# Applications to the Beilinson-Bloch Conjecture

Green

June 30, 2010

## 1 Green 1 - Applications to the Beilinson-Bloch Conjecture

California is like Italy without the art. - Oscar Wilde

Let $X$ be a smooth projective variety. There are two ways to look at it. One is to look at it as a compact Kähler manifold with a Hodge metric giving an projective embedding. The other is to look at $X$ as already a subset of $\mathbb{P}^n(k)$ given by explicit equations and work algebraically.

We use the Hodge metric to get a Hodge structure, which sits inside $\Gamma\backslash D$. A lot of what you get here comes from algebra, but aren't Hodge structures of varieties, and even occasionally from analysis.

We can take the field of definition of $X$, $k$, to be finitely generated over $\mathbb{Q}$, by noting that $X$ is defined over $\mathbb{Q}$ adjoin the coefficients of the equations. Now, some things we can do with $k$ as an abstract field, but others, we can do only when $k \subset \mathbb{C}$.

Let's look at $\mathbb{Q}(\pi)$. As $\pi$ is transcendental, we can represent elements of this field by $p(\pi)/q(\pi)$ where $p, q \in \mathbb{Q}[x]$. We can similarly describe $\mathbb{Q}(e)$. Abstractly, these fields are isomorphic. But they're different subfields of $\mathbb{C}$. We can think of these as both being $\mathbb{Q}(x)$, which is distinct from $\mathbb{Q}(\sqrt{7})$.

So look at the elliptic curves $y^2 = x(x-1)(x-\pi)$ and $y^2 = x(x-1)(x-e)$. As abstract curves over $\mathbb{Q}(x)$, they're isomorphic. However, over $\mathbb{C}$, the Hodge structures are not equivalent.

In mathematics, we've got a break between technology, which we need to prove theorems, and intuition, which we need to figure out theorems.

If we write $k$ as a field finitely generated over $\mathbb{Q}$, then $k = \mathbb{Q}(\alpha_1, \ldots, \alpha_t)[\beta_1, \ldots, \beta_r]$ where $\alpha_1, \ldots, \alpha_t$ are algebraically independent and $\beta_1, \ldots, \beta_r$ are algebraic over it.

Enter geometry. $\mathbb{Q}(x) = \mathbb{Q}(\mathbb{P}^1)$, so $k = \mathbb{Q}(S)$ for $S$ defined over $\mathbb{Q}$, and $\mathbb{Q}(S_1) \cong \mathbb{Q}(S_2)$ if $S_1$ and $S_2$ are birational over $\mathbb{Q}$.

A point $s \in S(\mathbb{C})$ is a geometric point, or a very general point of $S$ if $s$ does not lie on any proper $\mathbb{Q}$-subvariety of $S$ (that is, the Zariski closure of $s$ over $\mathbb{Q}$ is $S$)

**Example 1.1.** *For $S = \mathbb{P}^1$, $s$ is very general if and only if $s$ is transcendental.*

Now, let $k = \mathbb{Q}(\alpha)$ for $\alpha$ transcendental, and look at $y^2 = x(x-1)(x-\alpha)$. For every very general point of $S$, in fact, for $s \neq 0, 1, \infty$, we get a smooth elliptic curve. So we have $\mathcal{X} \to S \setminus E$, and $E$, the discriminant locus, is defined over $\mathbb{Q}$.

**Example 1.2.** *Let $k$ be a number field. Then $k = \mathbb{Q}(\alpha)$ and $\alpha$ has minimal polynomial $p(x) \in \mathbb{Q}[x]$. Now let $S$ be the variety defined by $p$, that is, a finite number of points. $S$ is defined over $\mathbb{Q}$, but the points are not, individually, defined over $\mathbb{Q}$. $X$ defined over $k$ means that we get $[k : \mathbb{Q}]$ complex varieties, one for each embedding into $\mathbb{C}$.*

**Example 1.3.** *Let $X$ be defined over $\mathbb{Q}$ and $p \in X$ a very general point. Then $X \to \mathbb{P}^n(\mathbb{Q})$ by $p \mapsto (p_0, \ldots, p_n)$ then $k = \mathbb{Q}(p_1/p_0, \ldots, p_n/p_0)$. We can take $S = X$, so transcendence degree is $\dim X$.*

**Example 1.4.** *Let $X$ be defined over $\mathbb{Q}$. Take $(p, q) \in X \times X$ a very general point. Then we get an embedding into $\mathbb{P}^n$, and the field's transcendence degree is $2 \dim X$.*

**Example 1.5.** *Look at hypersurfaces of degree $d$ in $\mathbb{P}^N$. If $F = \sum_{|I|=d} a_I x^I$, then if we look at $k = \mathbb{Q}(a_I)$ we get a much larger dimensional projective space for $S$, and we hafve $\mathcal{X} \to S$ the universal family of hypersurfaces.*

So, which computations do we actually need the complex embeddings for? Grothendieck learned to compute cohomology groups using just $k$, but for the Hodge structures, really need $\mathbb{C}$.

Big Point: Hodge structures require a complex embedding.

Set $y^2 = x(x-1)(x-\alpha) = f(x)$. If we differentiate, we get $2ydy = f'(x)dx$, so we have $\frac{2dy}{f'(x)} = \frac{dx}{y}$, and this lets us represent the holomorphic 1-form on $E$. Call it $\omega$.

Now, let $\lambda$ be a simple closed homotopically nontrivial curve in $E$. Look at $\int_\lambda \omega$. We can write $\lambda = \sigma_1 + \ldots + \sigma_n$ a bunch of lines, and so we need to integrate $\omega$ along each of these, and $\int_{\sigma_m} \frac{dx}{y} = \int_{\sigma_m} \frac{dx}{\sqrt{f(x)}}$, and we need $\int_C \frac{dx}{\sqrt{x(x-1)(x-\alpha)}}$. We can expand as a power series and integrate.

Now, the point is that hte integral lattice $H^r(X, \mathbb{Z}) \subset H^r(X, \mathbb{C})$ is going to depend transcendentally on $s$, on complex embedding $k \to \mathbb{C}$.

Now, take $X$ a smooth variety defined over $k$. We'll be wanting $\Omega^1_{X(k)/k}$ to be the Kähler differentials over $k$. That is, the module $d(f+g) = df + dg$, $da = 0$ for $a \in k$ and $d(fg) = fdg + gdf$. We also have $\Omega^1_{X(k)/\mathbb{Q}}$ where we only have $da = 0$ for $a \in \mathbb{Q}$.

Now, $da = 0$ for all $a \in k$ actually implies $da = 0$ for all $a \in \bar{k}$. Why? Look at the minimal polynomial of $\alpha$, $a_0\alpha^m + \ldots a_m = 0$. Take $d$ of this, and the Liebniz rule implies that we have some element of $k$ times $d\alpha = 0$, and so $d\alpha = 0$.

So $\Omega^1_{k/\mathbb{Q}}$ is a $k$ vector space with basis $d\alpha_1, \ldots, d\alpha_\ell$.

These Kähler differentials give us a complex $\Omega^*_{X(k)/\mathbb{Q}}$, and need to be careful, this complex doesn't end at $\dim X$.

## 2 Green 2

Let $X$ be a smooth variety defined over $k$ where $k$ is finitely generated over $\mathbb{Q}$. Then we defined last time $\Omega^1_{X(k)/k}$ and $\Omega^1_{X(k)/\mathbb{Q}}$, and defined the complex $\Omega^*_{X(k)/k}$

Now, we need to define hypercohomology. Start with a complex $A^*$, then $H^r(A^*) = \ker d/\mathrm{im} d$ on $A^r$.

Now, look at $A^{*,*}$ a double complex, and take the total complex. This is a complex if $d, \delta$ the horizontal and vertical operators anticommute, and then we use $D = d + \delta$, and hypercohomology is defined by $\mathbb{H}^r(A^{*,*}) = H^r(T^*, D)$.

Note: Hypercohomology is not on the list in "My Favorite Things"

Using Cech cohomology, we define $A^{p,q} = C^q(\mathfrak{U}, \Omega^p)$, and add a sign to the Cech differential, then we can take Hypercohomology.

Now, there's a theorem that tells us that for $i_s : k \subset \mathbb{C}$, $\dim_k \mathbb{H}^n(\Omega^*_{X(k)/k}) = \dim_{\mathbb{C}} H^n_{dR}(X_s, \mathbb{C})$.

**Theorem 2.1** (Grothendieck Comparison Theorem)**.**

1. $\mathbb{H}^n(\Omega^*_{X(k)/k} \otimes_{i_s} \mathbb{C} \cong H^n(X_s, \mathbb{C})$
2. $H^q(\Omega^p_{X(k)/k}) \otimes_{i_s} \mathbb{C} \cong H^{p,q}(X_s)$
3. $\mathbb{H}^n(\Omega^{\geq p}_{X(k)/k}) \otimes_{i_s} \mathbb{C} \cong F^p H^n(X_s, \mathbb{C})$

We can't read off the integral lattice, though.

$\mathrm{Gr}^m \Omega^*_{X(k)/\mathbb{Q}}$ is defined to by $F^m\Omega^*/F^{m+1}\Omega^*$ and that's $\Omega^m_{k/\mathbb{Q}} \otimes \Omega^{*-m}_{X(k)/k}$.

He compares the derivation of the long exact sequence on cohomology to X-rays...not good for you to see too many times.

Looking at spreads: take $\mathcal{X} \to S$, this gives a variation of Hodge structure over $S \setminus \Sigma$, the smooth locus. Looking at the Gauss-Manin connection $\nabla$, we have a complex $(\Omega^*_{k/\mathbb{Q}} \otimes \mathbb{H}^r(\Omega^*_{X(k)/k}, \nabla)$, and we can get Griffiths Transversality (also called the infinitesimal period relations)

Now, say that $k = \mathbb{Q}(x_1, \ldots, x_T)[y_1, \ldots, y_A]/(p_1, \ldots, p_B)$, then $\Omega^1_{k/\mathbb{Q}}$ is generated by $dx_1, \ldots, dx_A$.

**Example 2.2.** *Recall the example $y^2 = x(x-1)(x-\alpha)$. If we differentiate, but don't assume that $d\alpha = 0$, we get $2ydy = f'(x)dx - x(x-1)d\alpha$. And now, $\frac{2dy}{f'(x)} = \frac{dx}{y} - \frac{x(x-1)}{f'(x)y}d\alpha$. And this last term gives us problems lifting. Fortunately, the coboundary map is essentially a measure of how much lifting fails, so the last term can be thought of as an element of $\Omega^1_{k/\mathbb{Q}} \otimes H^1(\mathscr{O}_{X(k)})$, so the whole thing is $\nabla\omega$ for some 1-form.*

Now we introduct $Z^p(X(k))$, the codimension $p$ cycles defined over $k$ and $CH^p(X(k))$ the cycles mod rational equivalence defined over $k$.

So now, we define $K_p(\mathscr{O}_{X(k)})$ to be the Quillen $p$th K-group for $\mathscr{O}_{X(k)}$. Now, in the $k$ Zariski topology, we have $CH^p(X(k)) \cong H^p(K_p(\mathscr{O}_X))$. Modulo torsion, it's a theorem of Soulé that we can replace Quillen K-theory with Milnor K-theory.

$K_p^{Milnor}$ is generated multiplicatively by $f \in \mathscr{O}^*_{X(k)}$ elements $f_1 \otimes \ldots \otimes f_p$ modulo the Sternberg relations that if $f_i = 1 - f_j$ for some $i \neq j$, then $f_1 \otimes \ldots \otimes f_p = 0$.

(The lecture here became very difficult to typeset notes for, due to a sequence of complicated commutative diagrams. There are printed notes, and I will link to them as soon as they're online)

The key point is that we have an arithmetic cycle class $CH^p(X(k)) \to H^p(\Omega^p_{X(k)/\mathbb{Q}})$, and there is a criterion relating Hodge classes, and relating all of this to the Absolute Hodge conjecture.

# 3 Green 3

Remember, we have $k/\mathbb{Q}$ finitely generated, and this gives us $S$ with $\mathbb{Q}(S) \cong k$. We also start with $X$ defined over $k$ and get $\mathcal{X} \to S$.

We set $Z^p(X(k))$ the cycles on $X$ defined over $k$, take it mod rational equivalence on $X$ defined over $k$, and we get $CH^p(X(k))_\mathbb{Q} = CH^p(X(k)) \otimes_\mathbb{Z} \mathbb{Q}$.

There is a conjectural filtration $CH^p(X(k))_\mathbb{Q} = F^0 \supset F^1 \supset \ldots \supset F^{p+1} = 0$ with $F^1$ the homologically (but not rationally) trivial cycles.

For any $\Gamma \in Z^n(X \times Y(k))$ we get a map $CH^p(X(k)) \to CH^{p+n-\dim X}(Y(k))$ which should take $F^{n_1}$ to $F^{n_1+n_2}$.

**Example 3.1.** *$F^1CH^p(X(k))_\mathbb{Q}$. Let $Z \in Z^p(X(k))$. Then $[Z] \in \mathbb{H}^{2p}(\Omega^*_{X(k)/k})$, and then $Z \in F^1$ if and only if $[Z] = 0$.*

**Example 3.2.** *Now look at $F^2$. We expect that $F^2$ is the kernel of $F^1 \to J^p(X_s) \otimes \mathbb{Q}$, the Abel-Jacobi map for $X_s$. So we need that this kernel doesn't depend on which $s$ is chosen for very general $s$.*

Beilinson's Conjectural Formula: $\operatorname{Gr}^m CH^p(X(k))_\mathbb{Q} \cong \operatorname{Ext}^m_{MM_k}(\mathbb{Q}, H^{2p-m}(X)(p))$ where $MM_k$ is the category of mixed motives over $k$. So, in particular, we have no idea how to compute this Ext group in general.

Now, we're going to talk about $\operatorname{Ext}^1$ and the Abel-Jacobi map.

Let $X$ be a smooth projective variety defined over $k$. Take $Z_i$ smooth cycles defined over $k$, $f_i : Z_i \to X$ the inclusion, defined over $k$. Set $Z = \sum n_i f_{i*} Z_i$. Then we have $\Omega^*_{X(k)/k} \oplus \bigoplus_i \Omega^{*-1}_{Z_i(k)/k}$ and we have differential $\alpha \oplus \bigoplus \beta_i \mapsto d\alpha \oplus \bigoplus (d\beta_i - f_i^* \alpha)$.

Then $d^2 = 0$, so we have a complex.

We can get an exact sequence $0 \to k(-(m-p)) \to V \to H^{2m-2p+1}(\Omega^*_{X(k)/k}) \to 0$, and we can construct splittings that respect the Hodge filtration, and splittings that respect the integral lattice, but not one that does both! So, working out the set of extension classes, we get $J^p(X_s)$, so it is $\operatorname{Ext}^1_{MHS}(\mathbb{Z}, H^{2p-1}(X_s)(p))$.

# 4 Green 4

Today we're going to look at cycles over $k$ on a variety $X$ defined over $\mathbb{Q}$.

**Conjecture 4.1** (Deligne-Bloch-Beilinson). *Let $X$ be defined over $\mathbb{Q}$. Then $CH^p(X(\mathbb{Q}))_\mathbb{Q}$ is captured by cycles classes and $\mathcal{AJ}^p_X \otimes \mathbb{Q}$.*

That is, $F^2 = 0$.

The Conjecture in fact says that $F^m = 0$ for $m \geq$ the transcendence degree of $k$, plus two, for $X$ defind over $k$.

Now, if $X$ is defined over $\mathbb{Q}$ and $k$ a finitely generated extension of $\mathbb{Q}$, we can find $S$ with $k = \mathbb{Q}(S)$, and set $\mathcal{X} = X \times S$, and for any cycle $Z \in Z^p(X(k))$ we can spread it to $\mathcal{Z} \in Z^p(X \times S(\mathbb{Q}))$, and there will exist a $W \subset S$ a proper subvariety, defined over $\mathbb{Q}$ of lower dimension with $\mathcal{W} \in Z^{p-1}(X \times W)$ such that $\mathcal{Z} \to \mathcal{Z} + \mathcal{W}$.

If the conjecture on varieties over $\mathbb{Q}$ is ok, then $Z \cong 0$ in rational equivalence over $\mathbb{Q}$ for some $\mathcal{W}$, so $[\mathcal{Z} + \mathcal{W}]$ is torsion, and so its Abel-Jacobi image is also zero after tensoring with $\mathbb{Q}$.

(I lost track of the lecture here)