DIFFERENTIAL GEOMETRY HW 2

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Prove that the only orientation-reversing isometries of $\mathbb{R}^2$ are glide reflections, that any two glide-reflections through the same distance are conjugate by an orientation-preserving isometry, and that there are no other conjugacies among orientation-reversing isometries.

Proof. Suppose f is an orientation-reversing isometry of $\mathbb{R}^2$. Let ℓ_1 be the oriented x-axis (with arrow pointing towards $+\infty$) and let e_1, e_2 denote the standard basis vectors in $\mathbb{R}^2 = T_{(0,0)}\mathbb{R}^2$. Let O denote the origin and let $\ell' = f(\ell)$ (with orientation). Ignoring, for the moment, the degenerate case where ℓ and ℓ' are parallel, let p be the point of intersection of ℓ and ℓ'. Let θ be the angle between ℓ and ℓ' considered as oriented lines; let ℓ_1 be the oriented line that bisects the angle θ (i.e. the "positive" direction of ℓ_1 lies in the first "quadrant" determined by ℓ and ℓ'). Let R_1 be reflection across ℓ_1; then $R_1(\ell) = \ell'$. Let d be the oriented distance from $R_1(O)$ to $f(O)$. Let T_1 be the translation $(x, y) \mapsto (x - d, y)$, let $\ell_2 = T_1(\ell_1)$ and let R_2 be reflection across ℓ_2. Then, if T_2 is translation by the oriented distance d along ℓ_2, $T_2 \circ R_2(\ell) = \ell'$. Moreover, $T_2 \circ R_2(O) = f(O)$ and $d(T_2 \circ R_2)|_O(e_1) = df|_O(e_1)$; since both $T_2 \circ R_2$ and f are orientation-reversing, this implies that $d(T_2 \circ R_2)|_O(e_2) = df|_O(e_2)$ and, therefore, we see that $f \equiv T_2 \circ R_2$ which is, by construction, a glide reflection.

In the degenerate case where ℓ and ℓ' are parallel, we let ℓ_1 be vertical (i.e. the y-axis) if ℓ and ℓ' point in opposite directions and horizontal (the line midway between ℓ and ℓ') if they point in the same direction. If we then let $p = O$, the above construction again yields that $f \equiv T_2 \circ R_2$ and so is a glide reflection.

Now, suppose g_1 and g_2 are two glide reflections about the lines ℓ_1 and ℓ_2, respectively, where each translates through the same distance t. Assign arrows to ℓ_1 and ℓ_2 indicating the direction of the translation along the lines. Let f be any orientation preserving isometry mapping ℓ_1 to ℓ_2 and preserving the arrows. Then we claim that $g_1 = f^{-1} g_2 f$. To see this, let $p \in \mathbb{R}^2$; then p lies on some line ℓ parallel to ℓ_1 at a distance a. Since f preserves parallel lines, $f(p)$ lies on a line ℓ' parallel to ℓ_2 at a distance a. Then $g_2(f(p))$ lies on $g_2(\ell')$, which is a line parallel to ℓ_2 also at a distance a. Thus, $f^{-1}(g_2(f(p)))$ lies on $g_1(\ell)$ and is translated a distance t in the direction of ℓ_1, so $f^{-1}(g_2(f(p))) = g_1(p)$. The following picture illustrates this:

Now suppose g_1 and g_2 are glide reflections across ℓ_1 and ℓ_2 through different distances t_1 and t_2. Let $p \in \ell_1$. Then p minimizes $d(q, g_1(q))$ and $d(p, g_1(p)) = t_1$. If $g_1 = f^{-1} g_2 f$ for some isometry f, then $f(p)$ must minimize $d(q, g_2(q))$ since f is an isometry. However, $\min_{q \in \mathbb{R}^2} d(q, g_2(q)) = t_2$ (which is achieved by any point on ℓ_2), so we see that no such f exists. Therefore, we conclude that the conjugacies described above comprise all conjugacies of orientation-reversing isometries of $\mathbb{R}^2$. □

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Prove that every orientation-preserving isometry of the hyperbolic plane is either a rotation about a point, a limit rotation, or a translation.

Proof. Suppose f is an orientation-preserving isometry of the hyperbolic plane. Then f extends to a continuous map on the circle at infinity. We consider fixed points of f as a continuous map of the circle at infinity. In each case, we will consider the disc model D of the hyperbolic plane.

If there are zero fixed points on the circle at infinity then, by Brouwer's fixed point theorem, there must be a fixed point p in the interior of D. Since the hyperbolic plane is complete, the inverse of the exponential map induces an isometry of T_pD fixing the origin; since the only orientation-preserving isometries of the plane with a fixed point are rotations about the fixed point, we see that this isometry is a rotation. Then, passing through the exponential map, we see that f is a rotation of the hyperbolic plane.

If there is exactly one fixed point p on the circle at infinity, then geodesics of D passing through p must be mapped to other geodesics of D passing through p. We may as well assume that p is the point at infinity (in the upper half-plane model) since we can always conjugate by an appropriate rotation of D to make it so. Then the geodesics through p are just the vertical lines in the upper half-plane model; let ℓ be such a line and let $\ell' = f(\ell)$. Let R be the limit rotation taking ℓ to ℓ'. Then, since $f(\ell) = \ell'$, either $f \equiv R$ or $f \equiv T \circ R$ where T is a translation along ℓ'. Now, in the upper half-plane model, $R : z \mapsto z + a$ and $T : z \mapsto bz$ for real a and b. Hence, $f(z) = T \circ R(z) = b(z + a) = bz + ab$. However, unless $b = 1$ (i.e. T is the identity map) this map fixes the entire vertical line $x = \frac{-ab}{b-1}$, meaning

f fixes two points on the circle at infinity. Since we're in the case where f fixes exactly one point on the circle at infinity, we conclude that T must be the identity, so $f \equiv R$, which is a limit rotation.

If there are two fixed points on the circle at infinity, then f must preserve the geodesic γ connecting the fixed points. Moreover, f must map geodesics orthogonal to γ to other geodesics orthogonal to γ; this, combined with the fact that f is orientation-preserving, means that f preserves the curves at a fixed distance from γ. However, this is precisely the description of a translation along γ.

Finally, it may be that f has at least three fixed points on the circle at infinity. Since all isometries of $\mathbb{H}^2$ (now considering the upper half-plane model) are Möbius transformations and Möbius transformations are determined by where they map any three points in $\mathbb{C} \cup \infty$, this means that f is the identity.

Since the above cover all possible cases, we see that f must be the identity, a rotation, a limit rotation or a translation. □

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Prove that every orientation-reversing isometry of the hyperbolic plane is a glide reflection.

Proof. Suppose f is an orientation-reversing isometry of $\mathbb{H}^2$. Then we claim that f fixes to points on the circle at infinity. To see this, note that an orientation-reversing isometry of $\mathbb{H}^2$ induces an orientation-reversing homeomorphism of the circle at infinity; we claim that any such has exactly two fixed points. To see this, let $g : S^1 \to S^1$ be an orientation-reversing homeomorphism. Lift g to a map $\widetilde{g} : [0,1] \to \mathbb{R}$, where $g(0) = a$ (which we can do since $\mathbb{R}$ is the universal cover of S^1). Normalize so that $g(t)$ is parametrized by arc length. Then, since g is orientation-reversing, $\widetilde{g}(1) = a - 1$. Hence, the image of $g - Id$ is contained in $[a-2, a]$. If $a \notin \mathbb{Z}$, then $[a-2, a]$ contains 2 integers, which correspond to two fixed points of g. If $a \in \mathbb{Z}$, then $[a-2, a]$ contains three integers; however, $a - 2$ and a correspond to the same point in S^1 and so $a - 1$ and a correspond to two fixed points of g.

Thus, we see that f does indeed have two fixed points on the circle at infinity and so, if γ is the geodesic connecting these two fixed points, $f(\gamma) = \gamma$. Hence, $f|_\gamma$ is a translation; let $\phi : \mathbb{H}^2 \to \mathbb{H}^2$ be a translation along γ such that $\phi \circ f|_\gamma = Id$ and let $\psi : \mathbb{H}^2 \to \mathbb{H}^2$ be a reflection across γ. Then $\psi \circ \phi \circ f$ is an orientation-preserving isometry of $\mathbb{H}^2$ that fixes γ pointwise; thus, $\psi \circ \phi \circ f$ is the identity. Therefore, $f = \phi^{-1} \circ \psi$, which is, as constructed, a glide reflection. □

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Find some examples of inequivalent lattices in $\mathbb{R}^2$. Try to classify them all up to equivalence.

Answer: First, we may as well assume that our lattices Λ have a point at the origin; if not, conjugate by an appropriate translation. Now, if Λ is a lattice, let $p \in \Lambda$ be the point of the lattice closest to the origin. We may as well assume p lies on the x-axis; if not, conjugate by an appropriate rotation. Now, certainly, two lattices whose closest points to the origin lie at different distances from the origin are not equivalent, so, in the interest of classifying interesting non-equivalences, suppose the point p lies at distance one from the origin; that is, $p = (1, 0)$. Let q be the point of Λ that lies second-closest to the origin. Then $d(q, 0) \geq 1$, so q lies on the outside of the open unit disc. We can also ignore points q that lie outside the first quadrant, since an appropriate reflection or rotation will map any point outside the first quadrant to a point in the first quadrant at the same distance from the origin. Moreover, if $q = (x, y)$, it must be the case that $d((x-1, y), 0) \geq 1$ because, since $(1, 0)$ (and, thus, $(-1, 0)$) is a point in the lattice, $(x-1, 0)$ is also in the lattice.

Translating all of the above into visual format, we see that the potential locations for q are the shaded region (including boundaries) in the following picture:

Each point in this region represents a distinct lattice. Since we can't apply any orientation-preserving isometry without moving $(1, 0)$ and the origin and we've already taken care of the possibility of conjugation by an orientation-reversing isometry by restricting to the first quadrant, we see that each point in the above region in fact represents a lattice inequivalent to the others so represented, so this, up to scalings, gives a complete classification of inequivalent lattices.

♣

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Let Γ be generated by the above glide reflection, and also by the translation

$$t : (x, y) \mapsto (x, y + 1).$$

Find a presentation for Γ using the two generators r and t, and note that Γ is non-abelian. Show that the resulting surface $\mathbb{R}^2/\Gamma$ is a Klein bottle.

Proof. First of all,

$$r^k(x,y) = (x+k, (-1)^k y)$$

and

$$t^k(x,y) = (x, y+k),$$

so we see that r^k and t^k can never be the identity for $k \neq 0$, so both r and t are of infinite order. Hence, $\langle r \rangle$ and $\langle t \rangle$ are free, so any relations in Γ must be induced by relations between r and t. Now,

(1) $r \circ t(x,y) = r(x,y+1) = (x+1, -y-1) = t^{-1}(x+1,-y) = t^{-1} \circ r(x,y)$

and

(2) $t \circ r(x,y) = t(x+1,-y) = (x+1,-y+1) = r(x,y-1) = r \circ t^{-1}(x,y),$

so $r \circ t \equiv t^{-1} \circ r$ and $t \circ r \equiv r \circ t^{-1}$. Note that Γ is non-abelian. Moreover, we can derive (2) from (1):

$$r \circ t^{-1} = r \circ t^{-1} \circ r^{-1} \circ r = r \circ (r \circ t)^{-1} \circ r = r \circ (t^{-1} \circ r)^{-1} \circ r = r \circ r^{-1} \circ t \circ r = t \circ r$$

Since (1), (2) and their inverses yield all possible combinations of r, r^{-1}, t and t^{-1} and since r and t are of infinite order, these relations generate all of the relations in Γ. Since (1) generates (2), this means that

$$\Gamma = \langle r, t | rt = t^{-1} r \rangle = \langle r, t | trt = r \rangle.$$

Now, r and t map the (left and bottom) edges labeled by them in the below diagram to the (right and top) edges labeled by them in the below diagram:

Therefore, in $\mathbb{R}^2/\Gamma$, the edges with the same label are identified, so the above is a picture of the fundamental domain of $\mathbb{R}^2/\Gamma$. This, however, is precisely the Klein bottle. Thus we've confirmed that $\Gamma = \langle r, t | trt = r \rangle$, since this is exactly the fundamental group of the Klein bottle and Γ is the deck group of $\mathbb{R}^2/\Gamma$, which must be isomorphic to the fundamental group. $\square$

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Show that a discrete group of Euclidean isometries acts freely if and only if it is torsion-free.

Proof. Let Γ be a discrete group of Euclidean isometries.

($\Rightarrow$) Suppose Γ has torsion; that is, there exists $g \in \Gamma$ such that $g^k = Id_{\mathbb{R}^n}$. Let $p \in \mathbb{R}^n$. Consider the set $S = \{p, g(p), g^2(p), \ldots, g^{k-1}(p)\}$. Let the point q be the center of mass of S. Since the powers of g simply permute the elements of S and since centers of mass are preserved by isometries, $g(q)$ is the center of mass of $g(S) = S$, which means that $g(q) = q$. Hence, we see that g has a fixed point and so does not act freely. Thus, by contrapositive, we see that if Γ acts freely, then Γ is torsion-free.

($\Leftarrow$) On the other hand, suppose Γ does not act freely; that is, there exists some non-identity $g \in \Gamma$ such that $g(p) = p$ for some $p \in \mathbb{R}^n$. We may as well assume p is the origin (if not, then g is conjugate, via a translation, to an isometry fixing the origin), which means that g is an isometry of $\mathbb{R}^n$ fixing the origin; which is to say, an element of $O(n)$. Let $G = \langle g \rangle$, the subgroup of Γ generated by g. Then, since $g \in O(n)$, $G \subset O(n)$. Since Γ is discrete, so is G. Now, since $O(n)$ is compact, any discrete subset must be finite, meaning that G is finite and, therefore, there exists $k \in \mathbb{N}$ such that $g^k = Id_{\mathbb{R}^n}$. Thus, we see that Γ has torsion. By contrapositive, then, we conclude that if Γ is torsion-free, then it acts freely. $\square$

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