Final Exam, Question 9(b)

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Question 9(b): Consider $g : \mathbb{R}^2 \to \mathbb{R}$ defined by:

$$g(x, y) = \begin{cases} \frac{x^2y^2}{x^4+y^4}, & (x, y) \neq (0, 0); \\ 0 & (x, y) = (0, 0). \end{cases}$$

Show that the partial derivatives g_x and g_y both exist at $(0, 0)$. What are their values at $(0, 0)$?

Answer: We know that the partial derivative g_x is defined as

$$g_x(x_0, y_0) = \lim_{h \to 0} \frac{g(x_0 + h, y_0) - g(x_0, y_0)}{h}.$$

So, to show the partial derivative exists at $(0, 0)$, we have to show that the above limit exists at $(0, 0)$ i.e. when $(x_0, y_0) = (0, 0)$.
Now, we know that if $x \neq 0$ and $y = 0$,

$$\begin{aligned} g(x, y) &= \frac{x^2y^2}{x^4 + y^4} \\ &= \frac{0}{x^4} \quad \text{(because } y = 0\text{)} \\ &= 0. \end{aligned}$$

So, we know that $g(x, 0) = 0$ if $x \neq 0$.

Thus, if we have both $x_0 = 0$ and $y_0 = 0$, which is our desired point, we have

$$\begin{aligned} g_x(0,0) &= \lim_{h\to 0} \frac{g(0+h,0) - g(0,0)}{h} \\ &= \lim_{h\to 0} \frac{g(h,0) - g(0,0)}{h} \\ &= \lim_{h\to 0} \frac{0-0}{h} \\ &= \lim_{h\to 0} \frac{0}{h} \\ &= \lim_{h\to 0} 0 \\ &= 0. \end{aligned}$$

So, the partial derivative g_x exists at $(0, 0)$ and has value 0. Similarly, one can show g_y exists and also has value 0.