

# Hodge Loci and Absolute Hodge Classes

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## 1 Charles 1 - Hodge Loci and Absolute Hodge Classes

Let  $X$  be a compact Kähler manifold. We have Hodge classes  $H^{2p}(X, \mathbb{Z}) \cap H^{p,p}(X)$ .

**Conjecture 1.1** (Hodge Conjecture). *If  $X$  is projective, Hodge classes are algebraic.*

**Theorem 1.2** (Voisin). *This is false for Kähler.*

Now, let  $k$  be a field, embeddable in  $\mathbb{C}$ . For  $X$  smooth and projective over  $k$ , we have  $H_{dR}^i(X/k)$ . Now, take  $\sigma : k \rightarrow \mathbb{C}$  an embedding.

We get Betti-deRham isomorphisms which are compatible with the cycle class map.

**Definition 1.3** (Relative Hodge Class). *Let  $\alpha \in H_{dR}^{2i}(X/k)$ . We say that  $\alpha$  is a Hodge class relative to  $\sigma$  if  $\alpha \in F^i H_{dR}^{2i}(X/k)$  and the image of  $\alpha$  in  $H^{2i}(X_\sigma(\mathbb{C}), \mathbb{Q}(i)) \otimes \mathbb{C}$  lies in the real subspace.*

We say that  $\alpha$  is absolute Hodge if it is a Hodge class relative to any  $\sigma$ .

Remarks: First,  $X/\mathbb{C}$ , we can define what it means for a class to be an absolute Hodge class. Now, how dependent is this on  $k$ ? The cohomology classes of algebraic cycles are absolute Hodge.

Proof: If  $Z$  is an algebraic cycle in  $X$ , then for any  $\sigma : k \rightarrow \mathbb{C}$ , we have  $Z \times_\sigma \mathbb{C}$  algebraic cycle in  $X \times_\sigma \mathbb{C}$  So this gives a Hodge class relative to  $\sigma$ , and works for all  $\sigma$ .

Thus, we have two conjectures:

**Conjecture 1.4.** *Hodge classes are absolute.*

**Conjecture 1.5.** *Absolute Hodge classes are algebraic.*

These two conjectures imply the Hodge conjecture.

Let  $X/\mathbb{C}$  be smooth and projective of dimension  $d$ . then we have  $H^d(X \times X) = \bigoplus_i H^i(X) \otimes H^{d-i}(X)$  contains a class  $[\Delta] = \sum \pi_i$  via the Künneth formula.

It's a conjecture that the  $\pi_i$  are algebraic.

**Proposition 1.6.** *The  $\pi_i$  are absolute Hodge cycles.*

Conjecture: The inverse of the Lefschetz map is algebraic.

**Proposition 1.7.** *This is absolute.*

**Theorem 1.8** (Deligne). *Hodge classes on abelian varieties are absolute.*

Let  $X/\mathbb{C}$  be smooth and projective,  $Z$  an algebraic cycle of codimension  $i$  in  $X$ , homologically equivalent to zero. We have  $AJ(Z) \in J^i(X)$ , which gives us an extension of mixed Hodge structures  $0 \rightarrow H^{2n-1}(X, \mathbb{Z}(n)) \rightarrow H \rightarrow \mathbb{Z} \rightarrow 0$ .

Now,  $AJ(Z) = 0$  if and only if the sequence splits which is if and only if there exists a Hodge class  $H$  mapped to 1 in  $\mathbb{Z}$ .

This is related to the Bloch-Beilinson filtration on Chow Groups,  $F^2CH^i(X) = \ker AJ$ .

Now, we have out GM connection and a VHS structure.

Remark: If  $X/k$  and  $k = \bar{k}$ , and  $K \supset k$ , then  $CH^i(X)K \neq CH^i(X_k)$ . But, this does not for cohomological equivalence!

Indeed, take  $Z \subset X_K$  be an algebraic cycle. Then the cohomology class  $[Z]_{dR} \in F^i H^{2i}(X/K)$  actually comes from  $F^i H^{2i}(X/k)$ .

**Proposition 1.9.** *For  $\alpha \in F^i H^{2i}(X_K/K)$  an absolute Hodge cycle, then  $\alpha$  is defined over  $k$ .*

*Proof.* We can assume that  $K$  is finitely generated over  $k$ , and  $K$  is the function field of some quasiprojective  $S/k$ . We can assume also that  $\alpha$  extends to  $\bar{\alpha}$ , a section of  $F^i H_{dR}^{2i}(X/k) \otimes \mathcal{O}_S$ . We want to prove that  $\alpha$  is constant.

Now, pull everything back to  $\mathbb{C}$  by some  $k \subset \mathbb{C}$ . We want to check that  $\alpha_{\mathbb{C}}$  is constant.

But,  $\alpha$  being absolute, for every  $k \rightarrow \mathbb{C}$ ,  $\alpha_{\sigma}$  is going to lie the rational lattice of  $H_{dR}^i(X/\mathbb{C}) \otimes \mathcal{O}_S$ , and so will be constant.  $\square$

**Theorem 1.10** (Principle B of Deligne).  *$\pi : \mathcal{X} \rightarrow S$  a smooth projective morphism with  $S$  quasi-projective and connected. Let  $\alpha$  be a global section of  $R^{2i}\pi_*\mathbb{Q}(i)$ , which is Hodge everywhere. Then if  $\alpha_s$  is absolute Hodge for some  $s \in S(\mathbb{C})$ , it is for all  $s \in S$ .*

*Proof.* Just conjugate  $\alpha$ .  $\nabla = \nabla^{\sigma}$  for  $\sigma \in \text{Aut}\mathbb{C}$  Then  $\nabla\alpha^{\sigma} = \nabla^{\sigma}\alpha^{\sigma} = (\nabla\alpha)^{\sigma} = 0$ .  $\square$

Now, we need  $\alpha$  algebraic as a section of  $\mathcal{H}_{\mathcal{X}/S}$ . For this, we take the global invariant cycle theorem which says that  $\alpha$  is the restriction of a Hodge class in  $H^{2i}(Y, \mathbb{Q}(-1))$  where  $\mathcal{Y} \supset \mathcal{X}$  is a smooth compactification.

Application: Let  $H$  be a HS of weight 2 with  $h^{2,0} = 1$ . Then there exists  $H'$  a Hodge structure of weight 1, polarized, with  $H \rightarrow \text{End}H'$

Starting with  $\pi : \mathcal{X} \rightarrow S$  a family of polarized K3's, get  $\alpha : \mathcal{A} \rightarrow S$  a polarized abelian scheme, then we have  $R^2\pi_*\mathbb{Q}(1) \rightarrow \text{End}(R^1\alpha_*\mathbb{Q})$  for all  $S$ . connection.

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Let  $X/k$  be a smooth projective (maybe quasi-projective) variety and  $k$  of characteristic 0 and embeddable into  $\mathbb{C}$ .

**Definition 2.1.** Fix  $\sigma : k \rightarrow \mathbb{C}$  and  $\alpha \in H_{dR}^{2i}(X/k)$ , say that  $\alpha$  is Hodge with respect to  $\sigma$  if with the isomorphism  $H_{dR}^{2i}(X/k) \otimes \mathbb{C} \cong H_B(X_{\mathbb{C}}(\mathbb{C}), \mathbb{Q}(i)) \otimes \mathbb{C}$  then  $\alpha$  lies in the rational subspace and  $\alpha \in F^i H_{dR}^{2i}(X/k)$ .

We say that  $\alpha$  is absolute Hodge if  $\alpha$  is Hodge with respect to every  $\sigma$ .

Now, if  $X/\mathbb{C}$  and  $\alpha \in H_B^{2i}(X(\mathbb{C}), \mathbb{Q}(i))$  is absolute Hodge if it is a hodge class and its image in  $H_{dR}^{2i}(X/\mathbb{C})$  is absolutely Hodge, so Hodge with respect to all  $\sigma \in \text{Aut}(\mathbb{C})$ .

**Theorem 2.2** (Principle B). Let  $\pi : \mathcal{X} \rightarrow S$  be smooth, projective over  $\mathbb{C}$  with  $S$  quasiprojective, smooth and connected. Let  $\alpha$  be a global section of  $R^{2i}\pi_*\mathbb{Q}(i)$  such that  $\alpha$  is of type  $(i, i)$  at any point of  $S$  and for some  $s_0 \in S(\mathbb{C})$ ,  $\alpha_{s_0}$  is absolutely Hodge. Then  $\forall s \in S(\mathbb{C})$ , we have that  $S(\mathbb{C})$  is absolutely Hodge.

Remark: If we know  $\alpha_{s_0}$  is algebraic, then we can prove  $\alpha_s$  is.

We prove this before.

**Theorem 2.3** (Global Invariant Cycle Theorem). Let  $\pi : \mathcal{X} \rightarrow S$  as above,  $\rho : \pi_1(S(\mathbb{C}), s) \rightarrow \text{Aut}(H^{2i}(X_s, \mathbb{Q}(i)))$  the monodromy representation. Then the fixed points of  $\rho$  are those in the image of  $H^{2i}(\mathcal{Y}, \mathbb{Q}(i)) \rightarrow H^{2i}(X_s, \mathbb{Q}(i))$  where  $\mathcal{Y} \supset \mathcal{X}$  is a smooth compactification.

In our proof,  $\alpha$  is a global section of  $R^{2i}\pi_*\mathbb{Q}(i)$ , and so it is  $\pi_1$ -invariant and thus comes from  $\mathcal{Y}$ , so is algebraic. Then  $\forall s \in S(\mathbb{C})$ , we have that  $\alpha_s$  is absolute Hodge.

Remark: if  $X$  is defined over  $\mathbb{Q}$ , then Hodge classes on  $X$  are always absolute Hodge.

Question: How can we reduce questions in Hodge theory (Hodge cycles are absolute, Hodge conjecture) to questions over  $\mathbb{Q}$  and number fields?

Let  $X/\mathbb{C}$  be smooth projective. We can always find  $\mathcal{X} \xrightarrow{\pi} S/\mathbb{Q}$  with  $\pi$  smooth and projective,  $S$  smooth and quasiprojective and  $s \in S(\mathbb{C})$  such that  $X \cong X_s$  ( $X \subset \mathbb{P}_{\mathbb{C}}^n$  corresponds to a point of some Hilbert-scheme of subschemes of  $\mathbb{P}_{\mathbb{C}}^n$ )

What does it mean for Hodge classes fibers to be absolute?

Remark:  $\mathcal{X} \rightarrow S/\mathbb{Q}$ , so the Hodge bundle  $\mathcal{H}_{dR}^i(\mathcal{X}/S)$  are defined over  $\mathbb{Q}$ , and so is  $\nabla$ .

If  $\sigma \in \text{Aut}(\mathbb{C})$ , and  $\sigma$  permutes the complex points of the Hodge bundle, then

**Proposition 2.4.**  $\alpha \in F^i H_{dR}^{2i}(X_s/\mathbb{C})$  is an absolute Hodge class if and only if  $\alpha$  is rational and  $\sigma(\alpha)$  is an absolute Hodge class for all  $\sigma \in \text{Aut}(\mathbb{C})$ .

**Definition 2.5** (Hodge Locus). The locus of Hodge classes for  $\pi$  is the set of  $\alpha_t \in H_{dR}^{2i}(X_t/\mathbb{C})$  such that  $\alpha_t$  is a hodge class.

**Proposition 2.6.** *Hodge classes for the fibers of  $\pi$  are absolute iff the locus of Hodge classes is invariant under  $\text{Aut}(\mathbb{C})$ .*

The locus of Hodge classes is actually a countable union of analytic subvarieties of  $\mathcal{H}^{2i}(\mathcal{X}/S)$ .

Start with  $\alpha_t \in H_{dR}^{2i}(X_s/\mathbb{C})$ . We get a component of the locus of Hodge classes by taking parallel transport of  $\alpha_s$  over some small open subset and looking at the vanishing on  $\mathcal{H}^{2i}/F^i$ .

**Lemma 2.7.** *Let  $B$  be defined over  $\mathbb{Q}$  and  $Z \subset B(\mathbb{C})$  be a countable union of analytic subvarieties such that  $Z(\mathbb{C})$  is stable under the action of  $\text{Aut}(\mathbb{C})$ . Then  $Z$  is a countable union of algebraic varieties over  $\mathbb{Q}$ .*

The idea is to take a very general point of  $Z$  and look at the orbit of it under  $\text{Aut}(\mathbb{C})$ .

We know the geometric part of the conclusion:

**Theorem 2.8** (Deligne-Cattani-Kaplan). *Let  $\pi : \mathcal{X} \rightarrow S$  as before. Then the locus of Hodge classes in  $\mathcal{H}^{2i}(\mathcal{X}/S)$  is a countable union of algebraic subvarieties.*

We don't get information on the field of definition.

**Theorem 2.9.** *Let  $\mathcal{X} \rightarrow S/\mathbb{Q}$  as before. The Hodge locus of  $\pi$  is the image in  $S$  of the component of the locus of Hodge classes passing through  $\alpha$ . By DCK, this locus is algebraic. Let  $Z_\alpha$  be in this locus. Assume it is defined over  $\mathbb{Q}$ . Then the Hodge conjecture for  $(X, Z_\alpha)$  can be reduced to the HC for some  $\mathcal{X}/\mathbb{Q}$ .*

Proof follows from DCK and the global invariant cycle theorem.