DIFFERENTIAL GEOMETRY HW 9

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2.

Prove the following inequality on real functions (Wirtinger's inequality). Let $f : [0, \pi] \to \mathbb{R}$ be a real function of class C^2 such that $f(0) = f(\pi) = 0$. Then

$$\int_0^\pi f^2 dt \leq \int_0^\pi (f')^2 dt,$$

and equality occurs if and only if $f(t) = c \sin t$, where c is a constant.

Proof. For a geometric solution, let γ be a normalized geodesic joining p and $-p$ on the unit sphere S^2. Let $v(t)$ be a parallel field along γ with $\langle v, \gamma' \rangle = 0$ and $|v| = 1$. Let $V = fv$. Then the index form

$$\begin{aligned} I_\pi(V, V) &= \int_0^\pi \left[\langle V', V' \rangle - \langle R(\gamma', V)\gamma', V \rangle \right] dt \\ &= \int_0^\pi (f')^2 |v| dt - \int_0^\pi f^2 \langle R(\gamma', v)\gamma', v \rangle dt \\ &= \int_0^\pi (f')^2 dt - \int_0^\pi f^2 dt. \end{aligned}$$

Since $\gamma(\pi)$ is the first conjugate point to $\gamma(0)$, there are no conjugate points between $\gamma(0)$ and $\gamma(\pi)$, so, by the Morse Index Theorem, $I_\pi(V, V) \geq 0$. Hence, $\int_0^\pi f^2 dt \leq \int_0^\pi (f')^2 dt$. Moreover, only if $I_\pi(V, V) = 0$, which is true if and only if V is a Jacobi field. But the Jacobi fields on the sphere are precisely of the form $\sin tw$ for $|w| = 1$, so we see that equality in the above inequality implies that $f = c \sin t$ for some constant c.

Alternatively, we can prove the above result using Fourier series. Let $\sum a_n e^{int}$ be the Fourier representation of f. Then

$$f'(t) = \sum a_n i n e^{int}.$$

Then

$$\int_0^\pi f^2 dt = \int_0^\pi \sum a_n e^{int} dt = \sum |a_n|^2$$

since the e^{int} are L^p-orthonormal. On the other hand,

$$\int_0^\pi (f')^2 dt = \int_0^\pi \sum a_n i n e^{int} dt = \sum |n a_n|^2,$$

which makes it clear that $\int_0^\pi f^2 dt \leq \int_0^\pi (f')^2 dt$. If equality holds, then $a_n = 0$ for $n \neq \pm 1$, so

$$f(t) = a_1 e^{it} + a_{-1} e^{-it} = b \sin t + c \cos t$$

for some b and c. However, since $f(0) = 0$, $c = 0$, so $f(t) = b \sin t$. $\square$

3.

Let M^n be a complete simply connected Riemannian manifold. Suppose that for each point $p \in M$, the locus $C(p)$ of (first) conjugate points of p reduces to a unique point $q \neq p$ and that $d(p, C(p)) = \pi$. Prove that, if the sectional curvature K of M satisfies $K \leq 1$, then M is isometric to the sphere S^n with constant curvature 1.

Proof. Let J be a Jacobi field along a normalized geodesic $\gamma : [0, \pi] \to M$ joining p to q with $J(0) = J(\pi) = 0$ and $\langle J, \gamma' \rangle = 0$. Since $|\gamma'(0)| = 0$, we can complete to an orthonormal basis for the tangent space $e_1, \ldots, e_{n-1}, \gamma'$. Parallel transport to get fields $e_i(t)$. Then $J = \sum_{i=1}^{n-1} a_i e_i$. Define $K(t) = K(\gamma'(t), J(t))$. Then, since J is a Jacobi field,

$$\begin{aligned}
0 = I_\pi(J, J) &= \int_0^\pi \left[\langle J', J' \rangle - \langle R(\gamma', J)\gamma', J \rangle \right] dt \\
&= \int_0^\pi \sum_{i=1}^{n-1} (a_i')^2 dt - \int_0^\pi K(t) \sum_{i=1}^{n-1} a_i^2 dt \\
&\geq \sum_{i=1}^{n-1} \int_0^\pi a_i^2 (1 - K(t)) dt \\
&\geq 0,
\end{aligned}$$

where the first inequality follows from problem 2 above. Therefore, it follows that $\int_0^\pi a_i^2 (1 - K(t)) dt = 0$ for all i, meaning that $K(t) \equiv 0$. Therefore, since M is simply connected, M is the round sphere. $\square$

4.

Let $a : \mathbb{R} \to \mathbb{R}$ be a differentiable function with $a(t) \geq 0$, $t \in \mathbb{R}$ and $a(0) > 0$. Prove that the solution to the differential equation

$$\frac{d^2 \phi}{dt^2} + a\phi = 0$$

with initial conditions $\phi(0) = 1$, $\phi'(0) = 0$, has at least one positive zero and one negative zero.

Proof. Suppose the solution ϕ does not have a positive zero. Then $\phi(t) > 0$ for all $t > 0$ (since $\phi(0) = 1$). Hence, for $t > 0$,

$$\phi'(t) = \int_0^t \phi''(t) dt = -\int_0^t a\phi dt \leq 0.$$

Thus, $\phi(t) \leq \phi(0) = 1$. Since $\phi(t) \neq 0$ for $t > 0$, this implies, by the mean value theorem, that $\phi'(t)$ gets arbitrarily close to zero for large t. On the other hand, since $a(0) \neq 0$, there exists $\epsilon > 0$ such that $\phi(t)$ and $a(t)$ are both strictly positive for $0 \leq t < \epsilon$. If we let $c = -\int_0^\epsilon a\phi dt$, then we see that $\phi'(t) \leq c < 0$ for $t > \epsilon$. From this contradiction, then, we see that ϕ does have at least one positive zero.

Similarly, suppose ϕ does not have a negative zero. Then $\phi(t) > 0$ for all $t < 0$ and so, for $t < 0$,

$$\phi'(t) = \int_0^t \phi''(t)dt = -\int_0^t a\phi dt = \int_t^0 a\phi dt \geq 0.$$

Thus, $\phi(t) \leq \phi(0) = 1$. Again, by the mean value theorem, $\phi'(t)$ gets arbitrarily close to zero for large negative t, but $\phi(t)$ is bounded away from zero outside an ϵ ball where a and ϕ are both positive.

Therefore, we conclude that ϕ has at least one positive zero and one negative zero. □

5.

Suppose that M^n is a complete Riemannian manifold with sectional curvature strictly positive and let $\gamma : (-\infty, \infty) \to M$ be a normalized geodesic in M. Show that there exists $t_0 \in \mathbb{R}$ such that the segment $\gamma([-t_0, t_0])$ has index greater or equal to $n - 1$.

Proof. Let V be a parallel field along γ with $\langle \gamma', V\rangle - 0$ and $|V| = 1$. Let

$$\phi_V = \langle R(\gamma', V)\gamma', V\rangle$$

and

$$K(t) = \inf_V \phi_V(t).$$

Let $a : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that

$$0 \leq a(t) \leq K(t), \quad a < a(0) < K(0), \quad t \in \mathbb{R}.$$

Let ϕ be a solution of $\phi''+a\phi = 0$ with $\phi'(0) = 0$, $\phi(0) = 1$, and let $-t_1$, t_2 be the two zeros guaranteed by problem 4 above. Let $X = \phi V$. If $t \in [-t_1, t_2]$,

$$\begin{aligned}
I_t(X,X) &= -\int_{-t_1}^{t} \langle X'' + R(\gamma', X)\gamma', X\rangle dt \\
&= -\int_{-t_1}^{t} \langle(\phi''V + 2\phi'V' + \phi V'') + \phi R(\gamma', V)\gamma', \phi V\rangle dt \\
&\leq -\left[\int_{-t_1}^{t} (\phi'' + K(t)\phi)\phi dt + \int_{-t_1}^{t} \langle 2\phi'V' + \phi V'', \phi V\rangle dt\right] \\
&< -\int_{-t_1}^{t} (\phi'' + a\phi)\phi dt - \int_{-t_1}^{t} \langle 2\phi'V' + \phi V'', \phi V\rangle dt \\
&\leq -\int_{-t_1}^{t} (\phi'' + a\phi)\phi dt \\
&\equiv 0.
\end{aligned}$$

Since this holds for any such choice of V and there are $n-1$ linearly independent choices for V, we see that $\gamma([-t_1, t_2])$ has index at least $n-1$. If we let $t_0 = \min\{t_1, t_2\}$, then $\gamma([-t_0, t_0])$ has index at least $n-1$, as desired. □

6.

A *line* in a complete Riemannian manifold is a geodesic

$$\gamma : (-\infty, \infty) \to M$$

which minimizes the arc length between any two of its points. Show that if the secional curvature K of M is strictly positive, M does not have any lines. By an example show that the theorem is false if $K \geq 0$.

Proof. Let M^n be a complete Riemannian manifold with strictly positive curvature, and suppose $\gamma : (-\infty, \infty) \to M$ is a line. Then, by problem 5 above, there exists $t_0 \in \mathbb{R}$ such that $\gamma([-t_0, t_0])$ has index greater than or equal to $n-1$. By the Morse Index Theorem, then, there must be at least one conjugate point in $\gamma((-t_0, t_0))$. However, Jacobi's Theorem says that γ is minimizing on $[-t_0, t_0]$ only if there are no conjugate points in $(-t_0, t_0)$. From this contradiction, then, we conclude that there can be no lines on M.

This theorem is clearly false if $K \geq 0$, as we can see by letting $M = \mathbb{R}^2$. Here $K = 0$ and each straight line in the plane is a line in the sense defined above. □

(B)

Let a, b and c be three points in that order along a geodesic γ in a Riemannian manifold M. Suppose that a is conjugate to b and that also a is conjugate to c. Must b be conjugate to c?

Partial Answer: I haven't been able to construct a good counterexample, but I believe this is false in general for dimension 3 and higher.

In dimension 2 it is trivially true, since there is only 1 dimension of Jacobi fields orthogonal to γ.

Here is how I would go about constructing a counter-example in dimension 3: Let γ be some geodesic in, say, S^3 and let V_1, V_2 be unit vectors orthogonal to $\gamma'(0)$ and to each other. Let $V_i(t)$ be the parallel transport of V_i along γ. Deform the metric on S^3 so that $K(\gamma', V_1) \equiv 1$ while $K(\gamma', V_2) \equiv 1 + \epsilon$ for some small ϵ. Then let J_i be the Jacobi fields associated to V_i defined in the usual way: $J_i(t) = \frac{\sin(\sqrt{K}t)}{\sqrt{K}} V_i(t)$. Let $b = \gamma\left(\frac{\pi}{\sqrt{1+\epsilon}}\right)$ and $c = \gamma(\pi)$. Then $a = \gamma(0)$ is conjugate to both b and c. However, $d(b, c) = \pi - \frac{\pi}{\sqrt{1+\epsilon}} =: \delta$, which is small. Since the sectional curvature of this manifold is bounded above by, say, 2, we know, by Rauch, that the distance between conjugate points is at least $\frac{\pi}{\sqrt{2}}$. So long as ϵ (and, hence, δ) is small, this means that b and c cannot be conjugate.

Here's a picture:

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