DIFFERENTIAL GEOMETRY HW 9

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1

Let $x_1, \ldots, x_n$ be local coordinates on the Riemannian manifold M, let $g_{ij} = \langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle_M$ be the coefficients of the first fundamental form, and Γ_{ij}^k the corresponding Christoffel symbols.

Introduce corresponding local coordinates $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ on TM as done on page 62 of do Carmo, and then calculate the coefficients of the first fundamental form on TM. The answer involves both g_{ij} and Γ_{ij}^k.

Answer: Let $(p, v) \in TM$. Define α_i, β_j to be curves in TM with

$$\alpha_i : t \mapsto (p_i(t), v_i(t)), \quad \beta_j : s \mapsto (q_j(s), w_j(s)),$$

with $p_i(0) = q_j(0) = p$, $v_i(0) = w_j(0) = v$ and $\frac{\partial}{\partial x_i} = \alpha_i'(0)$, $\frac{\partial}{\partial y_j} = \beta_j'(0)$ for all $i, j \in \{1, \ldots, n\}$. Note that if we write

$$\sum_i a_i \frac{\partial}{\partial x_i} + \sum_j b_j \frac{\partial}{\partial y_j}$$

as $(a_1, \ldots, a_n, b_1, \ldots, b_n)$, then

$$\alpha_i'(0) = (p_i^{1\prime}(0), \ldots, p_i^{n\prime}(0), v_i^{1\prime}(0), \ldots, v_i^{n\prime}(0)) = \frac{\partial}{\partial x_i},$$

so $\frac{dp_i^k}{dt}(0) = \delta_{ik}$ and $\frac{dv_i^k}{dt}(0) = 0$. Hence,

$$\begin{aligned}
\left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle_{(p,v)} &= \left\langle d\pi\left(\frac{\partial}{\partial x_i}\right), d\pi\left(\frac{\partial}{\partial x_j}\right) \right\rangle_p + \left\langle \frac{Dv_i}{dt}(0), \frac{Dv_j}{dt}(0) \right\rangle_p \\
&= \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle_p + \left\langle \left[\frac{dv_i^k}{dt} + \Gamma_{l,m}^k v_i^m \frac{dp_i^l}{dt}\right] \frac{\partial}{\partial x_k}, \left[\frac{dv_j^r}{dt} + \Gamma_{su}^r v_j^u \frac{dp_j^s}{dt}\right] \frac{\partial}{\partial x_r} \right\rangle_p \\
&= g_{ij} + \left\langle \sum_{k,m} \Gamma_{im}^k v_i^m \frac{\partial}{\partial x_k}, \sum_{r,u} \Gamma_{ju}^r v_j^u \frac{\partial}{\partial x_r} \right\rangle_p \\
&= g_{ij} + \sum_{k,m,r,u} \Gamma_{im}^k \Gamma_{ju}^r v_i^m v_j^u g_{kr}.
\end{aligned}$$

Now, since $d\pi = [I \ 0]$, $d\pi\left(\frac{\partial}{\partial y_j}\right) = 0$. Also, note that, in coordinates,

$$\frac{\partial}{\partial y_j} = \beta_j'(0) = (q_j^{1\prime}(0), \ldots, q_j^{n\prime}(0), w_j^{1\prime}(0), \ldots, w_j^{n\prime}(0)),$$

so $\frac{dq_j^k}{ds}(0) = 0$ and $\frac{dw_j^k}{ds}(0) = \delta_{jk}$. Therefore,

$$\begin{aligned}
\left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_j} \right\rangle_{(p,v)} &= \left\langle d\pi\left(\frac{\partial}{\partial x_i}\right), d\pi\left(\frac{\partial}{\partial y_j}\right) \right\rangle_p + \left\langle \frac{Dv_i}{dt}, \frac{Dw_j}{ds} \right\rangle_p \\
&= 0 + \left\langle \sum_{k,m} \Gamma_{im}^k v_i^m \frac{\partial}{\partial x_k}, \sum_r \left[\frac{dw_j^r}{ds} + \sum_{s,u} \Gamma_{su}^r w_j^u \frac{dq_j^s}{ds} \right] \frac{\partial}{\partial x_r} \right\rangle_p \\
&= \left\langle \sum_{k,m} \Gamma_{im}^k v_i^m \frac{\partial}{\partial x_k}, \sum_r \frac{dw_j^r}{ds} \frac{\partial}{\partial x_r} \right\rangle_p \\
&= \left\langle \sum_{k,m} \Gamma_{im}^k v_i^m \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_j} \right\rangle_p \\
&= \sum_{k,m} \Gamma_{im}^k v_i^m g_{kj}.
\end{aligned}$$

Finally,

$$\begin{aligned}
\left\langle \frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j} \right\rangle_{(p,v)} &= \left\langle d\pi\left(\frac{\partial}{\partial y_i}\right), d\pi\left(\frac{\partial}{\partial y_j}\right) \right\rangle_p + \left\langle \frac{Dw_i}{dt}, \frac{Dw_j}{dt} \right\rangle_p \\
&= 0 + \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle \\
&= g_{ij}.
\end{aligned}$$

Therefore, since the inner product is symmetric, we can represent the first fundamental form by the $2n \times 2n$ matrix with $n \times n$ blocks as below:

$$\begin{pmatrix} \left(g_{ij} + \sum_{k,m,r,u} \Gamma_{im}^k \Gamma_{ju}^r v_i^m v_j^u g_{kr}\right) & \left(\sum_{k,m} \Gamma_{im}^k v_i^m g_{kj}\right) \\ \left(\sum_{k,m} \Gamma_{jm}^k v_j^m g_{ki}\right) & (g_{ij}) \end{pmatrix}$$

♣

2

Let $M = S^2$ be the round 2-sphere of radius 1. Let $US^2 \subset TS^2$ be the unit tangent bundle, and give US^2 the Riemannian metric induced by its inclusion in TS^2. Whe know that US^2 is diffeomorphic to $\mathbb{RP}^3$. Is it metrically a round $\mathbb{RP}^3$?

Proof. Done by Scott in class. □

3

Let $M = S^3$ be the round 3-sphere of radius 1, and US^3 its unit tangent bundle. We know that US^3 is diffeomorphic to $S^3 \times S^3$. Show that the Riemannian metric on US^3 is *not* the product metric. Find *all* the geodesics in US^3, not just the horizontal ones.

Proof. Note that we can think of S^2 as contained in $\mathbb{R}^3$, which has basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ and we can think of S^3 as contained in $\mathbb{R}^4$, which has basis $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$. Now, let $\alpha : \mathbb{R} \to US^3$ be given by $\alpha(t) = (p(t), v(t))$, where

$$p : t \mapsto \cos t + \mathbf{j} \sin t, \quad v : t \mapsto p(t)\mathbf{i}.$$

Let $U = \alpha'(0)$. Then, if $\pi : US^3 \to S^3$ is the projection, then $d\pi(U) = p'(t)$. Note that $\|p'(t)\| = 1$. On the other hand,

$$\frac{dv}{dt}(0) = \frac{d}{dt}(p(0)\mathbf{i}) = \left.\frac{d}{dt}\right|_{t=0} (\mathbf{i} \cos t - \mathbf{k} \sin t) = -\mathbf{k} \in T_1 S^3$$

since this vector is purely imaginary. Hence, it is its own projection onto the tangent space; that is,

$$\frac{Dv}{dt}(0) = \frac{dv}{dt}(0) = -\mathbf{k}.$$

Therefore, in the metric on US^3,

$$\begin{aligned} \langle U, U\rangle_{(1,\mathbf{i})} &= \langle d\pi(U), d\pi(U)\rangle_1 + \left\langle \frac{Dv}{dt}(0), \frac{Dv}{dt}(0) \right\rangle_1 \\ &= \|p'(0)\|^2 + \|-\mathbf{k}\|^2 \\ &= 2. \end{aligned}$$

On the other hand, if we think of α as a map into $S^3 \times S^2$, then α traces out the great circle in the $1\mathbf{j}$-plane on S^3. Along this curve, we can give S^2 coordinates induced by the coordinates $x\mathbf{i}, x\mathbf{j}, x\mathbf{k}$. Then $v(t) = p(t)\mathbf{i}$ is constant in S^2. Hence, if $\pi_1 : S^3 \times S^2 \to S^3$ is the first projection and $\pi_2 : S^3 \times S^2 \to S^2$ is the second projection, then $d\pi_1(U) = p'(t)$ and $d\pi_2(U) = v'(t) = 0$. $\|p'(t)\|$ is still 1, so in the product metric on $S^3 \times S^2$,

$$\begin{aligned} \langle U, U\rangle_{(1,i)} &= \langle d\pi_1(U), d\pi_1(U)\rangle_1 + \langle d\pi_2(U), d\pi_2(U)\rangle_{\mathbf{i}} \\ &= \langle p'(0), p'(0)\rangle_1 + \langle v'(0), v'(0)\rangle_{\mathbf{i}} \\ &= 1. \end{aligned}$$

Thus, we see that, although US^3 and $S^3 \times S^2$ are diffeomorphic, the Riemannian metric on US^3 is not the product metric. □

4

Let G be the 2-dimensional Lie group of all orientation-preserving affine maps $x \mapsto mx + b$, $m > 0$, of the real line to itself. Show that G does *not* admit a bi-invariant Riemannian metric. Give G a left-invariant Riemannian metric, and contrast the one-parameter subgroups of G with the geodesics through the identity.

Proof. Note that we can denote points in G by (b, m) where $b, m \in \mathbb{R}$, $m > 0$, so this looks like the upper half-plane. Since the group operation is function composition, if $f(x) = mx + b$ and $g(x) = nx + d$, then

$$(f \circ g)(x) = m(nx + d) + b = mnx + md + b,$$

which corresponds to the point $(md+b, mn)$. Hence, multiplication on G is defined by

$$(b,m)(d,n) = (md+b, mn).$$

Note that the identity element of G is $(0,1)$. Therefore, if we consider $T_{(d,n)}G$ as a copy of $\mathbb{R}^2$,

$$dL_{(b,m)} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$$

and

$$dR_{(b,m)} = \begin{pmatrix} 1 & b \\ 0 & m \end{pmatrix}.$$

Now, suppose $\langle , \rangle_{(b,m)}$ is a bi-invariant metric on G. Let $u, v \in T_{(0,1)}G$. Note that

$$(m,b)\left(\frac{-b}{m}, \frac{1}{m}\right) = (0,1).$$

Then

$$\langle u,v\rangle_{(0,1)} = \langle dL_{(b,m)}(u), dL_{(b,m)}(v)\rangle_{(b,m)} = \langle dR_{\frac{-b}{m},\frac{1}{m}} dL_{(b,m)}(u), dR_{\frac{-b}{m},\frac{1}{m}} dL_{(b,m)}(v)\rangle_{(0,1)}$$

However, $dL_{(b,m)}(u) = mu$ and $dl_{(b,m)}(v) = mv$, so this implies that

$$\langle u,v\rangle_{(0,1)} = \langle dR_{(\frac{-b}{m},\frac{1}{m})}(mu), dR_{(\frac{-b}{m},\frac{1}{m})}(mv)\rangle_{(0,1)}.$$

If we write $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, then we have that

$$dR_{\frac{-b}{m},\frac{1}{m}}(mu) = \begin{pmatrix} mu_1 + bu_2 \\ u_2 \end{pmatrix}$$

and similarly for v. Hence, if we let $u = v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then

$$\langle u,v\rangle_{(0,1)} = \langle mu, mv\rangle_{(0,1)} = m^2\langle u,v\rangle_{(0,1)},$$

which is impossibly since $m > 0$ and $u \neq 0$. Thus, we conclude that there is no bi-invariant metric on G.

On the other hand, we can define a left-invariant metric on G by letting $\langle , \rangle_{(0,1)}$ be the standard inner product on $\mathbb{R}^2$ and defining

$$\langle u,v\rangle_{(b,m)} = \langle dL_{(\frac{-b}{m},\frac{1}{m})}(u), dL_{(\frac{-b}{m},\frac{1}{m})}(v)\rangle_{(0,1)} = \left\langle \frac{1}{m}u, \frac{1}{m}v \right\rangle_{(0,1)} = \frac{1}{m^2}\langle u,v\rangle_{(0,1)}.$$

Note that $g_{11}(b,m) = g_{22}(b,m) = \frac{1}{m^2}$, $g_{12}(b,m) = 0$, so this is our old friend, the metric we defined on the hyperbolic plane. In this metric, we know that the geodesics through $(0,1)$ are simply the circles centered on the x-axis (b-axis) passing through $(0,1)$ along with the y-axis (m-axis).

Now, suppose $H \subset G$ is a one-parameter subgroup with $h : \mathbb{R} \to H$ a smooth map. Then, by definition,

$$h(s+t) = h(s)h(t)$$

with $h(0) = (0,1)$; suppose $h'(0) = (a,b) \in T_{(0,1)}G$. In coordinates,

$$(h_1(s+t), h_2(s+t)) = h(s+t) = h(s)h(t) = (h_2(s)h_1(t)+h_1(s), h_2(s)h_2(t)).$$

Hence,

$$h_2(s+t) = h_2(s)h_2(t),$$

so $h_2(t) = Cr^{pt}$ for some $C, r, p \in \mathbb{R}$. Now,

$$\begin{aligned} h_1'(t) = \lim_{\epsilon\to 0} \frac{h_1(t+\epsilon) - h_1(t)}{\epsilon} &= \lim_{\epsilon\to 0} \frac{h_2(t)h_1(\epsilon) + h_1(t) - h_1(t)}{\epsilon} \\ &= \lim_{\epsilon\to 0} \frac{h_2(t)a\epsilon}{\epsilon} \\ &= ah_2(t) \\ &= aCr^{pt}, \end{aligned}$$

since, for very small ϵ, $h_1(\epsilon) \approx \epsilon h_1'(0) = \epsilon a$. Therefore,

$$h(t) = \left(\frac{aC}{p} r^{pt} + d, Cr^{pt}\right)$$

where $d \in \mathbb{R}$, so H is just a straight line passing through $(0,1)$ (though with a weird parametrization, admittedly). Thus, we see that, except for the vertical line, the one-parameter subgroups of G and the geodesics through $(0,1)$ are distinct. □

5

Let $x_1, \ldots, x_n$ be local coordinates on the Riemannian manifold M, let $g_{ij} = \langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle$ be the components of the first fundamental form, and Γ_{ij}^k the corresponding Christoffel symbols.

(1) using the Einstein summation convention, show that the gradient of a smooth real-valued function f is given by

$$\text{grad}(f) = \nabla f = g^{ij} \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_i}.$$

Proof. Let $p \in M$. By definition, $\langle \nabla f, v \rangle_p = df_p(v)$ for any $v \in T_pM$. So it suffices to show that the expression given above satisfies this equation. Furthermore, since the inner product is bilinear, it suffices to show that the expression is satisfied for $v = \frac{\partial}{\partial x_k}$, where $x_1, \ldots, x_n$

is a coordinate system in a nieghborhood of p. To that end,

$$\begin{aligned}\left\langle g^{ij}\frac{\partial f}{\partial x_j}\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_k}\right\rangle_p &= g^{ij}\frac{\partial f}{\partial x_j}\left\langle\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_k}\right\rangle_p \\ &= g^{ij}\frac{\partial f}{\partial x_j}g_{ik} \\ &= g^{ij}g_{ki}\frac{\partial f}{\partial x_j} \\ &= \frac{\partial f}{\partial x_k}\end{aligned}$$

since

$$\sum_{i,j} g^{ij}g_{ki} = \delta_{jk}.$$

On the other hand,

$$df_p\left(\frac{\partial}{\partial x_k}\right) = \frac{\partial f}{\partial x_k}$$

so we see that

$$\left\langle g^{ij}\frac{\partial f}{\partial x_j}\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_k}\right\rangle_p = df_p\left(\frac{\partial}{\partial x_k}\right),$$

as desired. □

(2) If the smooth vector field V on M is given locally by $V = v^j\frac{\partial}{\partial x_j}$, show that its divergence is given by

$$\operatorname{div}(V) = \nabla\cdot V = \frac{\partial v^i}{\partial x_i} + \Gamma^i_{ij}v^j.$$

Proof. Recall that, by definition, $\operatorname{div}(V)$ is the trace of the map $Y(p)\mapsto \nabla_Y V(p)$. Let $x_1,\ldots,x_n$ be a system of local coordinates in a nieghborhood of $p\in M$. Suppose on this neighborhood that $V = \sum_j v^j\frac{\partial}{\partial x_j}$. If $Y = \frac{\partial}{\partial x_i}$ for some $i\in\{1,\ldots,n\}$, then the above maps Y to

$$\begin{aligned}\nabla_Y v^j\frac{\partial}{\partial x_j} &= v^j\nabla_Y\frac{\partial}{\partial x_j} + Y(v^j)\frac{\partial}{\partial x_j} \\ &= v^j\Gamma^k_{ij}\frac{\partial}{\partial x_k} + \frac{\partial v^j}{\partial x_i}\frac{\partial}{\partial x_j}\end{aligned}$$

Hence, the trace of the first term is $v^j\Gamma^i_{ij}$ and the trace of the second term is $\frac{\partial v^i}{\partial x_i}$; since trace is additive, this implies that

$$\nabla\cdot V = v^j\Gamma^i_{ij} + \frac{\partial v^i}{\partial x_i}.$$

□

6

Let x_1, x_2, x_3 be local coordinates on the 3-dimensional Riemannian manifold M, and let g_{ij} be the components of the first fundamental form. Let $g = \det(g_{ij})$.

(1) Given two vector fields $V = v^i \frac{\partial}{\partial x_i}$ and $W = w^j \frac{\partial}{\partial x_j}$, show that their cross product is given by

$$V \times W = \sigma^{rsk} g^{-1/2} g_{ri} g_{sj} v^i w^j \frac{\partial}{\partial x_k},$$

where the symbol σ^{rsk} is defined to be $+1(-1)$ if rsk is an even (odd) permutation of 123, and 0 otherwise.

Proof. By definition, if $v, w \in \mathbb{R}^3$, then $v \times w$ is the unique vector u such that $\langle u, z\rangle = \det([v\ w\ z])$. We would like to define the cross product in the same way on an arbitrary 3-manifold, but, in order to pick out the coordinates of a vector field, we have to be a little bit more careful. Note that, if $V = v^i \frac{\partial}{\partial x_i}$,

$$\begin{aligned}\left\langle V, g^{km}\frac{\partial}{\partial x_m}\right\rangle &= \left\langle v^i \frac{\partial}{\partial x_i}, g^{km}\frac{\partial}{\partial x_m}\right\rangle \\ &= v^i g^{km}\left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_m}\right\rangle \\ &= v^i g^{km} g_{im} \\ &= v^k,\end{aligned}$$

since $g^{km} g_{im} = \delta_{ki}$. Therefore, the matrix $[V_1\ V_2\ V_3]$ is given by

$$\left(\left\langle V_i, g^{jm}\frac{\partial}{\partial x_m}\right\rangle\right)^t.$$

Thus, we define the cross product by

$$\langle V_1 \times V_2, V_3\rangle = \sqrt{g}\det\left(\left\langle V_i, g^{jm}\frac{\partial}{\partial x_m}\right\rangle\right)^t.$$

Note that in $\mathbb{R}^3$, $\sqrt{g} = 1$, so this reduces to the familiar definition in Euclidean space; on a general manifold, we need this factor to make the cross product well-defined. To see why it really is well-defined, suppose y_1, y_2, y_3 are another set of coordinates. Then

$$\frac{\partial}{\partial y_i} = f^{ij}\frac{\partial}{\partial x_j}$$

for some smooth f^{ij}. If $\widetilde{g}_{ij} = \left\langle \frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}\right\rangle$, then

$$(\widetilde{g}_{ij}) = (f_{ik})(g_{kl})(f_{jl})^t,$$

so

$$\widetilde{g} := (\widetilde{g}_{ij}) = (\det(f_{ij}))^2 g.$$

Hence, if $F = \det(f_{ij})$, $\sqrt{\widetilde{g}} = F\sqrt{g}$. Therefore,

$$\begin{aligned}\sqrt{g}\det\left(\left\langle V_i, \widetilde{g}^{jm}\frac{\partial}{\partial x_m}\right\rangle\right)^t &= \sqrt{g}\det\left((f_{ij})^t\left\langle V_i, \widetilde{g}^{jm}\frac{\partial}{\partial y_m}\right\rangle\right)^t \\ &= \sqrt{g}F\det\left(\left\langle V_i, \widetilde{g}^{jm}\frac{\partial}{\partial y_m}\right\rangle\right)^t \\ &= \sqrt{\widetilde{g}}\left(\left\langle V_i, \widetilde{g}^{jm}\frac{\partial}{\partial y_m}\right\rangle\right)^t.\end{aligned}$$

Thus, we see that our definition of the cross product is well-defined regardless of the chosen coordinates.

Now, we can compute explicitly what $V\times W$ is. As defined above,

$$\begin{aligned}\left\langle V\times W, g^{kl}\frac{\partial}{\partial x_l}\right\rangle &= \sqrt{g}\sum_{\sigma\in S_3}\operatorname{sgn}(\sigma)\left\langle V, g^{\sigma(1)r}\frac{\partial}{\partial x_r}\right\rangle\left\langle W, g^{\sigma(2)s}\frac{\partial}{\partial x_s}\right\rangle\left\langle g^{kl}\frac{\partial}{\partial x_l}, g^{\sigma(3)m}\frac{\partial}{\partial x_m}\right\rangle \\ &= \sqrt{g}\sum_{\sigma}\operatorname{sgn}(\sigma)\left\langle V, \frac{\partial}{\partial x_r}\right\rangle\left\langle W, \frac{\partial}{\partial x_s}\right\rangle g^{\sigma(1)r}g^{\sigma(2)s}g^{k\sigma(3)} \\ &= \sqrt{g}\sigma^{rsk}v^ig_{ri}w^jg_{sj}\det(g^{nm})_{n,m} \\ &= \sqrt{g}\sigma^{rsk}v^iw^jg_{ri}g_{sj}\frac{1}{g} \\ &= \sigma^{rsk}\frac{1}{\sqrt{g}}g_{ri}g_{sj}v^iw^j.\end{aligned}$$

Therefore,

$$V\times W = \sigma^{rsk}\frac{1}{\sqrt{g}}g_{ri}g_{sj}v^iw^j\frac{\partial}{\partial x_k}.$$

□

(2) Check that $V\times W = -W\times V$ and that $V\times W$ is orthogonal to both V and W.

Proof. Note that, by part (1) above, for any $Z\in\mathfrak{X}(M)$,

$$\langle V\times W, Z\rangle = \sqrt{g}\det([V\ W\ Z])$$

where the matrix $[V\ W\ Z]$ is expressed in terms of some basis $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}$ of the tangent space associated with some coordinate chart x_1, x_2, x_3. Now,

$$\det([W\ V\ Z]) = -\det([V\ W\ Z])$$

since the determinant is alternating, so we see that $W\times V = -V\times W$. Also,

$$\det([V\ W\ V]) = \det([V\ W\ W]) = 0$$

since the determinant is alternating, so we see that

$$\langle V\times W, V\rangle = \langle V\times W, W\rangle = 0,$$

meaning $V \times W$ is orthogonal to both V and W. □

(3) Check that $|V \times W|^2 = |V|^2|W|^2 - \langle V, W\rangle^2$.

Proof. Let $p \in M$; then we can choose a coordinate chart x_1, x_2, x_3 around p that is orthogonal at p. Then $g_{ij} = \delta_{ij}$ and $g = 1$, so

$$V \times W = \sigma^{rsk} \frac{1}{\sqrt{g}} g_{ri} g_{sj} v^i w^j \frac{\partial}{\partial x_k} = \sigma^{ijk} v^i w^j \frac{\partial}{\partial x_k}.$$

Hence,

(1)
$$\|V \times W\|^2 = \langle V \times W, V \times W\rangle = \sigma^{ijk}\sigma^{rsm} v^i w^j v^r w^s g_{km} = \sum_{i \neq j} \left[(v^i)^2 (w^j)^2 - v^i w^i v^j w^j \right].$$

On the other hand,

(2)
$$\|V\|^2 \|W\|^2 = \langle V, V\rangle \langle W, W\rangle = \sum_{i,j} (v^i)^2 (w^j)^2$$

and

(3)
$$\langle V, W\rangle^2 = (v^i w^j g_{ij})^2 = \sum_{i,j} v^i w^i v^j w^j$$

Putting (1), (2) and (3) together, we see that

$$\|V \times W\|^2 = \|V\|^2 \|W\|^2 - \langle V, W\rangle^2$$

at p. Since our choice of $p \in M$ was arbitrary, we see that this identity holds on all of M. □

(4) Show that the curl of the vector field $V = v^i \frac{\partial}{\partial x_i}$ is given by

$$\text{curl}(V) = \nabla \times V = \sigma^{ijk} g^{-1/2} \frac{\partial (g_{km} v^m)}{\partial x_j} \frac{\partial}{\partial x_i}.$$

Proof. The only way I know how to define the curl abstractly is as the map that makes the diagram in problem 7 below commute. Since I showed in problem 7 that this expression for curl does indeed make the diagram commute, the curl is given by the above expression. □

(5) Check these formulas in rectangular, cylindrical and spherical coordinates in Euclidean 3-space.
Answer: This is trivial in rectangular coordinates.

♣

(6) Prove the following vector identities in any oriented Riemannian 3-manifold, in which we write $A \cdot B$ instead of $\langle A, B\rangle$ for the inner product of two vectors or vector fields.
(a): $\nabla(fg) = (\nabla f)g + f\nabla g$.

Proof. Using the result proved in problem 5 above,

$$\begin{aligned}\nabla(fg) &= g^{ij}\frac{\partial fg}{\partial x_j}\frac{\partial}{\partial x_i}\\ &= g^{ij}\left[f\frac{\partial g}{\partial x_j}\frac{\partial}{\partial x_i} + g\frac{\partial f}{\partial x_j}\frac{\partial}{\partial x_i}\right]\\ &= f\nabla g + (\nabla f)g.\end{aligned}$$

□

(b): $\nabla \cdot (fV) = (\nabla f)\cdot V + f\nabla \cdot V$.

Proof. Using the results proved in problem 5 above,

$$\begin{aligned}\nabla\cdot(fV) &= \frac{\partial fv^i}{\partial x_i} + \Gamma^i_{ij}fv^j\\ &= f\frac{\partial v^i}{\partial x_i} + v^i\frac{\partial f}{\partial x_i} + f\Gamma^i_{ij}v^j\\ &= f\nabla\cdot V + \left\langle g^{ik}\frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_k}, v^i\frac{\partial}{\partial x_i}\right\rangle\\ &= f\nabla\cdot V + (\nabla f)\cdot V.\end{aligned}$$

□

(c): $\nabla\cdot(V\times W) = (\nabla\times V)\cdot W - V\cdot(\nabla\times W)$.

Proof. Using part (1) above,

$$\begin{aligned}\nabla\cdot(V\times W) &= \nabla\cdot\left(\sigma^{rsk}g^{-1/2}g_{ri}g_{sj}v^iw^j\frac{\partial}{\partial x_k}\right)\\ &= \frac{\partial}{\partial x_m}\left(\sigma^{rsk}g^{-1/2}g_{ri}g_{sj}v^iw^j\right) + \Gamma^m_{mn}\sigma^{rsm}g^{-1/2}g_{ri}g_{sj}v^iv^j\\ &= \sigma^{rsk}\frac{1}{\sqrt{g}}g_{ri}v^i\frac{\partial g_{sj}w^j}{\partial x_k} + \sigma^{rsk}\frac{1}{\sqrt{g}}g_{sj}w^j\frac{\partial g_{ri}v^i}{\partial x_k}\\ &\quad +\sigma^{rsk}g_{ri}g_{sj}v^iw^j\frac{\partial\left(\frac{1}{\sqrt{g}}\right)}{\partial x_k} + \Gamma^m_{mn}\sigma^{rsm}\frac{1}{\sqrt{g}}g_{ri}g_{sj}v^iw^j. \qquad (4)\end{aligned}$$

Since

$$\frac{\partial}{\partial x_k}\left(\frac{1}{\sqrt{g}}\right) = \frac{-1}{2}g^{-3/2}\frac{\partial}{\partial x_k}(\det(g_{ij})) = \frac{-1}{2}g^{-3/2}\frac{\partial}{\partial x_k}(g_{ll}) = \frac{-1}{2}g^{-3/2}2\Gamma^l_{lk}.$$

Now, if we consider x_1, x_2, x_3 orthonormal at $p\in M$, then $g=1$ at p so this expression reduces to $-\Gamma^l_{lk}$. Hence, (4) reduces to

$$\sigma^{rsk}g_{ri}v^i\frac{\partial g_{sj}w^j}{\partial x_k} + \sigma^{rsk}g_{sj}w^j\frac{\partial g_{ri}v^i}{\partial x_k} \qquad (5)$$

Now,

$$(\nabla\times V)\cdot W = \left\langle \sigma^{ijk} g^{-1/2} \frac{\partial(g_{km}v^m)}{\partial x_j}\frac{\partial}{\partial x_i}, w^l \frac{\partial}{\partial x_l}\right\rangle = \sigma^{ijk}\frac{\partial(g_{km}v^m)}{\partial x_j} w^l g_{il} \tag{6}$$

and

$$V\cdot(\nabla W) = \sigma^{ijk}\frac{\partial(g_{km}w^m)}{\partial x_j} v^l g_{il}. \tag{7}$$

Therefore, putting (5), (6) and (7) together, we see that

$$\nabla\cdot(V\times W) = (\nabla\times V)\cdot W - V\cdot(\nabla\times W).$$

□

(d): $\nabla\times(fV) = (\nabla f)\times V + f(\nabla\times V)$.

Proof. Using the above results,

$$\begin{aligned}\nabla\times(fV) &= \sigma^{ijk} g^{-1/2}\frac{\partial(g_{km} f v^m)}{\partial x_j}\frac{\partial}{\partial x_i}\\ &= f\sigma^{ijk} g^{-1/2}\frac{\partial(g_{km}v^m)}{\partial x_j}\frac{\partial}{\partial x_i} + \sigma^{ijk} g^{-1/2} g_{km} v^m \frac{\partial f}{\partial x_j}\frac{\partial}{\partial x_i} \qquad (8)\end{aligned}$$

On the other hand,

$$\begin{aligned}(\nabla f)\times V &= (g^{ij}\frac{\partial f}{\partial x_j}\frac{\partial}{\partial x_i})\times V\\ &= \sigma^{rsk} g^{-1/2} g_{ri} g_{sj} g^{il}\frac{\partial f}{\partial x_l} v^j \frac{\partial}{\partial x_k}\\ &= \sigma^{lsk} g^{-1/2} v^j g_{sj}\frac{\partial f}{\partial x_l}\frac{\partial}{\partial x_i} \qquad (9)\end{aligned}$$

and

$$f(\nabla\times V) = f\sigma^{ijk} g^{-1/2}\frac{\partial(g_{km}v^m)}{\partial x_j}\frac{\partial}{\partial x_i}. \tag{10}$$

Putting (8), (9) and (10) together, we see that

$$\nabla\times(fV) = (\nabla f)\times V + f(\nabla\times V).$$

□

(7) Check the following vector identities.

(e): $\nabla\cdot(\nabla\times V) = 0$.

Proof. This follows from the result proved in problem 7 below.

□

(f): $\nabla\times(\nabla f) = 0$.

Proof. This follows from the result proved in problem 7 below. We can also demonstrate this by the following simple computation:

$$\begin{aligned}\nabla \times (\nabla f) &= \nabla \times \left(g^{ij} \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_i} \right) \\ &= \sigma^{ilk} g^{-1/2} \frac{\partial \left(g_{km} g^{mj} \frac{\partial f}{\partial x_j} \right)}{\partial x_l} \frac{\partial}{\partial x_i} \\ &= \sigma^{ilk} g^{-1/2} \frac{\partial^2 f}{\partial x_l \partial x_j} \frac{\partial}{\partial x_i} \\ &= 0,\end{aligned}$$

since mixed partials commute. □

7

There is a sense in which, on an oriented Riemannian 3-manifold,

gradient of a function, curl of a vector field, divergence of a vector field

corresponds to

exterior derivatives of a 0-form, 1-form and 2-form, respectively.

Make this analogy into a precise theorem and prove it.

Theorem 7.1. *The following diagram commutes:*

$$\begin{array}{ccccccc} C^\infty(M) & \xrightarrow{grad} & \mathfrak{X}(M) & \xrightarrow{curl} & \mathfrak{X}(M) & \xrightarrow{div} & C^\infty(M) \\ \downarrow Id & & \downarrow \vee & & \downarrow \beta & & \downarrow * \\ \Omega^0(M) & \xrightarrow{d} & \Omega^1(M) & \xrightarrow{d} & \Omega^2(M) & \xrightarrow{d} & \Omega^3(M) \end{array}$$

where for $V \in \mathfrak{X}(M)$, $V^\vee$ is the dual form to V, $\beta(V) = i(X)\nu$ and, for $f \in C^\infty(M)$, $*f = f\nu$ where ν is the volume form on M. Moreover, $^\vee$, β and $*$ are isomorphisms and the rows form chain complexes.

Proof. That $^\vee$ and $*$ are isomorphisms is clear. As for β, if $f, g \in C^\infty(M)$ and $X, Y, Z, W \in \mathfrak{X}(M)$, then

$$\begin{aligned}\beta(fX + gY)(Z, W) = i(fX + gY)\nu(Z, W) &= \nu(fX + gY, Z, W) \\ &= f\nu(X, Z, W) + g\nu(Y, Z, W) \\ &= f\beta(X)(Z, W) + g\beta(Y)(Z, W),\end{aligned}$$

since ν is multilinear. In local coordinates, if $X \in \mathfrak{X}(M)$ and $\eta \in \Omega^2(M)$, $X = \sum_{i=1}^3 f_i \frac{\partial}{\partial x_i}$ and $\eta = \sum_{i \neq j} h_{ij} dx_i \wedge dx_j$, so both these spaces are of dimension 3. Finally, if $X, Y \in \mathfrak{X}(M)$ such that $\beta(X) = \beta(Y)$, then

$$\nu(X, Z, W) = \beta(X)(Z, W) = \beta(Y)(Z, W) = \nu(Y, Z, W)$$

for all $Z, W \in \mathfrak{X}(M)$, so $X = Y$. Therefore, we can conclude that β is an isomorphism of vector spaces.

Now, working locally, let x_1, x_2, x_3 be a coordinate system orthonormal at $p \in M$. Let V be a vector field on M. Then

$$d(V^\vee) = d(v^i dx_i) = \sum_{i \neq j} \frac{\partial v^i}{\partial x_j} dx^j \wedge dx^i. \tag{11}$$

On the other hand,

$$\begin{aligned}\beta(\nabla \times V) = \beta\left(\sigma^{ijk} g^{-1/2} \frac{\partial(g_{km} v^m)}{\partial x_j} \frac{\partial}{\partial x_i}\right) \\ &= \beta\left(\sigma^{ijk} \frac{\partial v^k}{\partial x_j} \frac{\partial}{\partial x_i}\right) \\ &= \sum_{j \neq k} \frac{\partial v^k}{\partial x_j} dx_j \wedge dx_k,\end{aligned}$$

which is the same as (11), so we see that the middle square of the above diagram commutes.

Also, if $x_1, \ldots, x_n$ are coordinates on M,

$$\begin{aligned}(\mathrm{grad}(f))^\vee (V) &= \left(g^{ij} \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_i}\right)^\vee (V) \\ &= g_{ij} \frac{\partial f}{\partial x_j} dx_i \left(v^k \frac{\partial}{\partial x_k}\right) \\ &= g_{ij} v^i \frac{\partial f}{\partial x_j}\end{aligned}$$

On the other hand,

$$df(V) = \langle \mathrm{grad}(f), V \rangle = \left\langle \frac{\partial f}{\partial x_j} dx_j, v^i \frac{\partial}{\partial x_i} \right\rangle = v^i \frac{\partial f}{\partial x_j} g_{ij}$$

so the first square commutes. As for the third square, we showed on the last homework that, for $X \in \mathfrak{X}(M)$,

$$d\beta(X) = d(i(X)\nu) = \mathrm{div}\ X\nu = *\mathrm{div}\ X.$$

Since the diagram commutes and the bottom square is a chain complex, so is the top row. □

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