Hodge Structures and Mixed Hodge Structures

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1 ElZein 1 - Hodge Structures and Mixed Hodge Structures

Work of Deligne, then Griffiths

Hodge decomposition is a geometric invariant: this means that if $f : X \to Y$ is a morphism of compact Kähler manifolds, then we have a map $f^* : H^i(Y, \mathbb{Z}) \to H^i(X, \mathbb{Z})$, we have $H^i(Y, \mathbb{C}) = \oplus_{p+q=i} H^{p,q}$ and $f^*(H^{p,q}(Y)) \subset H^{p,q}(X)$. If f is analytic, then we have $f^* : H^{p,q}(Y) \to H^{p,q}(X)$.

The Hodge decomposition is a linear structure on the cohomology.

Theorem 1.1 (Deligne)**.** *The cohomology groups of algebraic varieties carry mixed Hodge structures.*

Definition 1.2 (Hodge Structure)**.** *A hodge structure of weight m is defined by a finitely generated group $H_{\mathbb{Z}}$, a decomposition of $H_{\mathbb{C}} = H_{\mathbb{Z}} \otimes \mathbb{C}$ into a direct sum $H_{\mathbb{C}} = \oplus_{p+q=m} H^{p,q}$ where $H^{p,q}$ is a complex subspace such that $\overline{H^{p,q}} = H^{p,q}$.*

Example 1.3. *Let $H_{\mathbb{C}} = \mathbb{C}^2$ and $\mathbb{Z}^2 = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$. We can make a Hodge structure by $H^{1,0} = \mathbb{C}(e_1 - ie_2)$ and $H^{0,1} = \mathbb{C}(e_1 + ie_2)$, but not by trying to make $H^{1,0} = \mathbb{C}e_1$.*

1.1 Hodge Structures of Weight 1

$H = H_{\mathbb{Z}}$, then $H_{\mathbb{C}} = H^{1,0} \oplus H^{0,1}$. We have $H_{\mathbb{Z}} \to H_{\mathbb{R}} \to H_{\mathbb{C}} = H^{1,0} \oplus H^{0,1} \to H^{0,1}$ gives an isomorphism $H_{\mathbb{R}} \cong H^{0,1}$, and the image $H_{\mathbb{Z}} \to H_{\mathbb{R}}$ gies a lattice, so $H^{0,1}/H_{\mathbb{Z}}$ is a torus.

When X is Kähler, $H^{0,1} = H^1(X, \mathscr{O}_X)$, we have an exact sequence of sheaves using the exponential map $0 \to \mathbb{Z}_X \to \mathscr{O}_X \to \mathscr{O}_X^* \to 0$ which gives a long exact sequence including $H^1(X, \mathbb{Z}) \to H^1(X, \mathscr{O}_X) \to H^1(X, \mathscr{O}_X^*) \to H^2(X, \mathbb{Z})$ where the last map takes line bundles to the first Chern class. So then $H^1(X, \mathscr{O}_X)/H^1(X, \mathbb{Z})$ is the kernel of Chern map, and this is a complex torus called the Picard torus.

1.2 Algebraic Operations on Hodge Structures

If H and H' are Hodge structures of the same weight then so is $H \oplus H'$.

$\hom_{\mathbb{C}}(H_{\mathbb{C}}, \mathbb{C})$ with $\hom_{\mathbb{Z}}(H_{\mathbb{Z}}, \mathbb{Z})$ is a Hodge structure of weight $-m$.

If H and H' are Hodge structures of weight m and m', then $H \otimes H'$ is a Hodge structure of weight $m + m'$

$\bigwedge^n H$ is a Hodge structure of weight mn

Definition 1.4 (Tate Structure)**.** *Take $H_{\mathbb{Z}} = 2\pi i\mathbb{Z}$ and $H_{\mathbb{C}} = \mathbb{C} = H^{-1,-1}$ of weight -2.*

We define $T(m) = \otimes_m H_{\mathbb{Z}}$, and this gives a Hodge structure of wieght $-2m$ with $H_{\mathbb{C}} = H^{-m,-m}$.

Definition 1.5 (Hodge Structure)**.** *A Hodge structure of weight m is defined by $H_{\mathbb{Z}}$ and a finite, decreasing filtration F by subspaces $H_{\mathbb{C}} \supset \ldots \supset F^p \supset F^{p+1} \supset \ldots$ such that for all p, $F^p \oplus \bar{F}^{m-p+1} = H_{\mathbb{C}}$.*

Proposition 1.6. *The two definitions of Hodge structure are equivalent.*

Proof. Start with a decomposition. Define $F^p H_{\mathbb{C}} = \oplus_{i \geq p} H^{i,m-i} \subset H_{\mathbb{C}}$. Then $\bar{F}^p = \oplus_{i \leq m-p} H^{i,m-i}$, so the property of filtrations follows. For the other direction, define $H^{p,q} = F^p \cap \bar{F}^q$. □

Definition 1.7 (Morphism of Hodge Structures)**.** *A morphism of Hodge structures $L : H \to H'$ is a map defined on the abelian groups such that, after complexification, satisfies $L(F^p) \subset F'^p$.*

1.3 Polarization

Definition 1.8 (Polarization)**.** *A polarization of a Hodge structure H of weight m is a bilinear form $S : H_{\mathbb{Q}} \otimes H_{\mathbb{Q}} \to \mathbb{Q}$ which is symmetric for m even and skew-symmetric for m odd such that the complex extension satisfies $S(H^{p,q}, H^{p',q'}) = 0$ unless $p = q'$ and $q = p'$, and that $i^{p-q} S(v, \bar{v}) > 0$ for $v \neq 0$.*

This gives a positive definite Hermitian form.

Definition 1.9 (Mixed Hodge Structure)**.** *A mixed Hodge structure H is defined to be a finitely generated group $H_{\mathbb{Z}}$, an increasing filtration W on $H_{\mathbb{Q}}$ and a decreasing filtration F such that $\operatorname{Gr}_m^W H_{\mathbb{Q}}$'s complexification has a Hodge structure of weight m induced from F.*

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Take a family of elliptic curves: $\mathcal{C}_\lambda$ given by $y^2 = x(x-1)(x-\lambda)$ for $\lambda \neq 0, 1$ in $\mathbb{C}^2$, for each λ. We need to compactify:

Look at $\bar{\mathcal{C}}_\lambda \subset \mathbb{P}^2$ given by $P_\lambda = Y^2 Z - X(X-Z)(X-\lambda Z) = 0$. The fibers are then projective curves of genus 1. Each one is homeomorphic to $\mathbb{R}^2/\mathbb{Z}^2$.

For each λ, we could look at the fiber as $R = \mathbb{C}[X, Y, Z]/P_\lambda$, but we're going to rely a lot on topology, so we'll be better served by using the $\mathbb{R}^2/\mathbb{Z}^2$ model.

Thus, we have a map $\bar{\mathcal{C}} \to \mathbb{C} \setminus \{0, 1\} \ni \lambda_0$ and by Ehresmann's lemma, we have that for a small neighborhood of the point, the preimage is diffeomorphic to $U \times \bar{C}_{\lambda_0}$.

Calling the projection map f, this tells us that $R^i f_* \mathbb{Z}|_{f^{-1}(U)} \cong \mathbb{Z}_U$. So for every λ in U, we have $\bar{C}_{\lambda_0} \xrightarrow{h} \bar{C}_\lambda$ a homeomorphism, and if we take a different path and get h', the two homeomorphisms are the same.

However, if we go around a hole, so that the two paths comprise a nontrivial loop, we can't homotope one to the other, and so we get a nontrivial map when we go around the loop. This gives us maps $H^i(\mathcal{C}_{\lambda_0}) \to H^i(\mathcal{C}_\lambda) \to H^i(\mathcal{C}_{\lambda_0})$, which is not necessarily the identity. We call this monodromy.

The monodromy transformation is not compatible with Hodge structure!

Lemma 2.1 (Deligne). *Let* $I^{p,q} = (F^p \cap W_{p+q}) \cap \big[(\bar{F}^q \cap W_{p+q}) + (\bar{F}^{q-1} \cap W_{p+q-2}) + \ldots\big]$. *Then under the projection* $W_m \to \mathrm{Gr}_m^W = W_m/W_{m-1}$ *we have* $I^{p,q} \to (\mathrm{Gr}_m^W H)^{p,q}$ *for* $p + q = m$.

Now, $W_m = \oplus_{p+q=m} I^{pq}$ and $F^p = \oplus_{i \geq p} I^{pq}$. However, $\bar{I}^{p,q} \neq I^{q,p}$. However, it is mod W_{p+q-2}. If it's exactly true, we call the mixed Hodge structure split.

Definition 2.2 (Morphism of MHS). *A morphism of mixed Hodge structures is* $\mathscr{L} : (H, W, F) \to (H', W', F')$ *such that* $\mathscr{L} : H \to H'$, $\mathscr{L}(W_m) \subset W'_m$ *and* $\mathscr{L}(F_p) \subset F'_p$.

Lemma 2.3. $\mathscr{L}$ *is strict for* W *and for* F *and the kernel and cokernel are natural mixed Hodge structures.*

Proposition 2.4. *The category of MHS's is an abelian category.*

Corollary 2.5. *Let* $\ldots \to H^{n-1} \to H^n \to H^{n+1} \to \ldots$ *then the cohomology of this sequence is a MHS.*

Let X be a compact Kähler manifold of dimension n and Ω_X^* the holomorphic deRham complex. Then we can set $F^p \Omega_X^*$ to be $0 \to \Omega_X^p \to \ldots$, and (Ω_X^*, F) is a filtered complex. We take a resolution of Filtered complexes $\Omega_X^* \to \mathscr{E}_X^* = \mathcal{A}_X^*$ the differential forms, a quasi-isomorphism. Then we can define $F^p \mathscr{E}_X^*$ by $\ldots \to 0 \to \mathscr{E}_X^{p,0} \to \mathscr{E}_X^{p+1,0} \to \ldots$.

In degree k, $(F^p \mathscr{E}_X^*)^k = \oplus_{p+q=k, p' \geq p}$.

Now, $F^p \Omega_X^* \to F^p \mathscr{E}_X^*$ is a quasi-isomorphism, and $\mathrm{Gr}_F^p \Omega_X^* = F^p \Omega_X^* / F^{p+q} \Omega_X^* \to F^p \mathscr{E}_X^* / F^{p+q} \mathscr{E}_X^*$, and this gives a qis Ω_X^p to the Dolbeaut resolution.

Definition 2.6. $H^i(X, \mathbb{C}) = H^i(\mathscr{E}_X^*)$ *and* $F^p H^i(X, \mathbb{C})$ *is* $\mathrm{im} H^i(F^p \mathscr{E}_X^*) \to H^i(\mathscr{E}_X^*)$.

Define $\mathrm{Gr}_F^p H^i(X, \mathbb{C})$ ot be $F^p H^i / F^{p+1} H^i$. Then we have ${}_F E_1^{pq} \xrightarrow{\gamma} {}_F E_1^{p+1,q} \to H^q(X, \Omega_X^p) \xrightarrow{d} H^q(X, \Omega_X^{p+1})$.

Theorem 2.7 (Hodge)**.** *The spectral sequence degenerates at rank 1. That is,* $d = 0$.

Thus, $H^q(X, \Omega^p_X) \cong \mathrm{Gr}^p_F H^{p+q}(X, \mathbb{C})$.

But even more, $F^p H^m(X, \mathbb{C}) \oplus \bar{F}^{m-p+1} H^m \cong H^m$.

If X is projective, then $[\omega] \in H^2(X, \mathbb{Q})$ is a hyperplane section $[H]$.

The fundamental class of a subvariety Z of a compact complex manifold of codimension r in X where $\dim X = n$ is $[Z] \in H_{2n-2r}(X, \mathbb{Z})$ and then this sits inside $H_{2n-2r}(X, \mathbb{C})$, which is isomorphic to $H^{2n-2r}(X)^*$. This gives us a class ω and $\int_{[Z]} \omega \in \mathbb{C}$, and so ω is Poincaré dual to $\eta_Z \in H^{2r}$.

Lemma 2.8. $[\eta_Z] \in H^{r,r}(X) \cap \mathrm{Im}(H^{2r}(X, \mathbb{Z}) \to H^{2r}(X, \mathbb{C}))$ *and* $[\eta_Z] \neq 0$

Define $[\eta_Z]$ for Z a subvariety in X, including the possibility that it might be singular.

3 ElZein 3 - Mixed Hodge Complex (MHC)

3.1 Desingularization

Let X be a complex irreducible algebraic variety. Then there exists a Zariski open dense subset of smooth point $U_{smooth} \subset X$, and its complement is X_{sing} the singular locus.

Theorem 3.1 (Hironaka)**.** *There exists a diagram*

$$\begin{array}{ccccc} Y & \longrightarrow & X' & \longleftarrow & U' \\ \downarrow & & \downarrow & & \downarrow \\ Y_{sing} & \longrightarrow & X & \longleftarrow & U_{smooth} \end{array}$$

such that X' *is smooth,* Y *is a normal crossing divisor on* X' *and* $U' = X \setminus Y$ *is isomorphic to* U_{smooth}.

Consider X projective and smooth complete variety, $i : Z \to X$ closed in X irreducible of codimension r, then $\pi : Z' \to Z$ a desingularization. Then $[Z'] \in H_{2n-2r}(Z', \mathbb{Z})$ gives $i_* \pi_* [Z'] := [Z] \in H_{2n-2r}(X, \mathbb{Z})$. By Poincaré duality, we get $[\eta_Z] \in H^{2r}(X, \mathbb{Z})$.

Lemma 3.2. $[\eta_Z] \in H^{r,r}(X) \cap \mathrm{Im}(H^{2r}(X, \mathbb{Z}) \to H^{2r}(X, \mathbb{C}))$*, and we will call this space* $H^{r,r}(X, \mathbb{Z})$.

Proof. $\int_{[Z]} \omega = 0$ if $\omega \notin \mathscr{E}^{n-r,n-r}_X$, and so we take ω to be the Kähler (1,1)-form on X, then we can deduce that $\int_{[Z]} i_Z^* (\bigwedge^{n-r} \omega) \neq 0$, by positivity. □

We define $Z_r(X)$ to be the set of all formal integer linear combinations of irreducible subvarieties of X of codimension r. We have a map $c\ell : Z_r(X) \to H^{r,r}(X, \mathbb{Z})$.

Question: Is this map surjective?

Let X be a smooth, open algebraic variety, X singular.

Remark: If X is smooth and proper complex algebraic variety $H^i(X, \mathbb{C})$ carries a pure Hodge structure of weight i.

There exists X' projective and $\pi : X' \to X$ which induces an isomorphism on an open dense Zariski subset, and $\pi^* : H^i(X, \mathbb{C}) \to H^i(X', \mathbb{C})$.

Now, the Hodge filtration $F^p H^i$ is algebraically defined. We deduce that the Hodge filtration F on $H^i(X, \mathbb{Z})$ is a Hodge filtration of a Hodge structure.

3.2 The Hodge Complex

Definition 3.3 (Hodge Complex)**.** *A Hodge complex of weight n is defined as follows: a complex of groups $K_{\mathbb{Z}}$ bounded below, a filtered complex of complex vector spaces $(K_{\mathbb{C}}, F)$, F finite on each degree, and $K_{\mathbb{Z}} \otimes \mathbb{C} \to K_{\mathbb{C}}$, a quasi-isomorphism, and the differential on $K_{\mathbb{C}}$ is strict with respect to F.*

So, for all k, $H^k(K_{\mathbb{C}}) \cong H^k(K_{\mathbb{Z}}) \otimes \mathbb{C}$, and $H^k(K_{\mathbb{C}})$ with induced filtration F satisfies $F^p H^\ell(K_{\mathbb{C}}) = \mathrm{Im}(H^k(F^p K_{\mathbb{C}}) \to H^\ell(K_{\mathbb{C}}))$, and (H^k, F) is a Hodge structure of weight $n + k$.

Definition 3.4 (Cohomological Hodge Complex)**.** *Let X be a topological space, a CHC of weight n K on X is a complex of sheaves $K_{\mathbb{Z}}$ bounded below, a filtered complex of sheaves $(K_{\mathbb{C}}, F)$, a quasi-isomorphism $\alpha : K_{\mathbb{C}} \to K_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ such that $(\mathbb{R}\Gamma K_{\mathbb{Z}}, \mathbb{R}\Gamma(K_{\mathbb{C}}, F), \mathbb{R}\Gamma(\alpha))$ is a Hodge complex of weight n*

Definition 3.5 (Mixed Hodge Complex)**.** *A mixed Hodge complex is $K_{\mathbb{Z}}$ a complex of groups bounded below, $(K_{\mathbb{Q}}, W)$ a filtered complex of $\mathbb{Q}$ vector spaces with $K_{\mathbb{Q}} \cong K_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ and $(K_{\mathbb{C}}, W, F)$ bifiltered, $\alpha : (K_{\mathbb{Q}}, W) \otimes \mathbb{C} \to (K_{\mathbb{C}}, W)$ an quasi-isomorphism such that for all n we have $(\mathrm{Gr}_n^W K_{\mathbb{Q}}, (\mathrm{Gr}_n^W K_{\mathbb{C}}, F))$ along with $\mathrm{Gr}_n^W(\alpha)$ is a Hodge complex of weight n.*

The definition of a cohomological mixed Hodge complex is similar.

Theorem 3.6 (Deligne)**.** *Let K be a MHC then the filtration $W[n]$ on $H^n(K_{\mathbb{Q}}) \cong H^n(K_{\mathbb{Z}}) \otimes \mathbb{Q}$ and the filtration F on $H^n(K_{\mathbb{C}})$ define a MHS.*

Proof. Take the spectral sequence ${}_W E_r^{pq}$. It has ${}_W E_1^{pq} = H^{p+q}(\mathrm{Gr}_{-p}^W K)$. These are HS of weight q. Look at d_1. We must prove that this map is compatible with F. Then $E_2^{p,q}$, the cohomology, is a Hodge structure of weight q. And then, d_2 can be shown to be zero, because it is a morphism of Hodge structures of different weights. □

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Theorem 4.1 (Deligne)**.** *Let X be a smooth complex algebraic variety. Then the cohomology groups carry a functorial MHS.*

You cannot define F on $\mathbb{Q}$, but you REALLY need to define W on $\mathbb{Q}$, before you get to $\mathbb{C}$.

Consider Y a normal crossing divisor in a smooth proper algebraic variety X such that $X^* \cong X \setminus Y$. For $y \in Y$, there exists $U \ni y$ in X with $z_1, \ldots, z_n$ such that $z_1 \ldots z_r = 0$ is a local equation for Y.

We can write $Y = \cup_i Y_i$ with the indices coming from an ordered set and the Y_i are irreducible. Let σ by $\sigma_1 < \ldots < \sigma_a$ in I, then $Y_\sigma = Y_{\sigma_1} \cup \ldots \cup Y_{\sigma_a}$ be a smooth proper algebraic variety.

Let $(\mathscr{E}^*_{X^*}, F)$ be the filtered complex of differential forms on $X* = X \setminus Y$ then take the global sections and compute the cohomology. This F will be cpute the correct Hodge filtration!

The deRham complex with logarithmix singularities along Y, $\Omega^*_X\langle Y\rangle = \Omega^*_X(\log Y) \subset \gamma_* \Omega^*_{X^*}$ where $\gamma : X^* \to X$ is the inclusion.

Locally, $\Omega^k_X(\log Y)|_U$ is the free $\mathscr{O}_U$-module generated by $\frac{dz_{i_1}}{z_{i_1}} \wedge \ldots \wedge \frac{dz_{i_\ell}}{z_{i_\ell}} \wedge dz_{j_1} \wedge \ldots \wedge dz_{j_k}$ where $i_r \leq r$, $j_s > r$ and $\ell + m = k$.

The weight filtration W with σ of length a form a set S^a with $Y^a = \coprod_{\sigma \in S^a} Y_a$ and $W_m \Omega^p_X(\log Y) = \sum_{\sigma \in S^m} \Omega^{p-m}_X \wedge \frac{dz_{\sigma_1}}{\sigma_1} \wedge \ldots \wedge \frac{dz_{\sigma_m}}{z_{\sigma_1}}$ contains a maximum of m of the $\frac{dz_i}{z_i}$.

W_m is an increasing filtration by subcomplexes, by construction.

Now, look at the residue map $\mathrm{Res} : \mathrm{Gr}^W_m \Omega^i_X(\log Y) \to \prod_\alpha \Omega^{i-m}_{Y^m}$ for each σ. $\alpha \wedge \frac{z_{\sigma_1}}{\sigma_1} \wedge \ldots \wedge \frac{dz_{\sigma_m}}{z_{\sigma_m}} \to \alpha|_{Y^\sigma}$.

Lemma 4.2. Res *is an isomorphism*

For any σ, there's an inverse ρ_σ taking α to wedgeing it with the $\frac{dz_i}{z_i}$ in σ.

Now, we get a map $\mathrm{Res} : (\mathrm{Gr}^W_m \Omega^*_X(\log Y), F) \to (\pi_* \Omega^*_{Y_m}[-m], F[-m])$ is a filtered isomorphism.

The associated gradeds are also isomorphic, then.

Corollary 4.3.

1. $H^p(\mathrm{Gr}^W_m \Omega^*_X(\log Y)) \cong \begin{cases} \pi_* \mathbb{C}_{Y^m} & p = m \\ 0 & \textit{else} \end{cases}$
2. $H^p(W_m \Omega^*_X(\log Y)) \cong \begin{cases} \pi_* \mathbb{C}_{Y^p} & p \leq m \\ 0 & \textit{else} \end{cases}$
3. $H^p(\Omega^*_X(\log Y)) \cong \pi_* \mathbb{C}_{Y^p}$ *for all* p

Exercise 4.4. $0 \to W_{m-1}\Omega^*_X(\log Y) \to W_m(\Omega^*_X(\log Y) \to \mathrm{Gr}^W_m \Omega^*_X(\log Y) \to 0$ *is exact.*

Now, we claim that for $y \in Y$ we have $H^p(W_m \Omega^*_X(\log Y))_y = H^p(U(y) \cap X^*, \mathbb{C})$ for $p \leq m$.

((Shamefully, I lost the thread here))

There is a mixed Hodge complex with $K_\mathbb{Z} = \mathbb{R}\Gamma(X, \mathbb{R}j_* \mathbb{Z}_{X \setminus Y})$ where $j : X \setminus Y \to X$ is the inlcusion, with $(K_\mathbb{Q}, W) = (\mathbb{R}\Gamma(X, \mathbb{R}j_* \mathbb{Q}_{X \setminus Y}), T = W)$ where T is the truncation functor, and with $(K_\mathbb{C}, W, F) = (\mathbb{R}\Gamma(X, \Omega^*_X(\log Y), W, F)$.

And then $(\mathrm{Gr}_m^W(K^{\mathbb{Q}}))$ and $(\mathrm{Gr}_m^W(K^{\mathbb{C}}), F)$ form a Hodge complex of weight m, and $(\pi_*\Omega^*_{Y^m}[-m], F[-m])$ is a Hodge complex of weight m.

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We're going to be working with the derived category, now. For $D^+(X)$, we take a complex and an injective resolution, then apply derived functors. We set $D^+F(X)$ to be the derived category of filtered complexes, so we require $F^pK \to F^pI$ qis and $\mathrm{Gr}_F^p K \to \mathrm{Gr}_F^p I$ an iso. Finally, we define $D^+F^q(X)$ the derived category of bifiltered complexes.

5.1 The Weight Filtration

Now $(\Omega^*_X(\log Y), W, F)$ is a MHS. Let $\Gamma : \mathcal{A} \to \mathcal{B}$ be a functor of abelian categories, we require that the functor be left exact. Let (K, F) be a filtered complex, then we have ${}_FE_1^{pq} = R^{p+q}\Gamma(\mathrm{Gr}_F^p K) = H^{p+q}(\mathbb{R}\Gamma(\mathrm{Gr}_F^p K))$.

Now, we look at ${}_WE_1^{pq} = H^{p+q}(\mathbb{R}\Gamma \,\mathrm{Gr}_{-p}^W K)$, but instead, we can use hypercohomology, and get ${}_WE_1^{pq} = \mathbb{H}^{p+q}(X, \mathrm{Gr}_{-p}^W \Omega^*_X(\log Y) = H^{p+q}(X, \pi_*\mathbb{C}_{Y^{-p}}[p])$, which is finally $\oplus_\sigma H^{2p+q}(Y^\sigma, \mathbb{C})$ where $Y^\sigma = Y_{\sigma_1} \cap \ldots \cap Y_{\sigma_k}$ with the induced Hodge filtration.

So, this has a Hodge structure of weight q.

${}_WE_1^{pq} = \oplus_{|G|=-p} H^{2p+q}(Y^\sigma)(P)$ has weight q, and we have d_1 which maps $H^{2p+q}(Y^\sigma) \to \oplus H^{2p+q+2}(Y^{\sigma'})(p+1)$, and if $Y^{\sigma'} \supset Y^\sigma$ then there is $s \in [1, -p]$ such that the Gysin map i_s (the Poincaré dual of the restriction map) allows us to compute that $E_2^{pq} = \mathrm{Gr}_{-p}^W H^{p+q}(X - Y, \mathbb{C})$, which is a HS of weight q.

And this gives us $W[n]$ on H^n.

5.2 Simplicial Resolutions

Let X be a variety (in particular, a topological space).

Definition 5.1 (Simplicial Variety)**.** *A simplicial variety X_* over X is a family of varieties $\pi_n : X_n \to X$ for all $n \in \mathbb{N}$ such that for each increasing map $f : [0, n] \to [0, m]$ we define a morphis $X(f) : X_m \to X_n$ over X satisfying the natural composition laws.*

Definition 5.2. *A sheaf F on X_* is a family of sheaves F_n on X_n satisfying that $F_*(f) : F_n \to X_*(f)_*F_m$ for all increasing functions $[0, m] \to [0, n]$ satisfying the natural conditions.*

So now, we take a complex $\pi_*F_0 \to \ldots \to \pi_*F_n \xrightarrow{d_n} \pi_*F_{n+1} \to \ldots$ with $d_n = \sum_{i=0}^{n+1}(-1)^i\delta_i$ where $\delta_i : [0, n] \to [0, n+1]$ is increasing function skipping i.

So now, let K^* be a complex of sheaves on X_*. This is a family of complex $K^{*,n}$ for each n. A resolution is $K^{*,n} \to I^{*,n}$ which must be compatible.

Definition 5.3. $\mathbb{R}\pi_* K = s(\pi_* I^{*,*})$ *with differential* d*.* $(\mathbb{R}\pi_* K^*)^n = \oplus_{p+q=n} \pi_*^q I^{pq}$ *and* $d(x^{pq}) = d_I(x^{pq}) + (-1)^p \sum_{i=0}^{n+1} \delta_i(x^{pq})$*.*

Definition 5.4 (Cohomological Descent)**.** *Let* $\pi : X_* \to X$ *be a simplicial variety over* X*. Then* π *is of cohomological descent if for all* F *on* X *we have* $F \xrightarrow{\cong} \mathbb{R}\pi_*(\pi^* F)$*.*

Theorem 5.5 (Deligne)**.** *For each separated complex variety* X *there exists a simplicial variety* X_* *which is proper and smooth and a normal crossing divisor* $Y_* \subset X_*$ *and a map* $\pi : U_i = X_i - Y_i \to X$ *satisfying the cohomological descent property.*

So by the descent property, we have $H^*(U_*, \mathbb{Z}) \cong H^*(X, \mathbb{Z})$ and we can give the former an MHS, so it gives the latter one.