Period Domains

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1 Carlson 1 - Period Domains

Period domains are parameter spaces for marked Hodge structures. We call $\Gamma\backslash D$ the period space, which is a parameter space of isomorphism classes of Hodge structures.

Let $X \to S$ be a morphism of varieties, X_s the fiber over s and Δ the subset of S where the fibers are singular. Then the base change to $S \setminus \Delta$ is a smooth family. This induces a period map $S \setminus \Delta \to \Gamma\backslash D$ by $s \mapsto H^n(X_s)$ with its Hodge structure. Really we lift to the universal cover, which maps into D and associates to $X_{\tilde{s}}$ its marked Hodge structure.

(Review of what a Hodge structure is, and then Primitive Cohomology)

So what is a marking on a Hodge structure? Take a Hodge structure $H_{\mathbb{Z}}$ and $H_{\mathbb{C}} = \oplus_{p+q=k} H^{p,q}$ with polarization Q. The lattice $H_{\mathbb{Z}}$ is isomorphic to $\mathbb{Z}^r$ for some r. A marking is an isomorphism $\mathbb{Z}^r \xrightarrow{m} H_{\mathbb{Z}}$ and Q_0 on $\mathbb{Z}^r$ such that m is an isometry.

Now, we set for curves, that is, rank 1 Hodge structures, $D = \{m^{-1}(H^{1,0}) \subset \mathbb{C}^{2g}\} \subset \{S \subset \mathbb{C}^{2g}\} = G(g, 2g)$ and the more general definition is similar.

Now let Γ be the isometry group of Q_0, and it is a discontinuous group, and $\Gamma\backslash D$ is an analytic space.

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So our set up is that D is a period domain, it's the set of $m^{-1}(H^{p,q})$'s, and we have a marked polarized Hodge structure H. We really want to describe the period domain in terms of the Hodge filtration.

Example 2.1. *In the weight 1 case, $D = \{m^{-1}(F^1) \subset \mathbb{C}^{2g}\} \subset G(g, 2g)$, and it needs to satisfy $Q(F^1, F^1) = 0$ so it is contained in a proper subvariety, but then there's also $i^{p-q}Q(x, \bar{x}) > 0$ where $x \in H^{p,q} \setminus 0$, so there's some open conditions. So $D \subset \check{D} \subset G(g, 2g)$.*

Example 2.2. *In the weight 2 case, we first note that for polarized Hodge structures, then F^2 determines F^1. So we have $D \subset \check{D} \subset G(p, 2p+q)$ where $p = \dim H^{2,0}$ and $q = \dim H^{1,1}_0$.*

Let's look at elliptic curves. Set $\mathscr{E} = \{(x,y)|y^2 = p(x)\}$ where $p(x)$ is a cubic with distinct roots. For example, $x(x-1)(x-t)$, so long as $t \neq 0, 1$ we get an elliptic curve, which is a nonsingular Riemann surface of genus 1.

The unique (up to scaling) abelian differential is $\omega = \frac{dx}{\sqrt{p(x)}} = \frac{dx}{y}$.

Exercise 2.3. *Show that ω is holomorphic at $x = 0, 1, t, \infty$.*

So, we can write $H^1(\mathscr{E}, \mathbb{C}) = \mathbb{C}\frac{dx}{y} \oplus \mathbb{C}\frac{d\bar{x}}{y}$. We define a marking $m : \mathbb{Z}^2 \to H^1(\mathscr{E}, \mathbb{Z})$, picking out a symplectic basis for the cohomology. Take the basis to be δ, γ, and the dual basis δ^*, γ^*.

We know that $\omega = A\delta^* + B\gamma^*$, and that $i\int_M \omega \wedge \bar{\omega} > 0$ and so $i(\omega \cup \bar{\omega})[M] > 0$, which means that $i(A\delta^* + B\gamma^*) \cup (\bar{A}\delta^* + \bar{B}\gamma^*) > 0$, and so we get that $i(A\bar{B} - B\bar{A}) > 0$.

Consequence of Riemann Bilinear relation is that $A \neq 0, B \neq 0$. So we can rescale ω so that the A-period is 1. So the period matrix $[AB]$ because $[1; Z]$. What can we say about Z? The relation is now $i(\bar{B} - B) > 0$, so we have that $\mathrm{Im}(Z) > 0$.

So we now know that $D = \mathbb{H}_1 = \{Z|\mathrm{Im}(Z) > 0\}$.

2.1 Period Map

Let $f : X \to S$ a morphism and f of maximal rank on $S \setminus \Delta$. The smooth part of the map is locally differentiably trivial. The period map is the map $p : \widetilde{S \setminus \Delta} \to D$ and also the map it induces $p : S \setminus \Delta \to \Gamma \backslash D$.

Look at the family $y^2 = x(x-1)(x-t)$ of elliptic curves. We claim that the period is defined and holomorphic on an open set away from $0, 1$ in $\mathbb{C}$. Take our marking to pick the basis δ, γ with intersection $+1$. Then $A(t) = \int_\gamma \frac{dx}{\sqrt{x(x-1)(x-t)}}$ and $B(t) = \int_\gamma \frac{dx}{\sqrt{x(x-1)(x-t)}}$.

These are integrals with fixed domains of integration, so we can take the $\bar{t}$ derivative of the inside to check holomorphicisty, and it's clear that the A-period is holomorphic, and so is the B-period for the same reasons.

So the period map is $Z(t) = B(t)/A(t)$. So we have $\tilde{P} : U \to D$ the period map. And then there's $\Gamma = \mathrm{Aut}(H^1(M, \mathbb{Z})) \cong \mathrm{Sp}(2, \mathbb{Z})$.

Exercise 2.4. *Show that* $\mathrm{Sp}(2, \mathbb{Z}) \cong \mathrm{SL}(2, \mathbb{Z})$.

So, we get a map $\mathbb{C} \setminus \{0, 1\} \to \Gamma \backslash \mathbb{H}$.

Let F be a fundamental region for Γ. Almost al of $\mathbb{H}$ is given by $\Gamma \cdot F$. No two points of F are equivalent under Γ, so if $x, y \in F$, then there is no g such that $gx = y$.

The space $\Gamma \backslash \mathbb{H}$ is an orbifold, not quite a manifold, it has a few special points.

Reference for Modular Forms: Serre's Course in Arithmetic.

Set $g_2(Z) = 60 \sum_{(m,n) \neq 0} \frac{1}{(m+nZ)^4}$ and say it has weight 4 and set $g_3(Z) = 140 \sum_{(m,n) \neq 0} \frac{1}{(m+nZ)^6}$ of weight 6. These are modular forms, they're not invariant under Γ, but transform in a controlled way. Define a polynomial in them by $\Delta = g_2^3 - 27g_3^2$ and set $j(Z) = \frac{g_2^3}{\Delta}$. We call Δ the discriminant.

If the elliptic curve is $y^2 = 4x^3 - g_2 x - g_3$ (and all complex elliptic curves can be put in this form by a change of variables) then $\Delta = 0$ if and only if the curve is singular.

Theorem 2.5. *The function j gives a biholomorphic map $j : \Gamma\backslash\mathbb{H} \to \mathbb{C}$.*

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3.1 Monodromy

Let $X \to S$, Δ the discriminant and X_s the fiber over $s \in S \setminus \Delta$ where X_s is nonsingular.

Let's look at the example $y^2 = x(x-1)(x-t)$ then $\Delta = \{0, 1, \infty\}$. Set $\phi_t(x)$ to be the flow where $\phi_t : X_0 \to X_t$ is a diffeomorphism. Then set $\rho(\gamma) = \phi_{1*} : H_n(X_0, \mathbb{Z}) \to H_n(X_0, \mathbb{Z})$, and so we get $\rho : \pi_1(S \setminus \Delta, 0) \to \mathrm{Aut}(H_n(X_0, \mathbb{Z}))$ the monodromy representation, and $\rho(\gamma) \in \mathrm{GL}(r, \mathbb{Z})$.

For n odd it preserves a skew-symmetric form, for n even a symmetric form, so we have $\mathrm{Sp}(r, \mathbb{Z})$ or $\mathrm{SO}(Q, \mathbb{Z})$.

And this gives $\tilde{f} : \widetilde{S \setminus \Delta} \to D$, and we can descend this to the quotient $S \setminus \Delta \to \Gamma\backslash D$ where Γ is at least as big as the image of the monodromy, at at most $\mathrm{Aut}(H_n(X, \mathbb{Z}))$.

To get a specific example we can compute, let's look at the family $y^2 = x^2 - t$. Let $t = \epsilon e^{i\theta}$. Following around this loop, we see that its image is $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and it's a Dehn twist.

For the family of cubics, we have a map $\rho : \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \to \mathrm{SL}(2, \mathbb{Z})$, in general, the image of ρ is large.

Beauville showed that for hypersurfaces, ρ is surjective or has finite index.

3.2 Asymptotics of the Period Map

Let $y^2 = x(x-1)(x-t)$. Recall that $\omega = \frac{dx}{y} = \frac{dx}{\sqrt{x(x-1)(x-t)}}$, and $A(t)$ and $B(t)$ as yesterday. We want to do the asymptotics of $A(t)$ as $t \to 0$, we get $\int_\delta \frac{dx}{x}\frac{1}{\sqrt{-t}} = \frac{2\pi}{\sqrt{t}}$, and for $B(t)$ we take $t >> 2$ and get $-2\int_1^t \frac{dx}{x\sqrt{x-t}} = \frac{4}{\sqrt{t}} \arctan \frac{\sqrt{1-t}}{\sqrt{t}}$, and this simplifies to $\frac{2i}{\sqrt{t}} \log t$ by using trig identities.

So the A-period is $2\pi i t^{-1/2}$ and the B-period is $2it^{-1/2} \log t$, and the period map is then $Z(t) = \frac{i}{\pi} \log t$.

$Z_{ij}(t) = t^a (\log t)^b$. For algebraic surfaces, $b \leq 2$. In general, $b \leq$ weight of the Hodge structure.

3.3 Period Domains and Hodge structures of weight 1

First, note that the number of moduli is $3g-3$ unless $g=1$, where it is 1.

Definition 3.1 (Abelian Variety)**.** *An abelian variety is a complex torus $\mathbb{C}^g/\Lambda$ thatis a projective variety.*

The moduli of abelian varieties of dimension g is $g(g+1)/2$.

Now, given a weight 1 Hodge structure, $J(H) = H_{\mathbb{C}}/H^{1,0} + H_{\mathbb{Z}}$ is a projective variety, by the Riemann Bilinear Relations and the Kodaira Embedding Theorem.

The Torelli Theorem says that the map $M \mapsto J(H^1(M))$ is injective, so we get a map $\mathcal{M}_g \to \mathcal{A}_g$, and finding equation for this image is called the Schottky problem.

What is D? In the general case of weight 1. We write down a matrix (A, B) of A-periods and B-periods, with the basis $\omega_1, \ldots, \omega_g$ a basis for $H^{1,0}$ and $\delta_1, \ldots, \delta_g, \gamma_1, \ldots, \gamma_g$ a symplectic basis for $H_1(\mathbb{Z})$.

We define $A_{ij} = \int_{\delta_j} \omega_i$ and $B_{ij} = \int_{\gamma_j} \omega_i$. The Riemann Bilinear Relations tell us that $i\int_M \omega \wedge \bar{\omega} > 0$ for ω a nonzero abelian differential, $\omega = \sum v_m \omega_m$ and $\omega_m = \sum A_{mi}\delta^i + \sum B_{nj}\gamma^j$.

Putting this all together, we get that we get that $i(v_m A_{mi}\bar{v}_n \bar{B}_{ni} - v_m B_{mj}\bar{v}_n \bar{A}nj) > 0$, which is just $iv(AB^* - BA^*)\bar{V} > 0$, and A, B nonsingular matrices. So we can find a basis such that the period matrix is $(1, Z)$. The first RB relation gives that $Z = Z^t$.

So, for weight 1, $D = \mathbb{H}_g = \{Z | Z^t = Z, \mathrm{Im}(Z) > 0\}$.

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Just a few more things about weight 1 before going on to weight 2.

$D = \mathbb{H}_g = \{Z | Z^t = Z, \mathrm{Z} > 0\}$. Another way to view it is that $\mathbb{H}_1 = \mathrm{SL}(2,\mathbb{R})/U(1)$. We can see this by noting that $\mathrm{SL}(2,\mathbb{R})$ acts transitively on $\mathbb{H}_1$, and the subgroup fixing i is $U(1)$.

More generally, $\mathbb{H}_g$ has a transitive action by $\mathrm{Sp}(2g,\mathbb{R})$. Let K be the isotropy group for H, then $\mathbb{H}_g = G/K = \mathrm{Sp}(2g,\mathbb{R})/U(g)$, and this is an example of a hermitian symmetric space, though higher weight period domains generally aren't.

4.1 Higher weight

Let H be a Hodge structure of weight k, D the period domain, then G is $\mathrm{Sp}(n,\mathbb{R})$ or $\mathrm{SO}(Q,\mathbb{R})$, and acts transitively on D, and let V be the isotropy group of H

In the weight 2 case, if $p = \dim H^{2,0}$ and $q = \dim H^{1,1}$, then $G = \mathrm{SO}(2p, q)$ and $V = U(p) \times \mathrm{SO}(q)$ with K, the maximal compact, $\mathrm{SO}(2p) \times \mathrm{SO}(q)$.

D is not in general a Hermitian symmetric space, but it is when $K = V$. For instance, if $p < h^{2,0} = 1$.

For instance, K3 surfaces.

Fact: K/V is compact complex subvariety of D.

Period map for hypersurfaces $X \subset \mathbb{P}^{n+1}$ has a holomorphic part and a horizontal part, F^p and F^{p-1}.

Poincaré Residue: Look at the cohomology of $\mathbb{P}^{n+1}\backslash X$. Grothendieck looked at this cohomology and its meromorphic forms with pole along X.

Let $z_0, \ldots, z_{n+1}$ be homogeneous coordinates on $\mathbb{P}^{n+1}$, let U_0 be $z_0 \neq 0$, then we have affine coordinates by dividing by z_0, and so $d(z_1/z_0) \wedge \ldots \wedge d(z_{n+1}/z_0) = \frac{\sum (-1)^i z_i dz_0 \wedge \ldots \wedge \hat{dz_i} \wedge \ldots \wedge dz_{n+1}}{z_0^{n+2}}$. Set the numerator as Ω.

Define $\Omega_A = A\Omega/Q^2$ where Q is the equation of X with $\deg A + n + 2 = 2 \deg Q$, so then $[\Omega]$ is acohomology class in $H^{n+1}(\mathbb{P}^{n+1} \setminus X)$.

We have a sequence $H^{n+1}(\mathbb{P}^n) \to H^{n+1}(\mathbb{P}^{n+1}\backslash X) \to H^n(X) \to H^{n+2}(\mathbb{P}^{n+1}$ so we define the local residue map in the usual way.

Claim: The period map is both holomorphic and horizontal.

To see that it's holomorphic, let X_t be given by $Q+tR = 0$. Then $\frac{\partial}{\partial t}\left(\frac{A\Omega}{(Q+tR)^2}\right) = 0$, so $\frac{\partial}{\partial t}\frac{1}{2\pi i}\int_\gamma \frac{A\Omega}{(Q+tR)^2} = 0$ and by the residue formula, the map is holomorphic. Horizontality then follows from Griffiths Transversality.

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Recall that $H^{n+1}(\mathbb{P}^{n+1} \setminus X) = \{A\Omega/Q^\ell\}$ the set of meromorphic forms with a pole at X. By the adjoint of the tube map, we have a residue map to $H^n(X)$.

This map is holomorphic, because if we take $\frac{\partial}{\partial t}\left(\operatorname{res}\frac{A\Omega}{Q^\ell}\right) = 0$, and for Horizontality, we take $\frac{\partial}{\partial t}\left(\operatorname{res}\frac{A\Omega}{(Q+tR)^\ell}\right) = -\ell \operatorname{res}\frac{RA\Omega}{(Q+tR)^{\ell+1}}$, and this is essentially Griffiths transversality for hypersurfaces.

Now, let D be a period domain of filtrations. It's an open subset of a closed subset of a product of Grassmannians $\prod_p Gr(F^p, \mathbb{C}^n)$. What are the tangent spaces of a Grassmannian?

Look at $Gr(r, \mathbb{C}^N)$. For $S \in Gr(r, \mathbb{C}^N)$, we have that $T_S Gr(r, \mathbb{C}^N) = \hom(S, \mathbb{C}^N/S)$.

So in general we have that $\frac{d}{dt}F_t^p \in \hom(F^p, \mathbb{C}^N/F^p)$, and this lies in $\hom(F^p, F^{p-1}/F^p)$.

Now take $G = G/V$ where G is $\mathrm{Sp}(n, \mathbb{R})$ or $\mathrm{SO}(Q, \mathbb{R})$, and $\mathfrak{g} = Lie(G)$, then $\mathfrak{g}_\mathbb{C} \subset \hom(H_\mathbb{C}, H_\mathbb{C})$ which is a HS of weight 0, inducing one on $\mathfrak{g}_\mathbb{C}$.

Let S be a V-module, these correspond to homogeneous vector bundles on G/V by taking $G \times_V S$, $\mathfrak{g}^-$ corresponds to the Holomorphic tangent bundle of $D = T_{hol}D$ and $\mathfrak{g}^{-1,1}$ is the horizontal part, and a family $\{X_t\}$ gives a family of Hodge structures, $H^n(X_t) \in \mathfrak{g}^{-1,1}$.

So then we have $D = G/V \to G/K$ with fiber K/V compact complex, and G/K a symmetric space. Then we set $\mathfrak{k}_\mathbb{C} = \oplus_{p \text{ odd}} \mathfrak{g}^{p,-p}$ and $\mathfrak{p}_\mathbb{C} = \oplus_{p \text{ even}} \mathfrak{g}^{p,-p}$.

5.1 Curvature of D

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It is fairly well known that $\mathbb{H}_g$ has holomorphic sectional curvature negative and bounded above by a negative constant. On D more general, the sectional curvatures can be positive or negative: the positive ones correspond to $\mathfrak{g}^{p,-p}$ for p even, and for p odd we get negative directions, and especially important is $\mathfrak{g}^{-1,1}$.

Distance Decreasing Principal: Let $f : M \to D$ a holomorphic, horizontal, negatively curved and bounded away from zero manifold, then this map is distance decreasing. Specifically, $f^* ds^2_D \leq ds^2_M$.

If M is a disk of radius R, then $ds^2_R = \frac{R^2 dz d\bar{z}}{(R^2 - |z|^2)}$.

Theorem 5.1. *A nontrivial variation of Hodge structures has at least 3 singularities.*

For elliptic curves, assume not. then we have $\mathbb{C}^* \to \mathbb{H}/\Gamma$, this lifts to the universal cover $\mathbb{C} \to \mathbb{H}$ the unit disc, and this gives an entire holomorphic function, which is a contradiction.

In generality, we look at $\mathbb{C}^* \xrightarrow{f} D/\Gamma$ and $\mathbb{C} \xrightarrow{\tilde{f}} D$. Now, we have $\tilde{f}^* ds^2_D \leq ds^2_R$ This is then $C dz d\bar{z} \leq \frac{dz d\bar{z}}{R^2}$ for $C \neq 0$ and in fact $C > 0$, and so $C \leq \frac{1}{R^2}$, which is a contradiction.

Application 2: The eigenvalues of a monodromy transformation T are roots of unity. Let f map the unit disc to D/Γ and $\tilde{f} : \mathbb{H} \to D$. We'll look at in. Then $d_D(\tilde{f}(in), \tilde{f}(\gamma \cdot in)) \leq d_{\mathbb{H}}(in, in+1)$. Now, $ds^2_{\mathbb{H}} = \frac{dx^2 + dy^2}{y^2}$, so the RHS is $\frac{1}{n}$.

We have $\tilde{f}(\gamma x) = \rho(\gamma)\tilde{f}(x)$, which rewrites LHS as $d_D(\tilde{f}(\gamma in), \rho(\gamma)\tilde{f}(in))$, and this is just gV when we write it as G/V, and so we have $d_D(g_n V, \rho(\gamma) g_n V) \leq \frac{1}{n}$, but we can use the homogeneous nature of our space to write $d_D(V, g_n^{-1}\rho(\gamma) g_n V) \leq \frac{1}{n}$.

So, the conjugacy class of $\rho(\gamma)$ has a limit point in V (compact), os the eigenvalues of $\rho(\gamma)$ have absolute value 1, and are algebraic integers, so then a theorem of Kronecker that implies that they are roots of unity.