

DIFFERENTIAL GEOMETRY HW 3

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32.

Determine the dihedral angles of the dodecahedron.

Answer: Let v be a vertex of the dodecahedron. Since the dodecahedron is regular, the choice of v general. Three sides intersect at v ; call them S_1 , S_2 and S_3 . For each S_i , let N_i be a unit normal vector from S_i . Since each of the N_i is a unit vector, we can view the N_i as points on the unit 2-sphere:

Let A , B and C be the sides of the triangle formed by the N_i as pictured above. Since the N_i are normal vectors to the S_i , the angle between N_i and N_j is equal to $\pi - \phi$, where ϕ is the dihedral angle between S_i and S_j (which is, of course, the same for all $i \neq j$); hence, since they lie on the unit sphere, the length of the arcs A , B and C is equal to $\pi - \phi$ as well. Therefore, if we can determine the angle θ between the sides of the spherical triangle ABC , we can use the spherical law of cosines to determine $A = B = C$. To that end, let S_2 and S_3 be as above and let S_4, S_5 and S_6 be the other sides adjoining S_1 and let N_i be the unit normals to the S_i . When transplanted to the sphere, the N_i form 5 triangles all with the common vertex N_1 :

Since all of the angles are equal, we see that $\theta = \frac{2\pi}{5}$. Therefore, by the spherical law of cosines,

$$\cos A = \cos B \cos C + \sin B \sin C \cos \theta$$

or, since $A = B = C$ and $\theta = \frac{2\pi}{5}$,

$$\cos A = \cos^2 A + \sin^2 A \cos \frac{2\pi}{5} = \cos^2 A + (1 - \cos^2 A) \cos \frac{2\pi}{5} = \cos \frac{2\pi}{5} + (1 - \cos \frac{2\pi}{5}) \cos^2 A.$$

Hence, by the quadratic formula,

$$\cos A = \frac{-1 \pm \sqrt{1 - 4(-1 + \cos \frac{2\pi}{5})(-\cos \frac{2\pi}{5})}}{2(-1 + \cos \frac{2\pi}{5})} = \frac{-1 \pm \sqrt{(2 \cos \frac{2\pi}{5} - 1)^2}}{-2 + 2 \cos \frac{2\pi}{5}} = \frac{1}{2 \cos \frac{2\pi}{5}} \mp 1.$$

Therefore $\phi = 180^\circ - A = 180^\circ - 63.435^\circ = 116.565^\circ$.



42.

Show that it is possible to put a hyperbolic structure on the 2-hole torus.

Proof. First, recall from Math 601 the 2-hole torus has the following cell structure:

To make the above into a manifold, we need to check that it's a manifold at all points. At points interior to the octagon, it's clear that this is a manifold. At points on the interior of an edge, it's still a manifold, since only two "sides" come together at any given edge. The only issue is what occurs at the vertices. It's easy to see that all the vertices are the exact same point; in order for a neighborhood of this point to be a disc, it must be the case that each of the interior angles of the octagon is exactly 45° . This is impossible in the euclidean plane, since the interior angles of a regular octagon are $\frac{6(180^\circ)}{8} = 135^\circ$. On the other hand, consider a regular geodesic

octagon in the hyperbolic plane:

When such an octagon is extremely small, the interior angles will be just slightly smaller than the interior angles of a euclidean octagon, so just under 135° . On the other hand, as the regular geodesic octagon gets very large, it approaches an ideal octagon, which has interior angles of 0° . Scaling the regular octagon gives a one-parameter family of regular geodesic octagons, which means that somewhere between the very tiny and very large octagons there is a regular geodesic octagon with interior angles of 45° . Since all the edges are geodesics, we can make the desired identifications to yield a 2-hole torus inheriting the hyperbolic structure from the hyperbolic plane. \square

43.

The Seifert-Weber space:

(a): In the Seifert-Weber space, show that the edges of the dodecahedron are grouped into 6 groups of 5.

Proof. This is easy to see by taking a dodecahedron and coloring the edges according to the identifications yielded in building the Seifert-Weber space; if you want to see the dodecahedron I did this on, just ask. \square

(b): Show that, at interior points of the edges, the space is a manifold.

Proof. We do this by showing that at any point on the interior of an edge, the cross section orthogonal to the edge is a disc. Since, by (a), at any edge 5 faces (after identification) come together, we want to show that, starting at a point on one of these faces and traversing around the edge, we come back to the same point on that face after passing through each of the 5 faces that come together at that edge. Again, this is easy to see on an actual dodecahedron. \square

(c): Show that, under the identifications, all 20 vertices are identified.

Proof. Starting at any vertex and marking it, say, red and then marking in the same color all other vertices identified to that vertex

under the identifications, it's easy to see that all 20 vertices will end up red. \square

(d): Show that the Seifert-Weber space is a manifold at the vertex.

Proof. To do so, we must show that a neighborhood of the vertex is a ball. At each vertex on the dodecahedron, draw oriented edges connecting the edges that meet at that vertex, labeled appropriately. In so doing, we end up with 20 2-simplices, one for each of the 20 vertices. If we can show that these 20 2-simplices, glued together according to the labeling induced by the identifications that determine the space, form a 2-sphere, then we know that these simplices form the boundary of a spherical neighborhood of the vertex in the Seifert-Weber space; meaning this neighborhood is a ball. Here is a complex formed by the 20 2-simplices:

Reading around the outside edges of this complex in a clockwise manner starting from the left, we get the word

$$GD^{-1}E^{-1}EDPM^{-1}BGG^{-1}B^{-1}MP^{-1}D^{-1}E^{-1}EDG^{-1},$$

which, after cancellations, is just the empty word; hence, this complex is just a 2-sphere. \square

(e): Justify giving a hyperbolic structure to the Seifert-Weber space.

Proof. Since we know from above that 5 faces come together at each edge, the following must be an end-on view of each edge in the

Seifert-Weber space:

Thus, we see that, if we are to use a regular dodecahedron, the angle between the faces must be $\frac{360^\circ}{5} = 72^\circ$. In a Euclidean dodecahedron, the dihedral angle is 116.565° , so we cannot give the Seifert-Weber a Euclidean structure; nor can we give it a spherical structure, since spherical dodecahedra have bigger dihedral angles than the Euclidean dodecahedron. However, hyperbolic dodecahedra have smaller dihedral angles than their Euclidean brethren, so we might hope to be able to get a hyperbolic regular geodesic dodecahedron with dihedral angle 72° . Using the same thought-process as in problem 42 above, we can describe a one-parameter family of regular geodesic dodecahedra by scaling. A very small such dodecahedron has dihedral angle just under 116.565° . An ideal dodecahedron has dihedral angle of 60° , so there exists a geodesic dodecahedron with dihedral angle 72° as needed. Then, by what we proved above in (a)-(d), we can indeed give the Seifert-Weber a hyperbolic space. \square

A2

Compute the fundamental group of the Gieseking manifold M^3 and of the complement of the figure-eight knot K in S^3 and show that $\pi_1(S^3 - K)$ is isomorphic to a subgroup of index 2 in $\pi_1(M^3)$.

Proof. To compute the fundamental group of the Gieseking manifold, we do a handle-body decomposition of M^3 as follows:

Then we can just read off the generating relationship of the fundamental group:

$$rs^{-1}s^{-1}r^{-1}sr = 1$$

or, equivalently,

$$r^{-1}s^2rs^{-1}r^{-1} = 1.$$

Multiplying on the left by r and on the right by rs yields the relation $s^2r = r^2s$, so we see that

$$\pi_1(M^3) \simeq \langle r, s | r^2s = s^2r \rangle.$$

Turning to the complement of the figure eight, we mark the arcs and choose preferred sectors as follows:

Then, computing the Wirtinger presentation of the knot complement yields the following relations:

$$x_3x_2x_1^{-1}x_2^{-1} = 1$$

$$x_2x_3x_4^{-1}x_3^{-1} = 1$$

$$x_1x_4x_1^{-1}x_3^{-1} = 1$$

$$x_4x_1x_4^{-1}x_2^{-1} = 1.$$

From the bottom two relations, we see that $x_2 = x_4x_1x_4^{-1}$ and $x_3 = x_1x_4x_1^{-1}$ so we may discard the last two relations. Also, from the first relation, we have that $x_2 = x_3x_2x_1^{-1}$. From this, we can deduce that

$$x_2 = x_3x_2x_1^{-1} = (x_1x_4x_1^{-1})(x_4x_1x_4^{-1})x_1^{-1} = (x_1x_4x_1^{-1})x_4(x_1x_4^{-1}x_1^{-1}) = x_3x_4x_3^{-1},$$

which is equivalent to the second relation, so we may discard it. Therefore, the generating relation of the fundamental group of the knot complement is given by

$$1 = x_3x_2x_1^{-1}x_2^{-1} = x_1x_4x_1^{-1}x_4x_1x_4^{-1}x_1^{-1}x_4x_1^{-1}x_4^{-1},$$

which can be re-written as

$$x_1^{-1}x_4x_1x_4^{-1}x_1^{-1} = x_4^{-1}x_1^{-1}x_4x_1x_4^{-1}.$$

Letting $a = x_1^{-1}$ and $b = x_4^{-1}$, we see that

$$\pi_1(S^3 - K) \simeq \langle a, b | (ab^{-1}a^{-1}b)a = b(ab^{-1}a^{-1}b) \rangle.$$

Now, to induce a homomorphism $\pi_1(M^3) \rightarrow \pi_1(S^3 - K)$, we simply need to know where the generators a and b map to and show that the relations are compatible. Define such a map by

$$a \mapsto rs \quad b \mapsto r^2.$$

Then

$$\begin{aligned} a^{-1}b^{-1}ab^{-1}a^{-1}bab^{-1}ab &= s^{-1}r^{-1}r^{-2}rsr^{-1}s^{-1}r^{-1}r^2rsr^{-2}rsr^2 \\ &= (r^2s)^{-1}sr^{-2}s^{-1}(r^2s)r^{-1}sr^2 \\ &= (s^2r)^{-1}sr^{-2}s^{-1}(s^2r)r^{-1}sr^2 \\ &= r^{-1}s^{-2}sr^{-2}s^{-1}s^2rr^{-1}sr^2 \\ &= r^{-1}s^{-1}r^{-2}s^2r^2 \\ &= r^{-1}(r^2s)^{-1}(s^2r)r \\ &= r^{-1}r \\ &= 1. \end{aligned}$$

However, this is just the generating relation for $\pi_1(S^3 - K)$, as we saw above, so we see that the relations are compatible. Therefore, $\pi_1(S^3 - K)$ is really isomorphic to a subgroup of $\pi_1(M^3)$. Now,

$$\pi_1(M^3)/\pi_1(S^3 - K) \simeq \langle r, s | r^2s = s^2r, rs = 1, r^2 = 1 \rangle.$$

Hence, in this group, $s = r^{-1}$ and

$$r = r^2r^{-1} = r^2s = s^2r = r^{-2}r = r^{-1},$$

so

$$\pi_1(M^3)/\pi_1(S^3 - K) \simeq \langle r | r^2 = 1 \rangle \simeq \mathbb{Z}/2,$$

so $\pi_1(S^3 - K)$ is isomorphic to a subgroup of $\pi_1(M^3)$ of index 2. \square