# Hodge Theory of Maps

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## 1 Migliorini 1 - Hodge Theory of Maps

The existence of a Kähler form give strong topological constraints via Hodge theory. Can we get similar constraints on algebraic maps?

Let $f : X \to Y$ a proper morphism. Our goal is going to be to linearize the problem.

For this lecture, we will assume $f$ is projective and smooth, to simplify the problem. In fact, we will assume $X$ and $Y$ are nonsingular.

So our map factors $X \to \mathbb{P}^N \times Y \to Y$ and $df$ is surjective. Equivalently, there exists a line bundle $L$ on $X$ which restricts on every fiber to an ample line bundle.

If $X \subseteq \mathbb{P}^n$ is a projective, nonsingular variety, tehn there's a universal hyperplane section $\{(x, H) | x \in H\} \subset X \times (\mathbb{P}^n)^*$ and call it $\mathcal{X}$, we have a map $\mathcal{X} \to (\mathbb{P}^n)^*$. This is not smooth, and we call this map $h$. Define $X^\vee$ to be the set of hyperplanes tangent to $X$. Then $h^{-1}((\mathbb{P}^n)^* \setminus X^\vee) \to (\mathbb{P}^n)^* \setminus X^\vee$, we have a smooth projective morphism.

**Lemma 1.1** (Ehresmann Lemma)**.** *Let $M, N$ be $C^\infty$ manifolds, $N$ connected, $f : M \to N$ a proper submersion. Then all of the fibers of $f$ are diffeomorphisc and $f$ is a $C^\infty$ locally trivial fibration.*

In fact, if $N$ is contractible, then $M$ is just $N \times F$.

If $\gamma$ is a look in $N$ starting at $n_0 \in N$, then you get in general a diffeomorphism of $f^{-1}(n_0)$ to itself that is not isotopic to the identity.

And example of this is to take a family of elliptic curves over $\mathbb{C}^*$, then a nontrivial loop in $\mathbb{C}^*$ gives a nontrivial (nonisotopically trivial, even) diffeomorphism $E \to E$.

As the diffeomorphisms depend only on the homotopy classes of looks, we get a natural representation $\pi_1(N, n_0) \to \mathrm{Aut}(H^k(f^{-1}(n_0), \mathbb{Z}))$.

Let $f : M \to N$. We define $R^i f_* \mathbb{Q}$ to be the sheaf on $N$ associated to the presheaf which sends $U$ to $H^i(f^{-1}(U), \mathbb{Q})$. These are locally constant sheaves, but not necessarily constant, due to monodromy, that is, the representation of $\pi_1$ mentioned above.

So what does all this have to do with Hodge theory?

**Example 1.2.** *Consider* $S^1 \to S^3 \to S^2$*, the Hopf fibration. Here* $R^0 f_*\mathbb{Q} = R^1 f_*\mathbb{Q} = \mathbb{Q}$.

*Also look at* $S^2 \times S^1 \to S^2$*. Then* $R^0 f_*\mathbb{Q} = R^1 f_*\mathbb{Q} = \mathbb{Q}$*. The Künneth formula tells us that* $H^*(S^2 \times S^1) = H^*(S^2) \otimes H^*(S^1)$*. But for the Hopf fibration, this fails:* $H^*(S^3)$ *is different.*

In general, for a fibration, knowledge the the $R^i f_*\mathbb{Q}$ does not suffice to compute the cohomology of the total space.

Now, if $f : M \to N$ is a fibration, there is a Leray Spectral Sequence with $E_2^{pq} = H^p(N, R^q f_*\mathbb{Q}) \Rightarrow H^{p+q}(M)$.

**Theorem 1.3** (Deligne)**.** *If* $f : X \to Y$ *is projective and smooth with* $Y$ *connected, then the Leray spectral sequence degenerates at* $E_2$.

If $Y$ is simply connected, then $H^*(X) = H^*(Y) \otimes H^*(F)$, as groups.

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Today we'll be following Chapter 4 of Volume II in C. Voisin's book.

Let $f : X \to Y$ a smooth projective morphism. We want to prove that the Leray Spectral Sequence degenerates at $E_2$. So then morally, $H^p(X) = \oplus_{a+b=p} H^q(Y, R^b f_*\mathbb{Q})$.

The key fact we'll need is the Hard Lefschetz theorem, and here it tells us that for $y_0 \in Y$, we have $H^k(f^{-1}(y_0)) = \oplus L^r H_0^{k-2r}(f^{-1}(y_0))$ where $L$ is the operator that takes the cup product with the Chern class of a relatively ample line bundle $\mathscr{L}$ on $X$, $H_0$ is the primitive cohomology, and $H_0^{n-k} = \ker L^{k+1}$.

Because $\mathscr{L}$ is globally define, this decomposition is monodromy invariant.

This decomposition is compatible with the differentials in the LSS. In fact, this decomposition, by Cattani's first set of notes, holds already at the level of forms.

$L \in H^2(X) = \hom(\mathbb{Q}, \mathbb{Q}[2])$, cupping with $L$ gives a morphism $Rf_*\mathbb{Q} \to Rf_*\mathbb{Q}[2]$.

EG, prove $d_2 = 0$. Enough to show on the primitive part.

$$\begin{tikzcd}
H^p(Y, R_0^{n-k} f_*\mathbb{Q} \arrow[r, "d_2"] \arrow[d, "L^{k+1}=0"] & H^{p+2}(Y, R^{n-(k+1)} f_*\mathbb{Q} \arrow[d, "L^{k+1}\cong H.L"] \\
H^p(Y, R^{n+k+2} f_*\mathbb{Q} \arrow[r, "d_2"] & H^{p+2}(Y, R^{n+(k+1)} f_*\mathbb{Q}
\end{tikzcd}$$

commutes. The same argument gives that $Rf_*\mathbb{Q} = \oplus R^k f_*\mathbb{Q}[-k]$.

This splitting, however, is not canonical.

Now, degeneration at $E_2$ implies that $H^p(X) \to H^p(Y, R^q f_*\mathbb{Q}) = H^p(f^{-1}(y_0))^{\pi_1} \to 0$

Now, if we have a projective map $\bar{f} : \bar{X} \to \bar{Y}$, $Y \to \bar{Y}$ the inclusion of an open sense subset such that $f : X \to Y$, the base change, is smooth, then there is a strong generalization of the above.

**Example 2.1.** *Let* $C$ *be a nodal cubic, given by* $Y^2Z = X^2(X - Z)$. *Then* $H^1(C) = \mathbb{Q}$, *and we can't expect* $H^1(C, \mathbb{Z}) = H^{1,0} \oplus \bar{H}^{1,0}$.

This can be fixed by looking at MHS, as discussed by ElZein. Here we have a finite increasing filtration $W$ and a finite decreasing filtration $F$, where $F$ induces a pure Hodge structure on each $W_k/W_{k-1}$ over $\mathbb{C}$ of weight $k$.

**Theorem 2.2.** *A map of MHS is strict with respect to* $W$ *and* $F$.

Let $\phi : H \to H'$ nad $a \in \mathrm{Im}(\phi) \cap W_kH'$ then $a \in \mathrm{Im}W_kH$.

The following theorem is actually true:

**Theorem 2.3** (Deligne)**.** *There is a functorial mixed Hodge structure on the cohomology groups of every open algebraic variety* $X$.

In general, $W_aH^\ell(X) = 0$ if $a < 0$ and $W_aH^\ell(X) = H^\ell(X)$ if $a \geq 2\ell$.

So on the nodal cubic, the $H^1$ classes are of type $(0, 0)$.

If $X$ is nonsingular, maybe no compact, then $W_aH^\ell(X) = 0$ if $a < \ell$ and if $X$ is compact, possibly singular, then $W_\ell H^\ell(X) = H^\ell(X)$.

So this implies that the Hodge structure guaranteed by the theorem of Deligne above is pure if $X$ is smooth and compact.

Weight Trick: Let $Y$ compact inside $X$ smooth lie inside $\bar{X}$, smooth and compact. Then $\mathrm{Im}(H(X) \to H(Y)) = \mathrm{Im}(H(\bar{X}) \to H(Y))$.

**Example 2.4.** $S^1 \subset \mathbb{R}^2 \setminus \{0\} \subset S^2$, *then* $H^1(\mathbb{R}^2 \setminus \{0\}) \to H^1(S^1)$ *is an isomorphism, but* $H^1(S^2) \to H^1(S^1)$ *is not, it's the zero map. The problem here is that the spaces are not algebraic. The Weight trick is a very algebraic phenomenon.*

In the algebraic setting, we have $W_\ell H^\ell(X) = \mathrm{Im}(H^\ell(\bar{X}) \to H^\ell(X))$ for $\bar{X}$ is any smooth compactification of $X$.

Now, let's prove the Weight trick:

We have $H^\ell(\bar{X}) \to H^\ell(X) \to H^\ell(Y)$. Now, any class of weight $\leq \ell$ in $H^\ell(Y)$ comes from $H^\ell(\bar{X})$.

From $E_2$ degeneration and MHS theory, we have that $H^\ell(f^{-1}(y_0))^{\pi_1} \subseteq H^\ell(f^{-1}(y_0))$ is a subHodge structure, and the invariant classes are compatible with the $(p, q)$ decomposition.

**Theorem 2.5.** *The monodromy representation is completely reducible, when* $f : X \to Y$ *is a smooth projective morphism to a quasiprojective variety.*

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Summary: $f : X \to Y$ a smooth projective morphism. Then $Rf_*\mathbb{R}_X$ is isomorphic in some sense to a complex wsith zero differentials and entries $R^\ell f_*\mathbb{Q}_X$, which are semisimple local systems and $L^k : R^{n-k}f_*\mathbb{Q}_X \to R^{n+k}f_*\mathbb{R}_X$ is an isomorphism, where $n = \dim X - \dim Y$.

What about these three facts for a general projective map $f : X \to Y$, $X$ nonsingular. Then $E_2$ degeneration fails! But yet, these three facts hold if we replace "local system" with "perverse sheaves".

Two famous theorems about surfaces:

1. Let $f : \tilde{X} \to X$ and $(X, x_0)$ a germ of a normal surface singularity, with $f$ a resolution of singularities. Then $f^{-1}(x_0) = \cup E_i$ with $E_i$ curves, and the intersection form $E_i \cdot E_j$ is negative definite. (Grauert-Mumford Theorem)
2. $X' \to C$ with $(C, y_0)$ is a germ of a smooth curve, then $X' \to C$ is a family of projective curves. Assume $f$ is smooth $f^{-1}(C \setminus y_0) \to C \setminus y_0$, the inverse image is $\cup E_i$, and the intersection form is negative semidefinite and the radical is generated by the class of the fiber (Zariski Lemma)

1 gives as a consequence the fact that $Rf_*\mathbb{Q}_{\tilde{X}} = \tau_{\leq i} R_{j*}\mathbb{Q}_{x_0}^{\oplus \# E_i}[-2]$, which has cohomological version $H^q(\tilde{X}) = H^q(X - x_0)$ if $q \leq 1$ and $H^2(\tilde{X}) = \mathbb{Q}^{\# E_i}$

Number 2 gives $Rf'_*\mathbb{Q}_{X'} = \mathbb{Q}_C \oplus \mathbb{Q}_C[-2] \oplus j_*R^1[-1] \oplus \mathbb{Q}_{y_0}^{\# E_i - 1}[-2]$ which is, cohomologically, $H^0(X') = \mathbb{Q}$, $H^2(X') = \mathbb{Q}^{\# E_i}$ and $H^1(X') = H^0(j_*R^1)$.

The first is the intersection cohomology complex of $X$. The second is the intersection cohomology complex of $R^1 f_*\mathbb{Q}$.

Now, let $X' \to \Delta \ni y_0$, which is $\Delta$ is small enough retracts onto $f^{-1}(y_0)$. Then $H^1(X') \cong H^1(f^{-1}(y_0))$ which both map to $H^1(X_\eta)$ the generic fiber. The image lands in $H^1(X_\eta)^{\pi_1(C \setminus y_0)}$ l.i.c.t.

Does this hold in general?

$H^*$(central fibre) $\to H^*$(generic fiber)$^\pi \to 0$, with the first a MHS.

**Theorem 3.1.** *On $H^*$(generic fibre) (smooth compact) it is possible to define a MHS such that the above map is a morphism of MHS.*

We call this the limit MHS.

R. Friedman "On the Clemens Schmid exact sequence"

Suppose that $T$, the monodromy operator, is unipotent (that is, $T - I$ is nilpotent) then $W$ is defined out of the Jordan form of $T - I$.

# 4 de Cataldo 1 - Hodge Theory of Maps

The goal of these two lectures is to state the decomposition theorem and to give examples.

**Theorem 4.1** (Decomposition Theorem)**.** *Let $f : X \to Y$ be a proper morphism of complex algebraic varieties. The direct image complex of the intersection cohomology complex of $X$ splits into a direct sum of intersection cohomology complexes on $Y$.*

Now, let $f : X \to Y$ a morphism and $F$ a complex of sheaves on $X$. Then let $F \to I$, where $I$ is a complex of injective sheaves on $X$ such that $\mathscr{H}^i F = \mathscr{H}^i I$. Now, define $Rf_*F = f_*I$.

So, if $c : X \to pt$, we can define $H^i(Rc_*F) = H^i(X, F)$.

So then we have $R^i f_*F = \mathscr{H}^i Rf_*F$.

**Exercise 4.2.** $(R^if_*F)_y = \varinjlim_{U\ni y} H^i(f^{-1}(U),F)$

Now, $Rf_*F$ contains information about the topology of the map, and we can extract it using the Leray Spectral Sequence.

**Theorem 4.3** (Comparison Theorem). $H_{sing}(X,\mathbb{Z}) = \check{H}(X,\mathbb{Z}) = H(X,\mathbb{Z}_X)$

**Exercise 4.4.** $\Delta^* \to \Delta$. *Then we have* $\mathbb{Z}_\Delta \to Rf_*\mathbb{Z}_{\Delta^*} \to H^1(\Delta^*)_\sigma[-1] \to 0$ *doesn't split.*

**Exercise 4.5.** $\Delta \to \Delta$ *by* $z \mapsto z^2$. *Then* $R^if_*\mathbb{Z}_X = 0$ *for* $i > 0$, *and we have* $0 \to \mathbb{Z}_Y \to f_*\mathbb{Z}_X \to P \to 0$ *doesn't split.*

However, with $\mathbb{Q}_X$, it does split, because the obstruction is divison by 2. So then we have that $f_*\mathbb{Q}_X$ is a direct sum of a local system and something which isn't a local system.

**Example 4.6.** *Take a rule nodal cubic,* $f$ *the normalization and* $j$ *the inclusion of the smooth locus. Then we have* $0 \to \mathbb{Z}_Y \to f_*\mathbb{Z}_X \to P_\sigma \to 0$ *and* $P$ *is a skyscraper sheaf at the singular point. We can show that this doesn't split.*

*However, we have* $0 \to \mathbb{Z}_Y \to f_*\mathbb{Q}_X \to \mathbb{Q}^3 \to 0$ *where* $\mathbb{Q}^3$ *is a three dimensional vector space at the origin.*

The decomposition theorem is far too much to hope for in anything other than complex algebraic geometry. For instance, look at the Hopf surface, which is fibered over $\mathbb{CP}^1$ by algebraic curves, but which is not algebraic. Then we don't have $E_2$ degeneration, even though the map is a submersion and $R^if_*\mathbb{Q}$ is constant on $\mathbb{CP}^1$, with $R^1f_*\mathbb{Q} \cong \mathbb{Q}^2$.

Now, let $C \subset \mathbb{CP}^2$, and look at the cone over the curve in $\mathbb{C}^3$. Blow it up at the origin, and we get a smooth surface which is the total space of $L = \mathscr{O}_{\mathbb{P}^2}(-1)|_C$.

But instead of looking at the blowup, look at the cone minux the origin (note, over $\mathbb{C}$, this doesn't disconnect the space). Call this map $j : U \to Y$. Then $Rj_*\mathbb{Q}_U$ has cohomology equal to $H(U)$, and to compute this we use the spectral sequence, and at that $E_2$ page is $\begin{array}{ccc} \mathbb{Q} & \mathbb{Q}^{2g} & \mathbb{Q} \\ \mathbb{Q} & \mathbb{Q}^{2g} & \mathbb{Q} \end{array}$ and all others zero, so the only differential is from the upper left to the bottom right, and is an isomorphism,and all other differentials vanish, so we get that the cohomology sheaves are $\mathbb{Q}_Y$, $\mathbb{Q}_0^{2g}$, $\mathbb{Q}_0^{2g}$ and $\mathbb{Q}_0$, where the subscript 0 means that we have skyscraper sheaves supported at the origin.

Now, $Rf_*\mathbb{Q}_X = \tau_{\leq 1}Rj_*\mathbb{Q}_U \oplus H^2(C)_0[-2]$. Why do we throw things away? We'll see later. This is an example of the decomposition theorem. The first summand is the intersection complex of the cone, $I_Y$ and the latter is $I_\sigma[-2]$.

Look at $C \times \mathbb{C}$ and then contract the curve. This isn't a holomorphic map! It's a real algebraic map, though. The decomposition theorem fails. We can tell by looking at the second term, because in the top space, the curve's cohomology class is trivial, because the bundle is trivial.

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Define $I_Y$ to be $\tau_{d-1} Rj_* \mathbb{Q}_U$ where $U$.

**Example 5.1.** *Let $Y = \mathbb{C}^d$ and $U$ the complement of the origin. Then we have $0 \to \mathbb{Q}_{\mathbb{C}^d} \to Rj_*\mathbb{Q}_U \to \mathbb{Q}_0[-(2d-1)] \to 0$.*

If $Y$ is smooth, then $I_Y = \mathbb{Q}_Y$. But for a nodal curve, if $j$ is the inclusion of the smooth locus, then $I_Y$ is $j_*\mathbb{Q}$.

On the other hand, cusps aren't seen, topologically.

Now let $C$ be a curve of genus $g$ and $C$ be the cone over it, and $U$ the complement of the origin. Then look at $\tau_{\leq 1} Rj_*\mathbb{Q}_U$, it is an extension of $\mathbb{Q}_0^{2g}[-1]$ by $\mathbb{Q}_Y$, but this doesn't split.

Recall the example $\Delta \to \Delta$ with $z \mapsto z^2$. At the origin, something happens: the preimage remains connected! So $f_*\mathbb{Q}/\mathbb{Q} = P$ is a local system, but it has stalk 0 at the origin, but is $-1$ over $\Delta^*$.

So why do we truncate at $d - i$ where $d$ is the dimension?

This is actually due to Poincaré duality! We have $H_c^{d-i} \cong (H^{d+i})^*$. So Poincaré duality implies that $b_{d-i} = b_{d+i}$.

Let $K$ be a complex on $Y$. Then there exists another complex, the dual $K^*$, such that $H^i(U, K^*) \cong H_c^{-i}(U, K)^*$.

The complex $\mathbb{Q}[d]$ is self-dual for a smooth variety, but NOT for a singular one! We set $IC_Y = I_Y[d]$, then $IC_Y$ is self-dual, for all varieties.

**Exercise 5.2.** *The rank of $\mathscr{H}^i(IC_Y^*)_0$ is equal to the rank of $H_c^{-i}(V_0, IC_Y)$.*

So $\tau_{\leq d-1} Rj_*\mathbb{Q}_U$ is motivated by Poincaré duality.

**Theorem 5.3** (Goresky-MacPherson). *$IH^{d-i} \cong (IH_c^{d+i})^*$*

Here, we'll find that the Lefschetz hyperplane theorem, Hard Lefschetz theorem, Hodge decomposition, primitive Lefschetz decomposition, and the Hodge-Riemann bilinear relations all hold for intersection cohomology groups.

**Theorem 5.4** (DT). *If $f : X \to Y$ is a proper morphism of complex algebraic varieties, then $Rf_*I_Y = \oplus_{b \in B} I_{\bar{Z}_b}(L_b)[\ell_b]$ where $B$ is a finite set, $Z_b \subset Y$ are locally closed nonsingular varieties, $L_b$ are simple local systems and $\ell_b \in \mathbb{Z}$.*