Chow Groups

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1 Murre 1 - Chow Groups

Conventions: k is an algebraically closed field, $X, Y, \ldots$ are varieties over k, which are projetive (at worst, quasi-projective), irreducible and smooth.

1.1 Algebraic Cycles

Let X/k be of dimension d. Let $0 \leq i \leq d$ and $q = d - i$. Then $Z_q(X) = Z^i(X) = \{Z | Z = \sum n_\alpha W_\alpha$ irreducible codimension i varieties$\}$ be the group of algebraic cycles on X of dimension q (codimension i).

Examples:

1. $Z^1(X)$ is the set of Weil divisors
2. $Z^d(X) = Z_0(X)$ is teh group of zero cycles, which are formal sums of points.
3. $Z^2(X) = Z_1(X)$ on a threefold is the free abelian group on the curves on X.

1.2 Operators on Cycles

There are three standard operations. The first is Cartesian product, if $V \in Z_{q_1}(X_1)$ and $W \in Z_{q_2}(X_2)$, then $V \times W \in Z_{q_1+q_2}(X_1 \times X_2)$.

The second is pushdown of cycles. If $f : X \to Y$ is a morphism and $Z \in Z_q(X)$, then $f_*(Z) \in Z_q(Y)$ is defined as follows: take $V \subset X$. If $f(V)$ is the set theoretic image of an irreducible subvariety, then $\dim f(V) \leq \dim V$. If $\dim f(V) < \dim V$, then $f_*(V) = 0$. If $\dim f(V) = \dim V$, then we have $k(V) \supset k(f(V))$ a finite extension of fields, and we set $f_*(V) = [k(V) : k(f(V))]f(V)$, where $f(V)$ is always taken set-theoretically.

The third operation is the intersection product of two cycles, but it is not always defined. Let V, W be subvarieties of X smooth, of codimension i and j. Then $V \cap W = \cup A_\ell$ where the A_ℓ are irreducible subvarieties of codimension at most $i + j$. The intersection is proper if the codimension is exactly $i + j$. We define the intersection multiplicity at A_ℓ to be $i(V \cap W, A_\ell) =$

Example 1.1. *If $X = S$ is a surface and V, W are curves and P a point on S, then we define the multiplicity at P to be* $\dim \mathscr{O}_{P,S}/(f,g)$ *where f, g are defining local equations for V and W. But this isn't the right definition if $d > 2$.*

In general, we take $i(V \cap W, A_\ell) = \sum (-1)^r \text{length}_{\mathscr{O}} \text{Tor}_r^{\mathscr{O}}(\mathscr{O}/\mathscr{I}_V, \mathscr{O}/\mathscr{I}_W)$ where $\mathscr{O} = \mathscr{O}_{A_\ell, X}$.

And so, we define $V.W = \sum_\ell i(V \cap W, A_\ell) A_\ell$ to be the intersection product if all the intersections are "good". This then extends by linearity to $Z^i(X) \times Z^j(X) \to Z^{i+j}(X)$.

There are other operations: let $f : X \to Y$ and $Z \in Z^i(Y)$, then $f^*(Z) \in Z^i(X)$ the pullback.

We also have operation by correspondences, in particular if $W \subset Y$, then we have that $f_*(W)$ is the image under the projection of $(X \times Z) \cdot \Gamma_f$.

Let X and Y be varieties of dimension d and ℓ and let $T \in Z^m(X \times Y)$ be a correspondence from X to Y. If $Z \in Z^i(X)$, then $T_*(Z)$ is just the pushforward along the projection to Y of $T \cdot (X \times Y$ in $Z^{i+m-\ell}$.

1.3 Good=adequate equivalence relations on cycles

Samuel in 1956 said that $\sim$ on the groups of algebraic cycles is an adequate relation if:

1. The set of cycles equivalent to zero should be a subgroup of $Z^i(X)$.

2. If Z, Z' are equivalent and $Z_1 \in Z^i(X)$, such that $Z \cdot Z_1$ and $Z' \cdot Z_1$ are defined, then they are equivalent.

3. If $Z \in Z^i(X)$ and $Z_1 \in Z^j(X)$, then there exists Z' equivalent to Z such that $Z' \cdot Z_1$ is defined. (This is called the moving lemma, which motivated it)

If $\sim$ is good, then $C^i_\sim(X)$, the set of equivalence classes of this equivalence relation, are defined and $C_\sim(X) = \oplus_{i=0}^d C^i_\sim(X)$ is a commutative ring with respect to the intersection product, and f_*, f^*, T are defined when f is proper.

Some commonly used adequate equivalence relations are

1. Rational equivalence (Chow, Samuel 1956)

2. Algebraic equivalence (Weil 1952)

3. Homological equivalence

4. Numerical equivalence

1.4 Rational Equivalence

This is a generalization of linear equivalence.

Linear equivalence is a relation on $\text{Div}(X) = Z^1(X)$. Here, for every $\phi \in k(X)^*$, we have $\text{div}(\phi) = sum_Y v_Y(\phi)Y$ for X smooth. If X is not smooth, need to use $\text{ord}_Y(\phi)$, which can be reviewed in Hartshorne.

For $0 \leq i \leq d$, and $Z \in Z^i(X)$, we say that Z is rationally equivalent to zero if $Z \in \langle \text{div}(\rho\rangle \subset Z^i(X)$ where $\langle\rangle$ denotes the subgroup generated by cycles of the form $\text{div}\phi$ where $\phi \in k(Y)^*$ for $\text{codim}\, Y = i-1$.

Equivalently, there exists a finite collection $\{Y_\alpha, \phi_\alpha\}$ of codimension $i-1$ subvarieties of X.

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2.1 Rational Equivalence

We have a cycle $Z \in Z^i(X) = Z_q(X)$ where $q = d - i$ and $d = \dim X$.

Lemma 2.1. *The following conditions on $Z \in Z^i(X)$ are equivalent:*

1. *Z is rationally equivalent to zero*
2. *there exists $T \in Z^i(\mathbb{P}^1 \times X)$ and $a, b \in \mathbb{P}^1$ such that $Z = T(a) - T(b)$.*

Proof. 1 implies 2 is easy. For the other direction, we use a theorem of Fulton, which says that if $f : X \to Y$ is proper and $\phi \in k(X)^*$ and $\dim X = \dim Y$, then $f_*(\text{div}(\phi)) = \text{div}(\text{Nm}(\phi))$.

Take the condition for $\lambda \in \mathbb{P}^1$. $\phi = (\pi_{\mathbb{P}^1})^*(T)$ and $T : X \to Y$. $Z = T(0) - T(\infty) = \text{div}\phi$. □

Rational equivalence is a "good" equivalence relation. $Z_{rat}(X)$ is a subgroup, for the second condition, let $Z, Z' \in Z^i(X)$ such that $Z \cdot W$ and $Z' \cdot W$ are defined for any given W. Then take $T \subset \mathbb{P}^1 \times X$ and $T(0) - T(\infty) = Z - Z'$. Because we can assume no horizontal components, it works.

The interesting part is the moving lemma. Let $Z \in Z^i(X)$ and choose finitely many W_j. We want to find $Z' \sim Z$ such that $Z' \cdot W_j$ are defined.

Case 1: $X = \mathbb{P}^N$, τ a general projective transforming on $\mathbb{P}^N$. $\tau(V)$ and V are rationally equivalent, so we can make sure $\tau(V) \cdot W$ is defined.

Case 2: $\text{codim}(V \cap W) = i + j - e$ where $e > 0$ is called the excess. Take a general linear space $L_{N-d-1} \subset \mathbb{P}^N$. Take $C_{L,V}$ the cone spanned by L and V. Then $\dim C_{L,V} = N - i$. $X \cdot C = V + V_*$, the excess intersection. Because L is general, the excess is $< e$, and things work out.

2.2 Chow Groups

We define $CH_q(X) = CH^i(X) = Z^i(X)/Z^i_{rat}(X)$.

Theorem 2.2 (Chow, Samuel 1956). *1. $CH(X) = \oplus_{i=0}^{d} CH^i(X)$ is a commutative ring with identity.*

2. $CH(X)$ behaves functorially with respect to $f : X \to Y$. More precisely:

(a) if f is proper, then f_* is

(b) if f arbitrary, f^* is

3. $T \in CH(X \times Y)$ gives $T_* : CH_q(X) \to CH_q(Y)$ is a homomorphism.

Theorem 2.3 (Localization Sequence). *Let $Y \xrightarrow{i} X \xleftarrow{j} U = X - Y$ inclusions. Then $CH_q(Y) \xrightarrow{i_*} CH_q(X) \xrightarrow{j^*} CH_q(U) \to 1$ is exact.*

Theorem 2.4 (Homotopy Property). *$X \times \mathcal{A}^n \to X$ induces $CH^i(X) \to CH^i(X \times \mathcal{A}^n)$ is an isomorphism for $0 \leq i \leq \dim X = d$.*

2.3 Algebraic equivalence

Let $Z \in Z^i(X)$, we say that Z is algebraically equivalent to zero if there exists $T \subset C \times X$ such that $Z = T(a) - T(b)$ where $a, b \in C$ and C is a curve.

It's easy to see that $Z_{rat} \subset Z_{alg}$, because $\mathbb{P}^1$ can be taken as C.

2.4 Homological Equivalence

Fixing a good cohomology theory, for instance, the usual ones, but also etale cohomology will work. Then we get the usual intersection theory in cohomology.

We require that there exist a cycle map $Z^i(X) \to CH^i(X) \to H^{2i}(X)$ such that the intersection product is compatible with the cohomological cup product. Set $Z_{hom}(X)$ to be the homologically trivial cycles. This is also a good equivalence relation.

$Z_{alg} \subset Z_{hom}$, by Matsusaka in 1956, it was shown for divisors. It is not true in general, proved in 1969.

2.5 Numerical Equivalence

Let X be a d-dimensional projective variety, $Z \in Z^i(X)$ and $W \in Z^{d-i}(X)$ such that $Z \cdot W$ is defined, and it's just a number of points, a zero cycle. We say that Z is numerically equivalent to zero if $\deg(Z \cdot W) = 0$ for all $W \in Z^{d-i}(X)$ where it is defined.

Is $Z_{hom}(X) \subseteq Z_{num}(X)$? It is known for divisor, and it's a conjecture for general i.

For X a variety over $\mathbb{C}$, the Hodge conjecture implies this one.

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3.1 Comparison

Let X be a smooth projective irreducible variety over $\mathbb{C}$, there are several topologies, in particular the Zariski topology and the étale topology, as well as teh underlying analytic space X_{an} which is a complex analytic manifold which is compact and connected in the classical topology.

We have a comparison theorem saying that $H^i(X_{an}, \mathbb{Q}) \otimes \mathbb{Q}_\ell \cong H^i_{et}(X, \mathbb{Q}_\ell)$ for $\ell \neq$ the characteristic.

Similarly, we have a theorem of Serre in 1956, GAGA, that $\mathscr{F} \mapsto \mathscr{F}_{\mathscr{O}_X} \mathscr{O}_{X_{an}}$ is an equivalence of categories between algebraic and analytic coherent sheaves, and the cohomologies are the same.

In particular, we find that the Picard groups are isomorphic, and so $CH^1(X)$ doesn't depend on whether we look analytically or algebraically.

3.2 Cycle Map

Let $Z \in Z^p(X)$ with $q = d - p$.

Proposition 3.1. *There exists a homomorphism $\gamma_{\mathbb{Z}} : Z^p(X) \to H^{2p}(X_{an}, \mathbb{Z})$ given by taking the inclusion $Z \to X$ and using the exact sequence of a pair for the analytic spaces.*

In particular, we have maps $T : H^0(Z, \mathbb{Z}) \to H^{2p}(X, Z, \mathbb{Z})$ and $\rho : H^{2p}(X, Z, \mathbb{Z}) \to H^{2p}(X, \mathbb{Z})$, and $H_{2p}(Z, \mathbb{Z}) \to H^0(X, \mathbb{Z})$. Taking $1 \in H_{2p}(Z, \mathbb{Z})$ to an element of $H^0(Z, \mathbb{Z})$, we define $\gamma(Z) = \rho T(1) \in H^{2p}(X, \mathbb{Z})$.

What is the position of $\gamma(Z) \in H^{2p}(X, \mathbb{C})$?

Lemma 3.2. *Let $j : H^{2p}(X, \mathbb{Z}) \to H^{2p}(X, \mathbb{C}) = \oplus_{r+s=2p} H^{r,s}(X)$ then $j\gamma_{\mathbb{Z}}(Z) \in H^{p,p}(X)$, and hence $\gamma_{\mathbb{Z}}(Z) = Hdg^p(X) = j^{-1}(H^{p,p})$.*

Proof. We have a nondegenerate pairing between $H^{r,s}$ and $H^{d-r,d-s}$. Set $\alpha = j\gamma_{\mathbb{Z}}(Z)$, and take a β. Look at $\langle j\gamma_{\mathbb{Z}}(Z), \beta\rangle = \int_Z \beta|_Z$, and this is zeor unless $\beta \in H^{q,q}(X)$. □

Reference for details: Griffiths and Harris, or Voisin.

3.3 Hodge Classes, Cycles and Conjecture

We have the theorem that the image of $\gamma_{\mathbb{Z}}$ is contained in $Hdg^p(X)$.

Conjecture 3.3 (Integral Hodge Conjecture)**.** *Is $\gamma_{\mathbb{Z}}$ surjective?*

This is true for $i = 1$, by the Lefschetz (1,1) Theorem. But for $i > 1$, this is no longer true! See Hirzebruch-Atiyah in 1962, Kollar in 1992 and Totaro 1998.

However, it is still wide open if it is true after tensoring with $\mathbb{Q}$.

But we do have:

Theorem 3.4 (Lefschetz (1,1) Theorem)**.** *$CH^1(X) \to Hdg^1(X)$ is onto.*

Sketch of proof, following Kodaira and Spencer:

Here we have that $CH^1(X) = \mathrm{Pic}(X)$, and we have the exponential sequence $0 \to \mathbb{Z} \to \mathscr{O}_{X_{an}} \to \mathscr{O}^*_{X_{an}} \to 1$ exact, so we get an exact sequence on cohomology. This gives us an exact sequence $H^1(X_{an}, \mathscr{O}^*_{X_{an}}) \stackrel{\gamma_{\mathbb{Q}}}{\to} H^2(X_{an}, \mathbb{Z}) \to H^2(X_{an}, \mathscr{O}_{X_{an}})$ and so we get that $D \mapsto \gamma_{\mathbb{Z}}(D)$, and every (1,1) class has image zero in $H^{0,2}(X)$, so the map is surjective onto them.

3.4 Intermediate Jacobians

Let $J^p(X)$ be the pth intermediate Jacobian of X over $\mathbb{C}$, then $J^p(X) = H^{2p-1}(X_{an}, \mathbb{C})/F^pH^{2p-1} + H^{2p-1}(X, \mathbb{Z})$. This is the Griffiths intermediate Jacobian.

Lemma 3.5. *$J^p(X)$ is a complex torus of dimension $\frac{1}{2}\dim H^{2p-1}(X_{an}, \mathbb{C})$.*

Proof. To see if $\alpha_1, \ldots, \alpha_m$ is a $\mathbb{Q}$ basis for $H^{2p-1}(X, \mathbb{Q})$, we need $v = \sum R_i\alpha_i \in F^p \Rightarrow v = 0$. We have that the $r_i \in \mathbb{Q}$, so $\bar{v} = v$, so $\bar{v} \in V$, and $F^p \cap V = -$. □

This is in general not an abelian variety. However, if $V \subset H^{p-1,p}$, then it is.

This will always happen for $i = 1$ and $i = d$ where $d = \dim X$, and these are the Picard variety of X and the Albanese variety of X, and, in the case of curves, these are the same and are the Jacobian of of X.

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We have $\mathcal{X}_d \subset \mathbb{P}^N$ an irreducible smooth variety. Look at the complex torus $J^p(X)$, the intermediate Jacobian of Griffiths. It won't in general be an abelian variety.

We define $J^p_{alg}(X) \subset J^p(X)$ to be the largest abelian subvariety of $J^p(X)$.

4.1 Abel-Jacobi Map

Let $Z \in Z^p_{hom}(X)$

Proposition 4.1. *There exists a homomorphism $\mathcal{AJ}^p : Z^p_{hom}(X) \to J^p(X)$ which factors through $CH^p_{hom}(X)$.*

an outlike of the construction, we note that $H^{2p-1}(X)$ and $H^{2d-2p+1}(X)$ are dual, and we take V, the universal cover of $J^p(X)$ and $F^{d-p+1}H^{2d-2p+1}$, which are also dual. Now, for $Z \in Z^p_{hom}(X)$, we have $Z = \partial C$ where C is a topological chain of dimension $2d - 2p + 1$.

So take $\alpha \in F^{d-p+1}H^{2d-2p+1}(X)$, and take the cohomology class of α representing this α, $v_C : \alpha \to \int_C \alpha$ for $v_C \in V$ does not depend on the choice of cohomology class, $\alpha = \alpha + d\beta$, and we can choose β such that it contains at least $d - p + 1$ dz's. Then $\int_C d\beta = \int_Z \beta = 0$.

Now, take C', and we have $\int_{C'} \alpha \in H^{2p-1}(X, \mathbb{Z})$ and $\partial(C' - C) = 0$, and $\int_{C'} - \int_C = \int_Z \alpha$. So we have a well defined element $\mathcal{AJ}^p(Z) \in J^p(X)$.

Example 4.2. *Let $X = C$ be a curve, then $J(C) = H^1(X,\mathbb{C})/H^{1,0} + H^1(X,\mathbb{Z})$, and if we take $D = \sum(p_i - p_i') \in \mathrm{Div}(X)$, $\alpha_1, \ldots, \alpha_g$ a basis for $H^0(X,\Omega) = H^{1,0}$, then $\mathcal{AJ}^1(D) = \left(\int_{T_i} \alpha_1, \ldots, \int_{T_i} \alpha_g\right) \in \mathbb{C}^g/\Lambda = J(X)$.*

Remark: Suppose that $Z \in Z^p_{alg}(X)$. We claim that $\mathcal{AJ}^p(Z) \in J^p_{alg}(X)$.

4.2 Algebraic Equivalence vs. Homological Equivalence

We look at $Z^i_{alg}(X) \subseteq Z^i_{hom}(X)$. For divisors $i = 1$, we have equality for every algebraically closed k by a theorem of Matsusaka in 1956.

Theorem 4.3 (Griffiths 1969)**.** *There exists $\mathcal{X}_d \subset \mathbb{P}^N$ smooth over $\mathbb{C}$ such that for certain $i > 1$ the $Z^i_{alg}(X) \neq Z^i_{hom}(X)$.*

We define $Gr^i(X) = Z^i_{hom}(X)/Z^i_{alg}(X)$ to be the Griffiths group, which is discrete.

Theorem 4.4. *There exist varieties such that for $i > 1$, $Gr^i(X) \neq 0$ and, in fact with $Gr^i(X) \otimes \mathbb{Q} \neq 0$.*

There exist Chow varieties, given $X_d \subset \mathbb{P}^N$ fixing the dimension q and degree m, then $Z^p(X)_{\deg m} = Ch(X, r, m)$ is the Chow variety.

Preparation:

Lefschetz hyperplane Theorem: $W = V \cap H \subset V \subset \mathbb{P}^N$ with H a hyperplane, $i^* : H^j(V,\mathbb{Z}) \to H^j(W,\mathbb{Z})$ is an isom for $j < \dim W$ and is injective for $j = \dim W$.

In particular, $W \subset \mathbb{P}^N$ a hypersurface section then $H^j(W) = 0$ for j odd and $j \neq \dim W$ and $H^{2j}(W,\mathbb{Z}) = \mathbb{Z}h$ where $h = d(WH)$ for $2j < \dim W$ and $2j > \dim W$.

We need the notion of a Lefschetz pencil: Let $V \subset \mathbb{P}^N$. Take H_0, H_1 hyperplanes in general position with respect to V, then $H_t = H_0 + tH_1$ for $t \in \mathbb{P}^1$ is a pencil, and we call $W_t = V \cap H_t$ a Lefschetz pencil.

Proposition 4.5.

1. *For all except finitely many t, W_t is smooth*
2. *W_t for $t \in S$ only are ordinary double points, where S is the locus of singular sections.*

Theorem 4.6 (Lefschetz)**.** *The action of Γ (the image of the monodromy representation) on $H^j(W,\mathbb{Q})$ is completely reducible.*

Fact: $H^j(W)_{van} = 0$ if $j \neq d = \dim W$ where the vanishing means it is in the kernel of $i_* : H^j(W) \to H^{j+2}(V)$.

Theorem 4.7. *Let $V_{d+1} \subset \mathbb{P}^N$ and $d + 1 = 2m$. Consider a Lefschetz pencil $\{W_t\}$ on V with $\dim W = 2m - 1 = d$, and assume that $H^{2m-1}(V) = 0$, but $H^{2m-1}(W) \neq H^{m,m-1} + H^{m-1,m}$. Let $Z \in Z^m(V)$ and set $Z_t = Z \cdot W_t \in Z^m(W_t)$. Assume that for very general t, the Z_t is algebraically equivalent to zero. Then Z is homologically equivalent to 0 on V.*

Indication of proof: Let $Z \in Z_{hom}(W_t)$, then step 1 is to look at $\mathcal{AJ}(Z_t) \in J^{2m-1}_{alg}(W_t)$, then to take a normal function $\nu : U \to \cup_{t\in U} J^{2m-1}(W_t)$, and step 2 is to show that $\nu = 0$ is homologically zero on V.

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Theorem 5.1 (Griffiths 1969)**.** *There exist varieties $X \subset \mathbb{P}^N$ over $\mathbb{C}$ smooth and $i > 1$ such that $Z^i_{alg}(X) \neq Z^i_{hom}(X)$.*

In fact, there exists $Z \in Z^i_{hom}(X)$ such that $Z \neq 0$ in $Griff^i(X) \otimes \mathbb{Q}$.

The key theorem is suppose that $V_{d+1} \subset \mathbb{P}^N$ with $d + 1 = 2m$ such that $H^{2m-1}(V) = 0$. Suppose there exists a Lefschetz pencil $\{W_t\} \subset V$ such that $H^{2m-1}(V_t, \mathbb{C}) \neq H^{m-1,m} \oplus H^{m,m-1}$. Assume that $Z \in Z^m(W_t)$ such that $ZW_t = Z_t$ in $CH^m_{alg}(W_t)$ for general t. Then $Z \sim_{hom} 0$ on V.

For instance, take $V_4 \subset \mathbb{P}^5$ a smooth quartic in $\mathbb{P}^5$. Then $H^3(V) = 0$, and we can take $W_t = VF_t$ where F_t is a hypersurface of degree r. Now, if $r \geq 5$, then $H^3(W_t) \neq H^{21} + H^{12}$.

Because $K_{W_t} > 0$ if $r \geq 5$, we have that $H^{3,0}((W_t) = H^0(W_t, \Omega^3) \neq 0$.

On V, there are two systems of planes $\{L\}$ and $\{L'\}$ and $H^*(V) = \mathbb{Z} + \mathbb{Z} = \mathbb{Z}(L + L') \oplus \mathbb{Z}(L - L')$. We can check that $(L - L')(L - L') = -2$ and so we take $Z = L - L'$, it's not homologically equivalent to zero.

$Z_t = ZW_t = C - C'$ where $C = LW_t$ and $C' = L'W_t$. Then Z_t is not homologically trivial on W_t. But Z_t is (and I didn't follow what happened next, looked like a proof that Z_t IS homologically trivial on W_t or something)

Theorem 5.2 (Clemens 1983)**.** *For sufficiently general quintic hypersurfaces in $\mathbb{P}^4$, $\dim Griff^2(X) \otimes \mathbb{Q} = \infty$.*

5.1 Albanese Kernel

Let $X = X_d$ be a smooth projective irreducible variety over an algebraically closed field k. For $i = 1$, we get divisors and $CH^1_{alg}(X) \cong \mathrm{Pic}^0(X)$ is an isomorphism of abelian varieties.

Now, look at $i = d$. These are zero cycles. We have $CH^{alg}_0(X) \to Alb(X)$ a surjective map.

What's the kernel? Let $T(X) = \ker(CH^{(0)}_0(X) \to Alb(X))$.

Theorem 5.3 (Mumford 1968)**.** *Let $X = S$ a surface over $\mathbb{C}$ with $p_g(S) \neq 0$, then $T(S) \neq 0$.*

In fact, it is "infinite dimensional" that is, $T(S)$ cannot be parameterized by an algebraic variety.

Equivalently, $\phi_m : \mathrm{Sym}^r S \times \mathrm{Sym}^r S \to CH^{(0)}_0(S)$ by $(z_1, z_2) \mapsto z_1 - z_2$ such that $p_g(S) \neq 0$, then there is no m such that ϕ_m is surjective.

5.2 Generalization of Bloch 1979

Let $X/k = \bar{k}$ and define $CH^i_{alg}(X)$ is "weakly" representable if there exists $(C, s \in C, T \in CH^i(C \times X))$ such that $T : CH^1_{alg}(C) = J(C) \to CH^i_{alg}(X)$ surjective.

Theorem 5.4 (Block 1979). *Fix a "good" cohomology theory. If $CH^2_{alg}(S)$ is weakly representable, then $H^2(S)_{tr} = 0$.*

Recall that $H^2(S) = H^2(S)_{alg} \oplus H^2(S)_{tr}$, with $H^2(S)_{alg}$ the image of $CH^1(S)$ and $H^2(S)_{tr}$ the orthogonal complement. Then the theorem implies the theorem of Mumford, because if $X = S/\mathbb{C}$, then $H^{2,0} \subset H^2(S)_{tr}$ is nonzero, so $H^2(S)_{tr}$ is nonzero.

Theorem 5.5 (Bloch 1979, Bloch-Srinivas 1983). $X_d/k = \bar{k}$ *Assume there exists $Y \subset X$ such that $CH^{(0)}_{alg}(Y_L) \to CH^{(0)}_{alg}(X_L)$ for all $L = Z \supset k$. Then there exists a divisor $D \subset X$ and two corredpondences Γ_1, Γ_2 in $CH^d(X \times X)$ with $|F_1| \subset Y \times X$ and $|\Gamma_2| \subset X \times D$ such that $N\Delta(X) = \Gamma_1 + \Gamma_2$ in $CH^d(X \times X)$ for some $N > 0$.*

Note: If $X = S$, and $CH^{(0)}_0(S)$ is weakly representable, the assumption if then fulfilled, so this implies Mumford.

Proposition 5.6. *If $\Delta(X)$ is as before, then $H^2(S)_{tr} = 0$.*

The Γ_i operate trivially on $H^2(S)_{tr}$ for $i = 1, 2$ and so $\Gamma : H^2(X) \to H^2(S)$ must restrict to zero on $H^2(S)_{tr}$.

Now, a sketch of the second Bloch theorem.

Take generaic point η of X. $k \subset k(y) \subset \overline{k(y)}$, for $y \in Y$. Then $\eta - y \in CH^0_0(X_{k(y)}) \to CH^0(X_{\overline{k(y)}})$ (couldn't read the last bit)