DIFFERENTIAL GEOMETRY HW 5

CLAY SHONKWILER

3.

Let M be a complete Riemannian manifold with non-positive sectional curvature. Prove that

$$|(d\exp_p)_v(w)| \geq |w|,$$

for all $p \in M$, all $v \in T_pM$ and all $w \in T_v(T_pM)$.

Proof. Let γ be a geodesic on M with $\gamma(0) = p$, $\gamma'(0) = v$. Define the Jacobi field $J(t) = (d\exp_p)_{tv}(tw)$. Then $J(0) = 0$ and

$$J'(0) = \frac{D}{dt}\left(t(d\exp_p)_{tv}(w)\right)\Big|_{t=0} = (d\exp_p)_{tv}(w) + 0\cdot\left(\frac{d}{dt}(d\exp_p)_{tv}(w)\right)\Bigg|_{t=0} = w.$$

Hence, $\langle J'(0), \gamma'(0)\rangle = \langle w, v\rangle$. Now, let $\widetilde{M} = T_pM$, which is linearly isometric to $\mathbb{R}^n$ with the usual inner product, so the sectional curvatures are all $\equiv 0$, which is greater than or equal to all sectional curvatures of M. Let $\widetilde{\gamma}(t) = tv$; then $\widetilde{\gamma}$ is a geodesic in $\widetilde{M}$ and, if we define $\widetilde{J}(t) = tw$, then $\widetilde{J}$ is a Jacobi field. Moreover, $\langle \widetilde{J}'(0), \widetilde{\gamma}'(0)\rangle = \langle w, v\rangle$. Therefore, by the Rauch Comparison Theorem, $|J(t)| \geq |\widetilde{J}(t)|$ for all t. In particular, for $t = 1$,

$$|(d\exp_p)_v(w)| = |J(1)| \geq |\widetilde{J}(1)| = |w|.$$

□

4.

(a): Let $C \subset \mathbb{R}^2$ be a regular curve. Show that the focal set $F(C) \subset \mathbb{R}^2$ of C is obtained by taking, on the positive normal n at $p \in C$ a length equal to $1/k$, where k is the curvature of C at p.

Proof. Let $x(s)$ parametrize C. Recall that $\exp^\perp(t, s) = x(t) + tn(s)$. If the tangent T and the normal n form a basis for $\mathbb{R}^2$, then

$$d\exp^\perp = \begin{pmatrix} \left\langle \frac{\partial x}{\partial s} + t\frac{\partial n}{\partial s}, \frac{\partial x}{\partial s}\right\rangle & \left\langle n, \frac{\partial x}{\partial s}\right\rangle \\ \left\langle \frac{\partial x}{\partial s} + t\frac{\partial n}{\partial s}, n\right\rangle & \langle n, n\rangle \end{pmatrix} = \begin{pmatrix} \left\langle \frac{\partial x}{\partial s} + t\frac{\partial n}{\partial s}, \frac{\partial x}{\partial s}\right\rangle & 0 \\ \left\langle \frac{\partial x}{\partial s} + t\frac{\partial n}{\partial s}, n\right\rangle & \langle n, n\rangle \end{pmatrix}.$$

Since $\langle n, n\rangle \neq 0$, $d\exp^\perp$ is singular (and C has a focal point) if and only if

$$\left\langle \frac{\partial x}{\partial s} + t\frac{\partial n}{\partial s}, \frac{\partial x}{\partial s}\right\rangle = 0.$$

By the Frenet equations, $\frac{\partial n}{\partial s} = -\kappa(s)T(s)$, so

$$\frac{\partial x}{\partial s} + t\frac{\partial n}{\partial s} = T(s) - t\kappa(s)T(s) = (1 - t\kappa(s))T(s)$$

which is zero precisely when $1 - t\kappa(s) = 0$: i.e., when $t = \frac{1}{\kappa(s)}$, exactly as predicted. Note that $\left|\frac{\partial x}{\partial s}\right| = 0$ precisely when $\kappa(s) = 0$, so we see that the above entirely characterizes the focal points of C. □

(b): Show that the focal set of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is given by

$$\left\{(x, y) \in \mathbb{R}^2; (ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}\right\}$$

Proof. Parametrize the ellipse by $\alpha(s) = (a\cos s, b\sin s)$ (note that, in spite of the notation, this curve is not parametrized by arc length). Then $\alpha'(s) = (-a\sin s, b\cos s)$. If we let $n(s) = (-b\cos s, -a\sin s)$, then n is normal to the ellipse. Now, $n'(s) = (b\sin s, -a\cos s)$, so

$$d\exp^{\perp} = \begin{pmatrix} a^2\sin^2 s + b^2\cos^2 s - tab & 0 \\ (b^2 - a^2)t\sin s\cos s & b^2\cos^2 s + a^2\sin^2 s \end{pmatrix},$$

which has a critical point when $t = \frac{a^2\sin^2 s + b^2\cos^2 s}{ab}$. Therefore, the focal locus of the ellipse is given by

$$\begin{aligned} \alpha(s) + \frac{a^2\sin^2 s + b^2\cos^2 s}{ab}n(s) &= \Bigg(a\cos s - \frac{a^2\sin^2 s + b^2\cos^2 s}{ab}b\cos u, \\ &\qquad b\sin s - \frac{a^2\sin^2 s + b^2\cos^2 s}{ab}a\sin s\Bigg) \\ &= \left(\frac{a^2 - a^2\sin^2 s - b^2\cos^2 s}{a}\cos s, \frac{b^2 - a^2\sin^2 s - b^2\cos^2 s}{b}\sin s\right) \\ &= \left(\frac{b^2 - a^2}{a}\cos^3 s, \frac{a^2 - b^2}{b}\sin^3 s\right). \end{aligned}$$

Since

$$\left((b^2 - a^2)\cos^3 s\right)^{2/3} + \left((a^2 - b^2)\sin^3 s\right)^{2/3} = (a^2 - b^2)^{2/3},$$

we see that the focal locus of the ellipse is given by

$$\left\{(x, y) \in \mathbb{R}^2 | (ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}\right\}.$$

□

(c): Show that the focal set of the curve

$$t \mapsto (\cos t + t\sin t, -\sin t + t\cos t)$$

is the circle $t \mapsto (\cos t, -\sin t)$.

Proof. Let $\alpha(s) = (\cos s + s\sin s, -\sin s + s\cos s)$ (again, despite notation, not necessarily parametrized by arc length). Then $\alpha'(s) = (s\cos s, -s\sin s)$. If we let $n(s) = (\sin s, \cos s)$, then n is normal to the spiral. Now,

$$n'(s) = (\cos s, -\sin s),$$

so

$$d\exp^{\perp} = \begin{pmatrix} s^2 + ts & 0 \\ 0 & 1 \end{pmatrix},$$

which has critical points when $t = -s$. Therefore, the focal locus of the spiral is given by

$$\alpha(s) - sn(s) = (\cos s + s\sin s - s\sin s, -\sin s + s\cos s - s\cos s) = (\cos s, -\sin s),$$

as expected. □

6.

What follows is a slight generalization of Sturm's Comparison Theorem. We present the theorem in geometric form.

Let M^2 be a complete Riemannian manifold of dimension 2, and let $\gamma : [0,\infty) \to M^2$ be a geodesic. Let $J(t)$ be a Jacobi field along γ with $J(0) = J(t_0) = 0$, $t_0 \in (0,\infty)$, and $J(t) \neq 0$, $t \in (0, t_0)$. Then J is a field normal to γ and can be written $J(t) = f(t)e_2(t)$, where $e_2(t)$ is the parallel transport of a unit vector $e_2 \in T_{\gamma(0)}M$ with $e_2 \perp \gamma'(0)$. Because J is a Jacobi field,

$$f''(t) + K(t)f(t) = 0,$$

where K is the Gaussian curvature of M^2. Assume that

$$K(t) \leq L(t),$$

where L is a diffferentiable function on $[0,\infty)$. Prove that any solution of the equation

$$\widetilde{f}''(t) + L(t)\widetilde{f}(t) = 0$$

has a zero on $[0, t_0]$, that is, there exists $t_1 \in [0, t_0]$ with $\widetilde{f}(t_1) = 0$.

Proof. Suppose $\widetilde{f}(t)$ is a solution to the given equation and that $\widetilde{f}(t) \neq 0$ for all $t \in [0, t_0]$. Since $f'' + Kf = 0$ and $\widetilde{f}'' + L\widetilde{f} = 0$,

$$\begin{aligned} 0 &= \int_0^{t_0} \left[\widetilde{f}(f'' + Kf) - f(\widetilde{f}'' + L\widetilde{f})\right] dt \\ &= \int_0^{t_0} (\widetilde{f}f'' - f\widetilde{f}'')dt + \int_0^{t} (K - L)f\widetilde{f}dt \\ &= [\widetilde{f}f' - f\widetilde{f}']_0^{t_0} - \int_0^{t_0} (\widetilde{f}'f' - f'\widetilde{f}')dt + \int_0^{t_0} (K - L)f\widetilde{f}dt \qquad (1) \\ &= \widetilde{f}(t_0)f'(t_0) - \widetilde{f}(0)f'(0) + \int_0^{t_0} (K - L)f\widetilde{f}dt. \end{aligned}$$

Now, since $\widetilde{f}(t) \neq 0$ for all $t \in [0, t_0]$, then either $\widetilde{f} > 0$ on this interval or $\widetilde{f} < 0$ on this interval. Similarly, since t_0 is the first conjugate point of $\gamma(0)$ along γ, $f > 0$ on $(0, t_0)$ or $f < 0$ on $(0, t_0)$. If $\widetilde{f} > 0$ and $f > 0$ on $(0, t_0)$, then, since $f(0) = 0 = f(t_0)$, $f'(0) > 0$ and $f'(t_0) < 0$. However, since $K \leq L$,

$$\int_0^{t_0} (K - L) f\widetilde{f} dt + \widetilde{f}(t_0)f'(t_0) - \widetilde{f}(0)f'(0) \leq \widetilde{f}(t_0)f'(t_0) - \widetilde{f}(0)f'(0) < 0,$$

contradicting (1) above. The other cases ($\widetilde{f} > 0$ and $f < 0$, etc.) follow similarly. □

7.

Let M^2 be a complete Riemannian manifold of dimension two and let $\gamma : [0, \infty) \to M^2$ be a geodesic with $\gamma(0) = p$. Let $K(s)$ be the Gaussian curvature of M^2 along γ. Assume that:

$$\int_t^{\infty} K(s) ds \leq \frac{1}{4(t+1)}, \qquad \text{for all } t \geq 0, \tag{2}$$

in the sense that the integral converges and has the bound indicated.

(a): Define

$$\omega(t) = \int_t^{\infty} K(s) ds + \frac{1}{4(t+1)},$$

and show that $\omega'(t) + (\omega(t))^2 \leq -K(t)$.

Proof. First, note that

$$\omega(t) = \int_0^{\infty} K(s) ds - \int_0^{t} K(s) ds + \frac{1}{4(t+1)},$$

so, by the Fundamental Theorem of Calculus,

$$\omega'(t) = -K(t) - \frac{1}{4(t+1)^2}.$$

Now,

$$(\omega(t))^2 = \left(\int_t^{\infty} K(s) ds\right)^2 + \frac{1}{2(t+1)} \int_t^{\infty} K(s) ds + \frac{1}{16(t+1)^2},$$

so

$$\begin{aligned}
\omega'(t) + (\omega(t))^2 &= -K(t) - \frac{1}{4(t+1)^2} + \left(\int_t^{\infty} K(s) ds\right)^2 + \frac{1}{2(t+1)} \int_t^{\infty} K(s) ds + \frac{1}{16(t+1)^2} \\
&= -K(t) - \frac{3}{16(t+1)^2} + \left(\int_t^{\infty} K(s) ds\right)^2 + \frac{1}{2(t+1)} \int_t^{\infty} K(s) ds \\
&\leq -K(t) - \frac{3}{16(t+1)^2} + \left(\frac{1}{4(t+1)}\right)^2 + \frac{1}{2(t+1)} \cdot \frac{1}{4(t+1)} \\
&= -K(t).
\end{aligned}$$

□

(b): For $t \geq 0$, put $\omega'(t) + (\omega(t))^2 = -L(t)$ (hence $L(t) \geq K(t)$) and define

$$\widetilde{f}(t) = \exp\left(\int_0^t \omega(s)ds\right), \quad t \geq 0.$$

Show that

$$\widetilde{f}''(t) + L(t)\widetilde{f}(t) = 0, \quad \widetilde{f}(0) = 1.$$

Proof. By definition,

$$\widetilde{f}'(t) = \frac{d}{dt}\left(\int_0^t \omega(s)ds\right)\exp\left(\int_0^t \omega(s)ds\right) = \omega(t)\exp\left(\int_0^t \omega(s)ds\right) = \omega(t)\widetilde{f}(t),$$

so

$$\widetilde{f}''(t) = \omega'(t)\widetilde{f}(t) + (\omega(t))^2\widetilde{f}(t).$$

Therefore,

$$\widetilde{f}''(t) + L(t)\widetilde{f}(t) = \omega'(t)\widetilde{f}(t) + (\omega(t))^2\widetilde{f}(t) + (-\omega'(t) + (\omega(t))^2)\widetilde{f}(t) = 0.$$

Also, $\widetilde{f}(0) = \exp(0) = 1$, as desired. □

(c): Observe that $\widetilde{f}(t) > 0$ and use the oscillation theorem of Sturm to show that there does not exist a Jacobi field $J(s)$ on $\gamma(s)$ with $J(0) = 0$ and $J(s_0) = 0$, for some $s_0 \in (0, \infty)$. Therefore, *the condition (2) implies that there do not exist conjugate points to p along* γ.

Proof. Since the exponential is always positive, $\widetilde{f}(t) > 0$. Now, suppose there exists a Jacobi field $J(s)$ on $\gamma(s)$ such that $J(0) = 0$ and $J(s_0) = 0$ for some $s_0 \in (0, \infty)$. Then, by problem 6 above, since $\widetilde{f}$ is a solution of the equation $\widetilde{f}''(s)+L(s)\widetilde{f}(s) = 0$ and $K(s) \leq L(s)$, $\widetilde{f}$ must have a zero on $[0, s_0]$. However, we just said that $\widetilde{f}$ is strictly positive, so this is impossible. Therefore, we conclude that there must not exist such a Jacobi field, meaning that there are no conjugate points to p along γ. □

(B)

Let $S^n(1)$ denote the n-sphere of radius 1 in $\mathbb{R}^{n+1}$. Consider the product metric on $S^3(1) \times S^1(1)$.

(a): What can one say about the curvature of this metric?

Answer: Let $M = S^3(1) \times S^1(1)$ and, for any $p \in M$ and $V \in T_pM$, let us denote by V_3 the component of V parallel to S^3 and by V_1 the component of V parallel to S^1. Now, for orthonormal $X, Y \in T_pM$,

$$K(X,Y) = \langle R(X,Y)X,Y\rangle = \langle R(X_1 + X_3, Y_1 + Y_3)(X_1 + X_3), Y_1 + Y_3\rangle.$$

Since everything in this expression is linear, we can completely split up all the terms. Moreover, since S^1 is only one-dimensional, X_1 and Y_1 must be parallel, so $R(X_1, Y_1) \equiv 0$. Also, if we think of M as a bundle over $S^3(1)$, X_1 is vertical and Y_3 is horizontal, so $R(X_1, Y_3) \equiv 0$ and similarly for $R(Y_1, X_3)$, $R(X_1, X_3)$, $R(Y_1, Y_3)$, etc. Therefore, by rearranging according to the usual formulas, all of the terms in the above expression vanish except for $\langle R(X_3, Y_3)X_3, Y_3\rangle$. Hence,

$$K(X, Y) = \langle R(X_3, Y_3)X_3, Y_3\rangle = |X_3 \wedge Y_3|^2,$$

which varies from 0 to 1. In essence, we're taking the unit square defined by X and Y, projecting it onto $S^3(1)$ and whatever the area of the projection is the sectional curvature of the plane spanned by X and Y.

As for scalar curvature, let z_1, z_2, z_3, z_4 be an orthonormal basis for T_pM, where $z_1, z_2, z_3 \in T_pS^3$ and $z_4 \in T_pS^1$. Then

$$K(p) = \frac{1}{12}\sum_{ij}\langle R(z_i, z_j)z_i, z_j\rangle;$$

the summands are zero whenever z_4 is involved and 1 otherwise. There are 6 terms involving z_4, so $K(p) = \frac{6}{12} = \frac{1}{2}$.

For Ricci curvature, the Ricci curvature in an arbitrary direction is 0 in the S^1 component and more in the S^3 directions. Without loss of generality, assume $v \in T_pM$ is given by $v = \left(\sqrt{1-d^2}, 0, 0, d\right)$ where the first three terms are the S^3 directions and the fourth is the S^1 direction. Complete this to an orthonormal basis: $w^1 = (0, 1, 0, 0)$, $w^2 = (0, 0, 1, 0)$, $w^3 = (-d, 0, 0, \sqrt{1-d^2})$. Then

$$\begin{aligned}
\mathrm{Ric}_p(v) &= \frac{1}{3}\left(\langle R(v, w^1)v, w^1\rangle + \langle R(v, w^2)v, w^2\rangle + \langle R(v, w^3)v, w^3\rangle\right) \\
&= \frac{1}{3}\left(\langle w_3^1, w_3^1\rangle\langle v, v\rangle - \langle v, w_3^1\rangle\langle w_3^1, v\rangle + \langle w_3^2, w_3^2\rangle\langle v, v\rangle - \langle v, w_3^2\rangle\langle w_3^2, v\rangle\right. \\
&\quad \left. + \langle w_3^3, w_3^3\rangle\langle v, v\rangle - \langle v, w_3^3\rangle\langle w_3^3, v\rangle\right) \\
&= \frac{1}{3}\left((1-d^2) - 0 + (1-d^2) - 0 + d^2(1-d^2) - d^2(1-d^2)\right) \\
&= \frac{2}{3}(1-d^2)
\end{aligned}$$

since only the parts in the S^3 direction are relevant and S^3 has constant sectional curvature.

♣

(b): What are the geodesics in this metric?

Answer: As in any product manifold, geodesics in M are of the form (γ_1, γ_3), where γ_1 is a geodesic in $S^1(1)$ and γ_3 is a geodesic in $S^3(1)$.

♣

(c): What is the first conjugate locus and cut locus of any point?

Answer: In the following schematic, the boundary of the picture forms the cut locus of the center point p, so $C_m(p) = S^3 \vee S^1$.

To determine the conjugate locus, we know that the conjugate locus of $S^3(1) \times S^1(1)$ is the image of the conjugate locus of the universal cover, $S^3(1) \times \mathbb{R}$. As we see in the following schematic picture, this means the conjugate locus of p is S^1:

♣

(C)

Visualize $\mathbb{RP}^2$ inside $\mathbb{CP}^2$.

Answer: Thinking of $\mathbb{CP}^2$ as complex lines in $\mathbb{C}^3 \simeq \mathbb{R}^6$, we know each complex line meets the unit sphere S^5 in a great circle, which gives an induced Hopf map $h : S^5 \to \mathbb{CP}^2$. Now, think of $z_i = x_i + iy_i$; then $x_1, x_2, x_3, y_1, y_2, y_3$ serve as coordinates of $\mathbb{R}^6$; if we consider the real 3-plane determined by x_1, x_2, x_3, then this 3-plane intersects the unit 5-sphere in a great 2-sphere S^2. Under the map h, antipodal points on this S^2 are identified, so $h(S^2) = \mathbb{RP}^2 \subset \mathbb{CP}^2$.

The isometries of $\mathbb{CP}^2$ certainly contain $U(3)/e^{i\theta} = SU(3)$, since $U(3)$ takes complex lines to complex lines and the circle action takes each complex line to itself. Moreover, "complex conjugation", that is, the map $(z_1, z_2, z_3) \mapsto (\bar{z_1}, \bar{z_2}, \bar{z_3})$ takes complex lines to complex lines, so we guess that the group of isometries of $\mathbb{CP}^2$ is $SU(3) \cup$ (complex conjugation). Note that this group of isometries is transitive and isotropic.

By the transitivity of the isometries of $\mathbb{CP}^2$, we can look at just a single point on $\mathbb{RP}^2$ to investigate further; let $p = (1 : 0 : 0)$. Recall from last week that $K(X, Y) = 1 + 3\cos^2\phi$ where $\cos\phi = \langle \bar{X}, i\bar{Y}$, where $\bar{X}$ and $\bar{Y}$ are horizontal lifts to TS^5 of $X, Y \in T\mathbb{CP}^2$. Specifying to the case of $\mathbb{RP}^2$, if $q \in h|_{S^2}^{-1}(p)$ (i.e. $q = (1, 0, 0)$ or $(-1, 0, 0)$), then we can just think of T_qS^2 as the 2-plane perpendicular to q (thought of as a vector in the 3-plane described above). Letting $q = (1, 0, 0)$, then T_qS^2 is spanned by $(0, 1, 0)$ and $(0, 0, 1)$. Hence,

$$\cos\phi = \langle (0, 1, 0), i(0, 0, 1)\rangle = \langle (0, 1, 0), (0, 0, i)\rangle = 0,$$

so the sectional curvature at p is $1 + 3\cos^2\phi = 1$.

Now, suppose $p \in \mathbb{CP}^2$ and $X, Y \in T_p\mathbb{CP}^2$ be orthonormal such that $K(X, Y) = 1$. Then, since the isometries of $\mathbb{CP}^2$ are transitive, there exists an isometry f of $\mathbb{CP}^2$ such that $f((1 : 0 : 0)) = p$. Since the isometries are isotropic, we may as well assume $f_* \circ h_*(0, 1, 0) = X$. Thinking of the horizontal part of T_qS^5 as the 4-plane perpendicular to the complex line associated with $h(q)$, then, since $\langle \bar{X}, i\bar{Y}\rangle = 0$, $\overline{(f^{-1})_*\bar{Y}} = \pm(0, 0, 1)$ or $\pm(0, 0, i)$. In the first case, the 2-plane spanned by X and Y is tangent to $f(\mathbb{RP}^2)$; in the second case, it's not. On the other hand, in the second case, the 2-plane spanned by X and Y is tangent to $f(\widetilde{\mathbb{RP}^2})$ where $\widetilde{\mathbb{RP}^2}$ is the $\mathbb{RP}^2$ given (as above) by intersecting the 3-plane spanned by x_1, x_2, y_3 with S^5, so it is tangent to an $\mathbb{RP}^2$.

♣

Extra Problem

A point in $\mathbb{CP}^2$ is chosen at random, and then a tangent 2-plane at the point is chosen at random. We know the sectional curvature can be any number between 1 and 4. Suppose we are told that it is either 1 or 4. Which of these two possibilities is more likely, and why?

Answer: Let $p \in \mathbb{CP}^2$. Then p corresponds to a complex line in $\mathbb{C}^3$. By a suitable rotation of $\mathbb{C}^3$, we may as well assume p corresponds to the complex line given by the first factor in $\mathbb{C}^3 = \mathbb{C} \times \mathbb{C} \times \mathbb{C}$. This plane corresponds to $(1 : 0 : 0) \in \mathbb{CP}^2$ in homogeneous coordinates, so we may as well assume $p = (1 : 0 : 0)$. Now, I like to visualize $T_p\mathbb{CP}^2$ as the set of all vectors in $\mathbb{C}^3$ perpendicular to the complex line corresponding to the point p. Hence, if $u, v \in T_p\mathbb{CP}^2$, then we can think of u, v as vectors in $0 \times \mathbb{C} \times \mathbb{C} \simeq \mathbb{C}^2$.

Now, we know from problem 12 on HW #4 that $K(u,v) = 1 + \cos^2\phi$ where $\cos\phi = \langle \bar{u}, i\bar{v}\rangle$ and $\bar{u}$ and $\bar{v}$ are the horizontal lifts of an orthonormal pair u and v, respectively, to TS^5. For $q \in S^5$, we can think of T_qS^5 as the 5-plane perpendicular to q thought of as a vector in $\mathbb{R}^6 \simeq \mathbb{C}^3$. Now, since $p = (1:0:0)$, we know that $q = (1,0,0)$ lies in the fiber over p and, in the above interpretation,

$$T_qS^5 = \{(iy_1, x_2 + iy_2, x_3 + iy_3) \in \mathbb{C}^3\}.$$

The vertical part of T_qS^5 is just $\{(iy_1, 0, 0)\}$ and the horizontal part is $\{(0, x_2+iy_2, x_3+iy_3)\}$. Hence, for $u, v \in T_p\mathbb{CP}^2$ thought of as $(u_1,u_2),(v_1,v_2) \in \mathbb{C}^2$, $\bar{u} = (0, u_1, u_2)$ and $\bar{v} = (0, v_1, v_2)$.

Now, suppose $u, v \in T_p\mathbb{CP}^2$ are an orthonormal pair such that $K(u,v) = 1$ or 4, so either $\langle \bar{u}, i\bar{v}\rangle = \cos\phi = 0$ or $\langle \bar{u}, i\bar{v}\rangle = \cos\phi = 1$. In other words, either u is parallel to iv in the above visualization, or u is perpendicular to iv. Since the space of vectors parallel to iv is 1-dimensional while the space of vectors perpendicular to iv is 3-dimensional, we expect that the latter should be much more likely and so $K(u,v) = 1$ should be more likely than $K(u,v) = 4$.

To see this rigorously, suppose $u = (a_1 + ib_1, a_2 + ib_2)$. If $v = \pm iu$, then

$$\begin{aligned}\cos\phi = \langle \bar{u}, i\bar{v}\rangle &= \langle \bar{u}, \mp\bar{u}\rangle \\ &= \mp 1.\end{aligned}$$

On the other hand, supposing $b_2 \neq 0$ (if it is, perform the below calculation with $v = \pm(0,i)$), then if we let $v = \pm\frac{1}{1+\frac{b_1^2}{b_2^2}}\left(1, -\frac{b_1}{b_2}\right)$, we see that

$$\begin{aligned}\cos\phi = \langle \bar{u}, i\bar{v}\rangle &= \left\langle (a_1, b_1, a_2, b_2), \pm\frac{1}{1+\frac{b_1^2}{b_2^2}}\left(0, 1, 0, -\frac{b_1}{b_2}\right)\right\rangle \\ &= \pm\frac{1}{1+\frac{b_1^2}{b_2^2}}(b_1 - b_1) \\ &= 0.\end{aligned}$$

Also, supposing $a_2 \neq 0$ (if it is, perform the below calculation with $v = \pm(0,1)$), then if we let $v = \pm\frac{1}{1+\frac{a_1^2}{a_2^2}}\left(i, -i\frac{a_1}{a_2}\right)$, we see that

$$\begin{aligned}\cos\phi = \langle \bar{u}, i\bar{w}_2\rangle &= \left\langle (a_1, b_1, a_2, b_2), \pm\frac{1}{1+\frac{a_1^2}{a_2^2}}\left(-1, 0, \frac{a_1}{a_2}, 0\right)\right\rangle \\ &= \pm\frac{1}{1+\frac{a_1^2}{a_2^2}}(-a_1 + a_1) \\ &= 0.\end{aligned}$$

The above three possibilities for v are pairwise orthonormal, so any vector orthonormal to u in $T_p\mathbb{CP}^2$ is a linear combination of them. Hence, we see that the only $v \in T_p\mathbb{CP}^2$ for which $K(u,v) = 4$ are $v = \pm iu$, whereas $K(u,v) = 1$ for any norm 1 linear combination of $\frac{1}{1+\frac{b_1^2}{b_2^2}}\left(1, -\frac{b_1}{b_2}\right)$ and $\frac{1}{1+\frac{a_1^2}{a_2^2}}\left(i, -i\frac{a_1}{a_2}\right)$. Thus, we conclude that, if we choose a tangent 2-plane to p, then it is much more likely that the sectional curvature at that 2-plane is 1 than 4.

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DRL 3E3A, University of Pennsylvania
E-mail address: shonkwil@math.upenn.edu