Variation of HS's, degenerations of HS's

Cattani

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1 Cattani 1 - Odds and ends

1.1 Riemann-Hodge Relations

Let $\alpha, \beta \in H^{n-k}$. We have $\alpha \cup \beta \in H^{2n-2k}$, and we define a number using $(-1)^{(n-k)(n-k+1)/2} \int_M \alpha \cup \beta \cup \omega^k$, then the Riemann-Hodge relations say that this gives a polarization on $H_0^{n-k}(M, \mathbb{C})$. We'll denote this by $Q(\alpha, \beta)$.

In dimension one, we have $H^1(M, \mathbb{C}) = H^{1,0} \oplus H^{0,1}$, and then $Q(\alpha, \beta) = \int_M \alpha \wedge \beta$, and $iQ(\alpha, \bar{\alpha}) > 0$ if $0 \neq \alpha$.

Subset: $G(g, H^1(M, \mathbb{C}))$, example 1.16.

1.2 Connections

Look at the sequence $0 \to \mathbb{Z} \to \mathscr{O}_X \to \mathscr{O}_X^* \to 0$, it gives the long exact sequence with the maps $H^1(X, \mathscr{O}_X) \to H^1(X, \mathscr{O}_X^*) \to H^2(M, \mathbb{Z})$, with the last map taking $\{g_{\alpha\beta}\}$ to $n_{\alpha\beta\gamma} = \frac{1}{2\pi i}(\log g_{\alpha\beta} - \log g_{\alpha\gamma} + \log g_{\beta\gamma}$.

We have $0 \to \mathbb{R} \to \mathcal{A}^0 \xrightarrow{d} Z^1 \to 0$ and $0 \to Z^1 \to \mathcal{A}^1 \to Z^2 \to 0$, and $\check{H}^2(M, \mathbb{R}) \cong H^1(M, Z^1) \cong H^0(M, Z^2)/d(H^0(M, \mathcal{A}^1))$ with the map taking $n_{\alpha\beta\gamma}$ to $\frac{1}{2\pi i} d \log g_{\alpha\beta}$.

Question: Can we find, in some natural way, 1-forms $\{\theta_\alpha \in A^1(U_\alpha)\}$ such that $\frac{1}{2\pi i} d \log g_{\alpha\beta} = \theta_\beta - \theta_\alpha$?

Let $L \to M$ be a line bundle, $U_\alpha \to L$ such that $|\sigma_\alpha|^2 = \rho_\alpha$ and $\rho_\beta = |g_{\alpha\beta}|^2 \rho_\alpha$.

$\frac{1}{2\pi i} \partial \log \rho_\beta = \frac{1}{2\pi i}(d \log(g_{\alpha\beta} + \partial \log \rho_\alpha)$

Taking $\theta_\beta = \frac{1}{2\pi} \partial \log \rho_\beta$, we get $-c(L) = \frac{1}{2\pi i} \bar{\partial} \partial \log \rho_\beta$, and to get $c(L)$ we just exchange ∂ and $\bar{\partial}$.

Suppose you have a coframe, that is, a basis for sections of $\mathscr{O}(U, E)$, s_i. We want a derivative map $D : \mathscr{O}(U, E) \to \Omega^1(U) \otimes \mathscr{O}(E)$. We need $D(fs) = df \otimes s + fDs$, then we'll have $Ds_j = \sum_i \theta_{ij} \otimes s_j$ where $\theta_{ij} \in \Omega^1(U)$.

These will need to satisfy some condition under change of coordinates in order to make sense. If $g_{\alpha\beta}$ is the matrix of transition functions, then if $\theta' = g_{\alpha\beta}^{-1} dg\alpha\beta + g_{\alpha\beta}^{-1} \theta g_{\alpha\beta}$. For any individual θ_α we have $\theta_\beta = g_{\alpha\beta}^{-1} dg_{\alpha\beta} + \theta_\alpha = d(\log g_{\alpha\beta})$.

Finding $\{\theta_\alpha\}$ such that $\theta_\beta - \theta_\alpha = \frac{1}{2\pi i} d \log g_{\alpha\beta}$ is the same as having a connection.

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Let $\phi : \mathcal{X} \to B$ be a holomorphic proper submersion of complex manifolds. So by Ehresmann's Theorem, we can assume that that around each b_0 we have a U such that $\phi^{-1}(U) \cong U \times X_{b_0}$.

For each $x \in X_{b_0}$, we have $U \to U \times X_{b_0}$ by $b \mapsto (b, x)$, and then we can send it along the inverse of the diffeomorphism to $\mathcal{X}$, and this map is holomorphic.

Now, assume that U is a polycylinder and that $b_0 = 0$.

The naive approach is that $X = X_0$ sits insider $\{t\} \times X \xrightarrow{G} X_t$, and this composition is the diffeomorphism g_t.

This gives $J_t = g_t^*(J_{X_t})$, with $J_t^2 = -I$, and so we have a family $\{J_t\}_{t\in\Delta}$ which give decompositions $[T_x(X)]_{\mathbb{C}} = T'_{x,t} \oplus T''_{x,t}$ for all $x \in X$. For $t = 0$, we get the usual decomposition for X

So in local coordinates, $(U, z_1^U, \ldots, z_n^U)$, we have $T''_{x,t}$ has a basis of the form $\frac{\partial}{\partial \bar{z}_j} - \sum_{i=1}^n \omega_{ij} \frac{\partial}{\partial z_i}$.

Thus, if $v \in T_0^h(\Delta)$, then we can write it out as $\sum_{i=1}^n v(\omega_{ij}(z,t)) \frac{\partial}{\partial z_i}$.

But of course, this is not all well defined, it is not global. Given $v \in T_0^h(\Delta)$, we can consider the expression $\sum_{i=1}^n \bar{\partial} v(\omega_{ij}(z,t)) \otimes \frac{\partial}{\partial z_i}$, and this will kill any holomorphic dependence. So what we get is a map $T_0^h(\Delta) \to A^{0,1}(T^h(X))$. But this is closed, it's not just a form, so we actually have $T_0^h(\Delta) \to H_{\bar{\partial}}^{0,1}(T^h(X)) = H^1(X, \mathscr{O}(T^h(X)))$. This is called the Kodaira-Spencer Map.

So for eahc $p \in \mathcal{X}$, we have $0 \to T_0(\phi^{-1}(\phi(p)) \to T_p\mathcal{X} \to T_{\phi(p)}(B) \to 0$, and these fit together into $0 \to T^h X \to T^h(\mathcal{X})|_X \to X \times T_0^h(B) \to 0$

This gives us a long exact sequence, and in particular, a map $H^0(X, \mathscr{O}(X \times T_0^h(B))) \to H^1(X, \mathscr{O}(T^h X))$. However, the first is simply $T_0^h(B)$, and so this gives us again the Kodaira-Spencer map.

Now, assume that the X_t are 1-dimensional. These are all diffeomorphic, so we can say that they are Riemann Surfaces of genus g. So then $H^1(X_t, \mathbb{C}) = H^{1,0}(X_t) \oplus H^{0,1}(X_t)$. Now, $H^1(X_t, \mathbb{C})$ is constant with respect to t, but the decomposition may not be. We still have all the maps $g_t : X \to X_t$, and they give isomorphisms $g_t^* : H^1(X_t, \mathbb{C}) \to H^1(X, \mathbb{C})$.

So we can view this as having $H^1(X, \mathbb{C}) = H_t^{1,0} \oplus H_t^{0,1}$, a fixed vector space with a varying decomposition, satisfying $H_t^{0,1} = \bar{H}_t^{1,0}$. But we also have the polarization form $Q = \int_X \cdot \wedge \cdot$ and $Q(H_t^{1,0}, H_t^{1,0}) = 0$ and $iQ(H_t^{1,0}, \bar{H}_t^{1,0}) > 0$.

So now we have $\Delta = \{W \in G(g, H^{-1}(X, \mathbb{C}))$ satisfying these conditions$\}$. So we can represent these by $2g$ by g matrices, satisfying the conditions. So the conditions guarantee we can make half into I and the other Z, and the condition then becomes ${}^tZ = Z$ and $\mathrm{Im}(Z) > 0$ (the imaginary part).

For a Riemann surface, we have that $H^{1,0}(X_t) = H^0(X_t, \Omega^1(X_t))$, so we just need to look at the subspace of $(1,0)$ forms among C^∞-forms.

So this whole approach is what we'll be generalizing to higher dimensional fibers.

In general, given $\phi : \mathcal{X} \to B$, we get $\pi_1(B, b_0) \xrightarrow{\rho} \mathrm{GL}(H^k(X, R))$ where $R = \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$. We also have $R^k\phi_* R$ to work with.

So how do these two things relate? Can we go between them without going back to the fibration? We can, via flat vector bundles. (Note: Riemann-Hilbert Correspondence)

Let $\tilde{B} \to B$ be the universal cover, and let π_1 act on the right, and assume we have a rep $\rho : \pi_1(B, b_0) \to \mathrm{GL}(V)$. Then we define $\tilde{B} \times_{\pi_1} V \to B$ where this is just the product modulo $\sim$ where $(\tilde{b}, v) \sim (\tilde{b}\gamma, \rho^{-1}(\gamma)v)$ for all $\gamma \in \pi_1$.

We say that a section of a bundle with connection is flat if $D\sigma = 0$.

Theorem 2.1 (Riemann-Hilbert)**.** *The follow are equivalent:*

1. *Representations of the fundamental group*
2. *Local systems*
3. *vector bundles with flat connection*

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Look in the appendix of the first set of lecture notes, A.5, on the Weight filtration. Also Flat Bundles.

Let $\phi : \mathcal{X} \to B$ projective, $\omega = R^2\phi_*\mathbb{Z}$. We can look at $R^k\phi_*\mathbb{C} \otimes \mathscr{O}_B$. Over a point, we have $\phi^{-1}(\Delta) \to \Delta$, and $\phi^{-1}(\Delta)$ is diffeomorphic via F to $\Delta \times X$, with inverse G, and these give $g_t : X \to X_t$ and $f_t : X_t \to X$.

If $\alpha \in H^k(X, \mathbb{C})$, we have $t \mapsto t_t^*\alpha$ the set of flat sections.

We look at the Gauss-Manin connection ∇. Then $\nabla($
$sum f_j s_j) = \sum df_j \otimes s_j$ for s_j flat.

Look at $H^k(X_t, \mathbb{C}) = \oplus_{p+q=k} H^{p,q}(X_t)$ then $H_t^{p,q} = g_t^*(H^{p,q}(X_t)) \subset H^k(X, \mathbb{C})$. These terms are all upper semicontinuous, so they cannot decrease in dimension, only increase, and as the total is constant, each must be.

Now, set $F_t^p = \oplus_{a \geq p} H_t^{k,k-a}$. For each p, we get a map $\Delta \to G(f^p, H^k(X, \mathbb{C})$ where $f^p = \dim F^p$, taking t to $F_t^p = g_t^* \left(\oplus_{a \geq p} H^{a,k-a}(X_t, \mathbb{C}) \right)$. This map $\mathcal{P}^p$ is smooth.

So back on the bundle, we get C^∞ subbundles $\mathbb{F}^p$.

Then the amazing fact is that $\mathcal{P}^p : \Delta \to G(f^p, H^k(X, \mathbb{C}))$ is holomorphic, so $\mathbb{F}^p$ is a holomorphic subbundle.

For notational convenience, we take $\dim \Delta = 1$. Then $\mathcal{P}^p_{*,0}\left(\frac{\partial}{\partial \bar{z}}\right) \in \hom(F^p(X), H^k(X, \mathbb{C})/F^p(X))$ is zero.

Once we have holomorphicity, there's Griffiths Transversality, which says that $\mathcal{P}^P_{*,0}\left(\frac{\partial}{\partial z}\right)$ belongs to $\hom(F^p(X), F^{p-1}(X)/F^p(X))$.

Now, we look at the weight 3 Hodge structure on H^3. We can break it up in two ways. If we take one half ot be $H^{3,0} \oplus H^{1,2}$, then we get the Weil filtration, and these are polarized, but Griffiths looked at $H^{3,0} \oplus H^{2,1}$, and lost the polarization in order to get things to vary holomorphically.

Sketch of Griffiths Transversality

Let $\alpha \in F^p(X) = \oplus_{a \geq p} H^{a,k-a}(X)$. Then we have $\alpha(t) \in F^p(X_t)$. We can look at $g_t^*(\alpha(t)) \in F_t^p$. Then we have $\left[\frac{\partial}{\partial t}|_{t=0} g_t^*(\alpha(t))\right] \in H^k(X, \mathbb{C})/F^p(X)$. Then there exists $\Theta \in \oplus_{a \geq p} A^{a,k-a}(\mathcal{X})$ such that $d\left(\Theta|_{X_t}\right) = 0$ and $[\Theta|_{X_t}] = \alpha(t)$.

$G^*\Theta = dt \wedge \phi + \psi$ and $d(G^*\Theta) = dt \wedge d\phi + dt \wedge \frac{\partial \psi}{\partial t} + d\bar{t} \wedge \frac{\partial \psi}{\partial \bar{t}}$ because the contractions of ϕ and ψ with $\frac{\partial}{\partial t}$ are both zero.

Then, it's a matter of counting the number of holomorphic and antiholomorphic terms.

We have a map $\mathcal{P}^p_{*,0} : T_0(B) \to \hom(H^q(X, \Omega^p), H^{q+1}(X, \Omega^{p-1})$ and a map $T_0^h(B) \to H^1(X, T_X)$. This gives us $H^q(X, \Omega^p) \times H^1(X, T_X) \to H^{q+1}(X, \Omega^{p-1})$ by contraction.

For the rest of these talks, we'll forget geometry and talk about abstract variations of Hodge structures.

Definition 3.1 (Variation of Hodge Structures)**.** *Let $H \to B$ be a local system of free $\mathbb{Z}$-modules, and let ∇ be the Gauss-Manin connection on $\mathbb{H} = H \otimes \mathcal{O}_B$. A variation of Hodge structures of weight k is an increasing flag of holomorphic subbundles $0 \subset \mathbb{F}^k \subset \mathbb{F}^{k-1} \subset \dots$ such that $\mathbb{F}^p \oplus \bar{\mathbb{F}}^{k-p+1} = \mathbb{H}$ and if $\nabla_v \Gamma(U, \mathbb{F}^p) \subset \Gamma(U, \mathbb{F}^{p-1})$ then $v \in T^{1,0}(B)$.*

A polarization of VHS is a flat bilinear form $\mathcal{Q}$ on $\mathbb{H}$ of parity $(-1)^k$ such that for each $x \in B$, $\mathcal{Q}_x$ polarizes the Hodge structure on $\mathbb{H}_x$.

Theorem 3.2 (Monodromy Theorem)**.** *For every $\gamma \in \pi_1(B, b_0)$, $\rho(\gamma)$ is quasi-unipotent, that is, there exists r such that $\rho(\gamma)^r$ is unipotent.*

In the geometric case, this theorem goes back to Langlands.

Let's call N the nilpotent part. So the monodromy theorem actually also says that $N^{k+1} = 0$.

In general, the first part goes back to Borel, and the second to Schmid.

The monodromy theorem requires that you have a polarization.

Tomorrow, we'll take this definition and we'll look how to map it into some nice space, a period domain, and look at the properties of this map, especially when $B = \Delta^*$, and look at limiting behavior as we go to 0.

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Last time, we defined an abstract VHS, starting from a local system $\mathcal{L}_{\mathbb{Z}}$, and the Gauss-Manin connection, which is the flat connection associated with a local system underling a PVHS.

4.1 Period Map

Let $\mathbb{V} \to (\Delta^*)^r \times \Delta^{n-r}$ be a PVHS.

Linear algebra:

1. $V_{\mathbb{Z}}$ a lattice, $V_{\mathbb{Q}} = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$

2. An integer k

3. A collection of hodge numbers $h^{p,q}$ with $p+q=k$ and $\sum h^{p,q} = \dim V_{\mathbb{C}}$, $h^{p,q} = h^{q,p}$ and $f^p = \sum_{a\geq p} g^{a,k-a}$.

4. Q a non-degenerate bilinear form over $\mathbb{Z}$ of parity $(-1)^k$.

Now, let D (of all this data) be the space of all Hodge structures with $V_{\mathbb{C}} = \oplus_{p+q=k} V^{p,q}$ polarized by Q and with $\dim V^{p,q} = h^{p,q}$.

Now, define $\check{D}$ to be the space of all filtrations F on $V_{\mathbb{C}}$ with $\dim F^p = f^p$ and $Q(F^p, F^{k-p+q}) = 0$.

Example 4.1 (Weight 1)**.** *Look at weight 1. Then* $\check{D} = \{F \subset G(n, \mathbb{C}^{2n}) | Q(F,F) = 0\}$.

Now, let $G_{\mathbb{C}} = \{M \in \mathrm{GL}(V_{\mathbb{C}}) | Q(Mu, Mv) = Q(u,v)\}$. Then $G_{\mathbb{C}}$ acts naturally on $\check{D}$. In fact, it acts transitively.

Now look on $V_{\mathbb{C}} = V^{2,0} \oplus V^{1,1} \oplus V^{0,2}$, then $-Q(u, \bar{u}) > 0$ for $u \in V^{2,0}$ and $Q(v, \bar{v}) > 0$ for $v \in V^{1,1}$, so we can write Q in the form $\begin{pmatrix} & & -I \\ & I & \\ -I & & \end{pmatrix}$.

That actually has a real structure, and if we look at $G = G_{\mathbb{C}} \cap \mathrm{GL}(V_{\mathbb{R}})$, then this group acts on D.

Now, by basic Lie theory, we have $\check{D} = G_{\mathbb{C}}/B$ and $D = G/B \cap G$.

So what are these groups? For weight 2, Q is positive definite on $V^{1,1}$ and negative definite on $V^{2,0}$, so we have $G = O(2h^{2,0}, h^{1,1})$. The subgroup fixing a given Hodge structure is $U(h^{2,0}) \times O(h^{1,1})$, and so $D = SO(2h^{2,0}, h^{1,1})/U(h^{2,0}) \times SO(h^{1,1})$, which sits inside:

$$\mathrm{SO}(2h^{2,0})/U(h^{2,0}) \to SO(2h^{2,0}, h^{1,1})/U(h^{2,0}) \times SO(h^{1,1}) \to SO(2h^{2,0}, h^{1,1})/SO(h^{2,0}) \times SO(h^{1,1})$$

and it turns out that every Hermitian symmetric space arises in this manner, though we don't always get something that nice.

So we have $G/V = D \subset \check{D} = G_{\mathbb{C}}/B$, and have $B \to G_{\mathbb{C}} \to \check{D}$, with B acting on $\mathfrak{g}/\mathfrak{b}$ and $T^h(\check{D}) = \check{D} \times_B \mathfrak{g}/\mathfrak{b}$.

If $V_{\mathbb{C}} = \oplus V^{p,q}$, then we can define $\mathfrak{g}^{a,-a} = \{X \in \mathfrak{g} | X(V^{p,q}) \subset V^{p+a,q-a}\}$ a weight 0 Hodge strcuture.

Let $b = \oplus_{a\geq 0} \mathfrak{g}^{a,-a}$. Then B preserves $b \oplus \mathfrak{g}^{-1,1}$ and $T^{-1,1}(\check{D}) = \check{D} \times_B (b \oplus \mathfrak{g}^{-1,1})$.

If we have a PVHS, then parallel translation to a fixed fiber defines a holomorphic map $B \to \Gamma \backslash D$ where Γ is the monodromy. We call this the period map.

the HS differential takes values on a horizontal subbundle.

We're going to discuss it from the largely analytic point of view.

So now, lets look at the local situation, a PVHS over $(\Delta^*)^r$. We have $\pi_1((\Delta^*)^r) = \mathbb{Z}^r$ with generators $c_1, \ldots, c_r$, which give us elements $\gamma_1, \ldots, \gamma_r \in$

$\mathrm{GL}(V_{\mathbb{Z}})$. We know that these are quasi-unipotent, and we'll assume that they're actually unipotent.

If we set N_i to be the log of the monodromy, we'll have $\gamma_j = e^{N_j}$ by definition, but then we'll have $[N_i, N_j] = 0$ and $N_i^{k+1} = 0$. So we can define $\psi(z_1, \dots, z_r) = \exp(-\sum z_i N_i)\tilde{\phi}(z_1, \dots, z_r)$ where we have $\phi : (\Delta^*)^r \to \Gamma \backslash D$ and $\tilde{\phi}$ lifting it onto the PVHS to D.

The map, ψ takes the PVHS to $\check{D}$ and we call the induced map on $(\Delta^*)^r$ also ψ.

Theorem 4.2 (Nilpotent Orbit Theorem)**.** *The map $\psi : (\Delta^*)^r \to \check{D}$ extends holomorphically to the origin.*

Note, this is not necessarily a Hodge structure! We just have $\check{D} \ni \psi(0)$, and call it the limit Hodge filtration.

Now, we have that $\exp(\sum z_j N_j)\psi(0) \in D$ if the imaginary parts of z_j are large enough. We call these period maps Nilpotent orbits.

This afternoon, we'll see how the limiting Hodge filtration along with the weight filtration from the monodromy determines a polarized MHS.

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We have local monodromy and we have the logarithms of the monodromy, $N_1, \dots, N_r$. The monodromy theorem tells us that the period map can be written as $\exp\left(\sum \frac{\log b_j}{2\pi i} N_j\right) \psi(t_1, \dots, t_r)$ where ψ extends holomorphically to $\psi : \Delta^r \to \check{D}$, with $F_{lim} = \psi(0)$.

But them we also have $(t_1, \dots, t_r) \mapsto \exp\left(\sum \frac{\log b_j}{2\pi i} N_j\right) F_{lim}$. So we want to find out what kinds of holomorphic maps we can take.

We take the weight filtration, it comes from the Jordan decompositions, and includes the fact that $N^\ell : \mathrm{Gr}_\ell^W \to \mathrm{Gr}_{-\ell}^W$. But what about when there are more than one monodromy operator?

Theorem 5.1. *The weight filtration determined by $\sum_{\lambda_j > 0} \lambda_j N_j$ is unique, and we denote it by $W(C)$, for the weight filtration of the cone. Moreover, $(W(C)[-k], F_{lim})$ is a MHS. In fact, each N is a $(-1,-1)$ morphism of MHS. Additionally, the converse holds.*

So a nilpotent orbit is just a MHS, with the weight filtration from the nilpotent cone, and with the polarization the obvious thing to try.

When you happen to have a MHS that splits over $\mathbb{R}$, then you should be able to extend everything to an $\mathfrak{sl}_2$-invariant picture.

Theorem 5.2 (SL_2-orbit)**.** *There exists a canonical (Schmid) splitting of the MHS $(W(N), F_{lim})$. More precisely, there is a natural way to produce another Hodge filtration F_0 such that $(W(C), F_0)$ splits over $\mathbb{R}$.*

Example 5.3. *Look at $i \in \mathbb{H}_1$. Then our space is $SL_2(\mathbb{C})/B$, and we have an induced Hodge filtration $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}_2(\mathbb{C}) = \mathfrak{sl}_2(\mathbb{C})^{-1,1} \oplus \mathfrak{sl}_2(\mathbb{C})^{0,0} \oplus \mathfrak{sl}_2(\mathbb{C})^{1,-1}$. If y, n_-, n_+ are the generators of the Lie algebra, then the first summand is generated by $(iy + n_- + n_+)$ and the second by $(n_+ - n_-)$.*

In the several variables situation, the key is to understand how to two splitting relate.

MHS's come in two types: split and nonsplit. Nonsplit give you nilpotent orbits, split give you $\mathfrak{sl}_2$ modules.

Now, look at $\mathfrak{g}_{\mathbb{C}} = I^{p,q}\mathfrak{g} = \{X \in \mathfrak{g} | X(I^{a,b} \subset I^{a+b,b+q}\}$. Let $b = \oplus I^{a,b}_{a \geq 0}\mathfrak{g}$.

Now, $\psi(t_1, \ldots, t_r) = \exp \Gamma(t_1, \ldots, t_r) F_{lim}$ where $\Gamma : \Delta^r \to \mathfrak{g}$ has $\Gamma(0) = 0$. Set $t_i = \exp(2\pi i z_i)$, then $\phi(z_1, \ldots, z_r) = \exp(\sum z_j N_j) \exp \Gamma(t_1, \ldots, t_r) F_{lim}$. Denote by $E(z) = \exp X(z)$.

Then horizontality is $E^{-1}dE = dX_{-1}$, and we call this Griffiths differential equation.

We have then that $d(E^{-1}dE) = 0$, as $dX_{-1} \wedge dX_{-1} = 0$. And so $X_{-1}(z) = \sum z_j N_j + \Gamma_{-1}$, where $\Gamma_{-1} : \Delta^r \to \oplus_b I^{-1,b}\mathfrak{g}$.