Deligne's Theorem on Abelian Varieties

Schnell and Kerr

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1 Schnell - Deligne's Theorem on Abelian Varieties, Part I

Theorem 1.1 (Deligne)**.** *On an abelian variety, all Hodge classes are absolutely Hodge.*

The proof breaks up into two parts:

1. reduce to the case of CM abelian varieties
2. Deal with CM case.

We'll deal with step 1 today.
Recall, in the case of weight 1:

Definition 1.2 (CM field)**.** *A CM field is a number field E of the form $E = F[t]/(t^2 - f)$ where $f \in F$ and F is totally real and under all embeddings $F \subset \mathbb{R}$, f is negative.*

Definition 1.3 (CM abelian variety)**.** *An abelian variety is CM if there exists a CM field $E \subset \operatorname{End}(A) \otimes \mathbb{Q}$ such that $\dim_E H^1(A, \mathbb{Q}) = 1$*

This implies that $[E : \mathbb{Q}] = \dim_{\mathbb{C}} H^1(A, \mathbb{Q}) = 2 \dim A$.
There is a nice criterion $MT(A) = MT(H^1(A, \mathbb{Q}))$ (MT means Mumford-Tate group)

Proposition 1.4. *If A is simple, then A is CM if and only if $MT(A)$ is abelian.*

Proposition 1.5. *Given any abelian variety A and a Hodge class α on A, there exists a family $\mathcal{A} \xrightarrow{\pi} B$ of abelian varieties with B irreducible and quasi-projective such that there exists $0 \in B$ with $\mathcal{A}_0 \cong A$ and the Hodge locus of α is B, and there is $t \in B$ where $\mathcal{A}_t$ is CM.*

Proof. Choose a polarization Q and let $G = \operatorname{Aut}(H^1(A, \mathbb{Q}), Q)$ and $M = MT(A)$ the smallest $\mathbb{Q}$-subgroup whose $\mathbb{R}$-points contain the image of $\phi : \mathbb{S}^1 \to G(\mathbb{R})$.

Abelian varieties of the same kind, along with a choice of basis for $H^1(A, \mathbb{Z})$, are parameterized by the period domain $D = G(\mathbb{R})/K$.

Note: points of D are classes of gH in terms of ϕ. $\phi_{gH} = g\phi g^{-1}$.

Main idea: family comes from the Mumford-Tate domain: $D_\phi = M(\mathbb{R})/M(\mathbb{R}) \cap K \subset D$.

This should have the properties that for all Hodge structures $H' \in D_\phi$,

1. $MT(H') \subset M$
2. any Hodge tensor for A is a Hodge tensor for H'
3. $\phi_{H'} = g\phi g^{-1}$ for $g \in M(\mathbb{R})$.

Finding CM points corresponds to finding points with abelian MT. $\phi(\mathbb{S}^1) \subset M(\mathbb{R})$ contained in some maximal $\mathbb{R}$-toruc T_0, and we can show that for $\xi_0 \in m_{\mathbb{R}}$ generic, T_0 is the stabilizer of ξ_0.

Nearby, there exists $\xi \in m_{\mathbb{R}}$ close to ξ_0, then if T is the stabilizedr of ξ, it is a $\mathbb{Q}$-torus. There exists $g \in M(\mathbb{R})$ such that $\xi = g\xi_0 g^{-1}$, and $g\phi g^{-1}$ has image in T. Then $MT(H_{g\phi g^{-1}}) \subseteq T$ is abelian.

Problem: family over quasi-proj base, not D_ϕ. Solution: Fix an $N >> 0$ and use a level N structure.

Define $\mathcal{M}_{g,Q,N}$ to be the moduli space of abelian varieties of dimension g with polarization Q and level N structure (a basis of the N-torsion points) and let $\mathcal{A}_{g,Q,N} \to \mathcal{M}_{g,Q,N}$ the universal family. OUr replacement for D_ϕ is to let $B \subset \mathcal{M}_{g,Q,N}$ be the Hodge locus of the Hodge tensors for $H^1(A, \mathbb{C})$ defining $MT(A)$.

B is algebraic by CDK, and finite etale over $\Gamma \backslash D_\phi$. In this case, things are ok. □

Proof that (for A simple), $MT(A)$ abelian implies A is CM.

We start with the fac tthat $E = \mathrm{End}(A) \otimes \mathbb{Q}$ is a division algebra, since A is simple. It is also the sety of $\mathbb{Q}$-endomorphisms that commute with $MT(A) = M$. So we know that M is abelian, and thus it acts on $H^1(A, \mathbb{Q})$, and we can write $H^1(A, \mathbb{C}) = \oplus_\chi H^1(A, \mathbb{C})_\chi$ for characters, and thus $E \otimes \mathbb{C} = \oplus_i \mathrm{End} H^1(A, \mathbb{C})_\chi$.

And so $\dim_{\mathbb{Q}} E \geq \dim H^1(A, \mathbb{Q}) = 2 \dim A$ is bounded above by $2 \dim A$. So $2 \dim A = \dim \mathbb{Q} E$ and thus E is a commutative field, so $\dim_E H^1(A, \mathbb{Q}) = 1$.

Now, use teh Rosati involution $\phi \mapsto \phi^t$ on E, and $Q(\phi h_1, h_2) = Q(h_1, \phi^t h_2)$, and F the fixed field. We claim that $[E : F] = 2$ and F is totally real.

We have that $F = \mathbb{Q}(\phi)$, with $\phi = \phi^t$ and take the minimal polynomial. Then λ_j, the roots, are the eigenvalues of the action of ϕ on $H^1(A, \mathbb{Q})$, and if we set $\lambda = \lambda_j$, and ϕ acts on $H^1(A, \mathbb{C})$ preserving $H^{1,0} \oplus H^{0,1}$, there exists $h \in H^{1,0}$ with $\phi(h) = \lambda h$, $\phi(\bar{h}) = \bar{\lambda}\bar{h}$. Look at $Q(\phi h, \bar{h}) = Q(h, \phi\bar{h})$, this is $\lambda Q(h, \bar{h}) = \bar{\lambda} Q(h, \bar{h})$ and so $\lambda = \bar{\lambda}$, so $\lambda \in \mathbb{R}$.

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Let A is a CM abelian variety, that is, an abelian variety such that $MT(H^1(A))$ is abelian.

Now, if $t \in H^{2p}(A^{an}, \mathbb{Q}) \cap F^p H^{2p}_{dR}(A)$ and $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ then we want to show that $t^\sigma \in F^p H^{2p}_{dR}(A^\sigma)$ lives in $H^{2p}(A^{an,\sigma}, \mathbb{Q})$.

Let $E/\mathbb{Q}$ be a CM field of degree $2e$ such that E is totally imaginary and there exists $p \in Gal(E/\mathbb{Q})$ with $p^2 = \mathrm{id}$, $\phi \circ p = \bar{\phi}$ for all $\phi \in \hom(E, \mathbb{C})$.

Now, take F to be the totally real fixed field, and ξ such that $E = F(\xi)$, and $\xi^2 \in F$ and $\sqrt{-1}\phi_i(\xi) > 0$ for $i = 1, \ldots, e$ with $\hom(E, \mathbb{C})$ generated by $\Phi = \{\phi_1, \ldots, \phi_e, \bar{\phi}_1, \ldots, \bar{\phi}_e\}$. We call (E, Φ) the CM type of E.

Now, consider $A/\mathbb{C}$ an abelian variety with $E \to \mathrm{End}(A) \otimes \mathbb{Q} = \mathscr{E}$. Then $V = H^1(A, \mathbb{Q})$ is an E-vector space of even dimension d and $\dim A = ed = D$.

Now, V is self-dual, and so E acts on $V^\vee$ and we have natrual quotient map $\bigwedge^d V^\vee \to \bigwedge^d_E V^\vee$, and the dual is an inclusion defined over E.

E is a $\mathbb{Q}$-vector space of dimension $2e$ and it acts on $E \otimes_{\mathbb{Q}} \mathbb{C} = \oplus_{\phi \in \hom(E,\mathbb{C})} \mathbb{C}_\phi$ adn similarly for V, adn we have

$$\begin{array}{ccc}
(\bigwedge^d_E V)_{\mathbb{C}} \cong \oplus \bigwedge^d_{\mathbb{C}} V_{\phi_i} & \oplus_{\sum d_i = d} & (\otimes_i \bigwedge^{d_i}_{\mathbb{C}} V_{\phi_i}) \cong (\bigwedge^d V)_{\mathbb{C}} \\
\uparrow & & \uparrow \\
\bigwedge^d_E V & \longrightarrow & \bigwedge^d V
\end{array}$$

The HS on V may be viewed as $\phi : \mathbb{U} \to GL(V)$ taking zz to the $\mathbb{C}$-linear endomorphism of multiplciation by z^{1-0} on $V^{1,0}$ and z^{0-1} on $V^{0,1}$ and this must commute with $v(E)$.

Therefore, $V_{\phi_i} = (V_{\phi_i} \cap V^{1,0}) \oplus (V_{\phi_i} \cap V^{0,1}) = V^{1,0}_{\phi_i} \oplus V^{0,1}_{\phi_i}$, and are of dimension a_i and b_i with $a_i + b_i = d_i$.

So the Hodge type of $\bigwedge^d_{\mathbb{C}} V_{\phi_i} \cong \bigwedge^{a_i}_{\mathbb{C}} V^{1,0}_{\phi_i} \otimes \bigwedge^{b_i}_{\mathbb{C}} V^{0,1}_{\phi_i}$ is (a_i, b_i).

Conclusion: If $\dim(V^{1,0}_{\phi_i}) = d/2$ for each $i = 1, \ldots, 2e$, then $\bigwedge^d_E V \subset \bigwedge^d_{\mathbb{Q}} V$ consists of Hodge classes (the Weil classes).

If A_0 is an abelian variety of dimension $d/2$ and $A = A_0 \otimes_{\mathbb{Q}} E = A_0 \times \ldots \times A_0$ $2e$ times, this is then $\mathbb{C}^{d/2} \otimes \mathbb{C}^{2e}/\Lambda \otimes \mathscr{O}_E$. Let $V = H^1(A, \mathbb{Q})$, this is just $H^1(A_0, \mathbb{Q}) \otimes_{\mathbb{Q}} E$, and so taking E to act on the factor of E, we get $V_{\phi_i} \cong V_0 \otimes \mathbb{C}_{\phi_i} \cong V_{i,\mathbb{C}}$.

This gives us that $\bigwedge^d V_{\phi_i} \cong \bigwedge^d V_{i,\mathbb{C}} = H^d(A_0, \mathbb{C}) \cong H^{d/2,d/2}(A_0)$.

Moreover, $\mathrm{Aut}(\mathbb{C})$ changes neither the product structure on A, the endomorphisms (which are defined by cycles in $A \times A$) nor the class of $[p]$ on A_0. Thus, $\bigwedge^d_E V$ in this cases consists of absolute Hodge clases.

Now, think of V as a fixed $\mathbb{Q}$-vector space of dimension D with nondegenerate alternating form $Q : V \times V \to \mathbb{Q}$.

Let ϕ be any weight 1 Hodge structure on V polarized by Q and $E \to \mathrm{End}(V, \phi)$ an isomorphism (in such a way that Q gives $V^{1,0}_{\phi_i}$ and $V^{0,1}_{\phi_i}$. We impose the condition that $\dim V^{1,0}_{\phi_i} = d/2$ for all i.

Then there exists a unique E-Hermitian form $\psi : V \times V \to E$ with $Q = \mathrm{tr}_{E/\mathbb{Q}}(\mathscr{E} \cdot \psi)$ and ϕ stabilizes ψ and commutes with $i(E)$. Hence, $M_\phi \subset \mathrm{Aut}_E V \cap Sp(V, \mathbb{Q}) = \mathrm{Res}_{F/\mathbb{Q}} U_E(V, \psi)$ and $X = M_\phi(\mathbb{R})^+$, $\phi \subset h^D$ is a MT domain which precisely classifies the abelian varieties (or HS's) satisfying the above conditions which are precisely that the HS for which $\bigwedge_E^d V \subset \bigwedge^d V$ consists of Hodge classes.

Now, $\mathcal{A} \to \Gamma\backslash X$ a torsion free congruence subgroup is by the Baily-Borel theorem a quasi-projective algebraic variety parameterizing such A.

Applying Principle B again leads to

Theorem 2.1. *Weil classes on "Veil algebraic varieties" are absolute Hodge*

The rest of Deligne's proof: Let M be cut out by Hg'_A and $\check{M}$ be cut out by $AH_{g'_A}$ (the Hodge and absolute Hodge tensors) then

Theorem 2.2 (Principle A). *If a tensor $t \in T^{k,\ell}H^1(A, \mathbb{Q})$ is fixed by $\check{M}$, then it is absolute Hodge.*

For CM abelian varieties, Deligne shows that $\check{M} \supseteq M$ is an equality by producing enough absolute Hodge classes to push $\check{M}$ inside M. He does this by looking at endomorphisms of the CM field, $A_{\sigma\Phi} \to A_\Pi$ and Weil Hodge classes.

This is dense on $\prod_{\Phi_i} A_{\Phi_i}$.