}{dt} = \frac{dV}{dt}$, that is to say, the covariant derivative coincides with the usual derivative of $V : I \to T_pM$.

Proof. Recall that, in local coordinates $(x_1, \dots, x_n)$,

$$\frac{DV}{dt} = \left[\frac{dv^k}{dt} + \Gamma_{ij}^k v^j \frac{dx_i}{dt}\right] \frac{\partial}{\partial x_k},$$

using the Einstein summation convention, where $V = \sum v^k \frac{\partial}{\partial x_k}$. However, since $f(t) = p$ for all $t \in I$, the x_i are constant along all of f; that is, they are the chosen coordinate vectors in T_pM. Hence, $\frac{dx_i}{dt} = 0$ for all i. Therefore,

$$\frac{DV}{dt} = \frac{dv^k}{dt}\frac{\partial}{\partial x_k} = \frac{dV}{dt},$$

by definition. □

8

Consider the upper half-plane

$$\mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2; y > 0\}$$

with the metric given by $g_{11} = g_{22} = \frac{1}{y^2}$, $g_{12} = 0$ (metric of Lobatchevski's non-euclidean geometry).

(a): Show that the Christoffel symbols of the Riemannian connection are: $\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^1_{22} = 0$, $\Gamma^2_{11} = \frac{1}{y}$, $\Gamma^1_{12} = \Gamma^2_{22} = -\frac{1}{y}$.

Proof. Recall that

$$\Gamma^m_{ij} = \frac{1}{2}\sum_k \left[\frac{\partial}{\partial x_i} g_{jk} + \frac{\partial}{\partial x_j} g_{ki} - \frac{\partial}{\partial x_k} g_{ij}\right] g^{km},$$

where (g^{km}) is the inverse of the matrix (g_{km}). Now,

$$(g_{km}) = \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix},$$

so

$$(g^{km}) = y^4 \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix} = \begin{pmatrix} y^2 & 0 \\ 0 & y^2 \end{pmatrix}.$$

Hence,

$$\Gamma^1_{11} = \frac{1}{2}\left[(0+0-0)\,y^2 + \left(0+0-\frac{-2}{y^3}\right)0\right] = 0$$
$$\Gamma^2_{11} = \frac{1}{2}\left[(0+0-0)\,0 + \left(0+0-\frac{-2}{y^3}\right)y^2\right] = \frac{1}{y}$$
$$\Gamma^1_{12} = \frac{1}{2}\left[\left(0+\frac{-2}{y^3}-0\right)y^2 + (0+0-0)\,0\right] = -\frac{1}{y}$$
$$\Gamma^2_{12} = \frac{1}{2}\left[\left(0+\frac{-2}{y^3}-0\right)0 + (0+0-0)\,y^2\right] = 0$$
$$\Gamma^1_{22} = \frac{1}{2}\left[(0+0-0)\,y^2 + \left(\frac{-2}{y^3}+\frac{-2}{y^3}-\frac{-2}{y^2}\right)0\right] = 0$$
$$\Gamma^2_{22} = \frac{1}{2}\left[(0+0-0)\,0 + \left(\frac{-2}{y^3}+\frac{-2}{y^3}-\frac{-2}{y^2}\right)0\right] = -\frac{1}{y}$$

□

(b): Let $v_0 = (0,1)$ be a tangent vector at point $(1,0)$ of $\mathbb{R}^{2+}$ (v_0 is a unit vector on the y-axis with origin at $(0,1)$). Let $v(t)$ be the parallel transport v_0 along the curve $x = t, y = 1$. Show that $v(t)$ makes an angle t with the direction of the y-axis, measured in the clockwise sense.

Proof. Since $v(t) = (v^1(t), v^2(t))$ is a parallel vector field, it satisfies the equations

$$0 = \frac{dv^k}{dt} + \sum_{i,j} \Gamma^k_{ij} v^j \frac{dx_i}{dt}$$

for $k = 1, 2$. Using the values for Γ^k_{ij} computed in part (a) above, these equations reduce to

$$\begin{cases} \frac{da}{dt} + \Gamma^1_{12} b & = 0 \\ \frac{db}{dt} + \Gamma^2_{11} a & = 0 \end{cases}$$

Hence, solutions of this system of linear first-order ODEs are of the form $a = \cos\theta(t)$, $b = \sin\theta(t)$. Along the given curve, $y = 1$, so $\Gamma^1_{12} = -1$ and $\Gamma^2_{11} = 1$, so

$$\begin{aligned} 0 &= \frac{da}{dt} - b = -\sin\theta(t)\frac{d\theta}{dt} - \sin\theta(t) \\ 0 &= \frac{db}{dt} + a = \cos\theta(t)\frac{d\theta}{dt} + \cos\theta(t) \end{aligned}$$

meaning that $\frac{d\theta}{dt} = -1$. Hence, $\theta(t) = C - t$ for some constant C. Now, since $v(0) = v_0 = (0, 1)$, so $0 = \cos(C - 0) = \cos C$, so $C = \frac{\pi}{2}$. Therefore, $t = \frac{\pi}{2} - \theta(t)$; since θ measures the angle $v(t)$ makes with the direction of the x-axis measured counter-clockwise, this means that t measures the angle $v(t)$ makes with the direction of the y-axis measured in the clockwise sense. □

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