Algebraic varieties and schemes over any scheme. Non singular varieties

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1 Lecture 1

Let k be a field and $k[x_1, \ldots, x_n]$ the polynomial ring with coefficients in k. Then we have two objects: polynomials $P \in k[x_1, \ldots, x_n]$ and polynomial functions obtained by turning polynomials into maps $k^n \to k$. For finite fields, these are very different, though they are the same for infinite fields.

A subset $E \subset k^n$ is called an algebraic set if it is equal to the zero locus of a collection of polynomials.

Theorem 1.1 (Finiteness Theorem of Hilbert)**.** *$k[x_1, \ldots, x_n]$ is Noetherian.*

So any collection of polynomials has zero locus determined by finitely many of them.

1.1 Zariski Topology

Closed sets are the algebraic sets, $\emptyset$ and k^n, and open sets are the complements. This is actually non-Hausdorff.

Theorem 1.2 (Hilbert Nullstellensatz)**.** *Let $k = \bar{k}$, then if P vanishes on an algebraic set $E = V(I)$, then some power of P is in the ideal I. In particular, this implies that $I(E) = \sqrt{I}$.*

Let $E = \cup E_i$ where the E_i are irreducible, that is, are not the union of two properly contained algebraic sets. Then, E_i is irreducible if and only if $I(E_i)$ is prime.

If $f = P|_E$, we call it an algebraic function, and there are also regular functions.

Theorem 1.3. *The set of algebraic functions on E is isomorphic to $k[x_1, \ldots, x_n]/I(E) = A(E)$.*

Definition 1.4 (Regular function)**.** *ϕ is regular at $x \in E$ if there exists a Zariski neighborhood of x such that $\phi = f/g$ on it.*

ϕ is regular on E if it is regular at each $x \in E$.

Theorem 1.5. *The set of regular functions on E is ring isomorphic to $A(E)$.*

Definition 1.6 (Morphism)**.** *A morphism of varietes $\phi : E \to F$ is a continuous function which pulls regular functions back to regular functions $A(F) \to A(E)$.*

1.2 Projective varieties

$\mathbb{P}^n_k = k^{n+1} \setminus \{0\}/\sim$ where $x \sim y$ if there exsits $\lambda \in k^*$ such that $x = \lambda y$. Now, we look at $R = k[x_0, \ldots, x_n]$ as a graded ring, with gradations the polynomials homogeneous of a given degree d.

Let S be a set of homogeneous polynomials. Then $V(S)$ the algebraic projective set of zeroes. These give a Zariski topology on $\mathbb{P}^n_k$, and we see that $I(E)$ is a homogeneous ideal and we define the coordinate ring to be $R/I(E)$.

For projective algebraic sets, a regular function is one that is locally f/g with $\deg f = \deg g$ and g not vanishing on the neighborhood.

An algebraic variety is then a Zariski open subset of a projective variety, and this gives us a category of algebraic varieties.

1.3 Sheaves

Let X be a topological space and look at the category of open sets. A presheaf is just a contravariant functor to the category of abelian groups.

A sheaf is a presheaf along with two conditions:

1. $s \in \mathscr{F}(U)$ such that its image in $\mathscr{F}(U_i)$ is zero for all i is zero.
2. If we have $s_i \in \mathscr{F}(U_i)$ which agree when restricted to $U_i \cap U_j$, then there exists s which restricts to each of them.

2 Lecture 2 - Sheaves

Let X be a topological space, $\mathscr{F}, \mathscr{G}$ sheaves. A morphism of sheaves $\mathscr{F} \to \mathscr{G}$ is a map for each open set compatible with the restriction maps.

We define the stalk at $x \in X$ of $\mathscr{F}$ to be $\varinjlim_{x \in U} \mathscr{F}(U)$ as abelian groups.

We define the associated sheaf to a given presheaf $\mathscr{F}$ to be the sheaf such that every map from the presheaf to any sheaf must factor through, and denote it $\tilde{\mathscr{F}}$. This is unique up to unique isomorphism.

The associated sheaf has the property that $\mathscr{F}_x \cong \tilde{\mathscr{F}}_x$ for all $x \in X$.

Now, let $f : X \to Y$ be a continuous map and $\mathscr{F}$ a sheaf on X. We define $f_*\mathscr{F}$ by $f_*\mathscr{F}(U) = \mathscr{F}(f^{-1}(U))$.

For $Y \subset X$, and $\mathscr{F}$ on X, we define $f^{-1}(\mathscr{F})$ to be $\mathscr{F}|_Y$.

2.1 Ringed Spaces

A pair $(X, \mathscr{O}_X)$ where $\mathscr{O}_X$ is a sheaf of rings on X is a ringed space. A morphism of ringed spaces $(X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a continuous map $f : X \to Y$ and a map $\mathscr{O}_Y \to f_*\mathscr{O}_X$.

A locally ringed space is a ringed space such that the stalks of the sheaf $\mathscr{O}_X$ are all local rings, and a morphism of locally ringed spaces is required to induce on the stalks maps $f^{-1}(\mathfrak{m}_{X,x}) = \mathfrak{m}_{Y,y}$.

Take $(E, \mathscr{O}_E)$ where E is an algebraic set and $\mathscr{O}_E$ is the sheaf of regular functions. This example is the fundamental one in algebraic geometry.

The locally ringed spcae $(\mathbb{C}^n, \mathscr{O}_{\mathbb{C}^n})$ with the sheaf being the sheaf of local holomorphic functions is the fundamental example in analytic geometry.

2.2 Local Analytic Spaces

For $U \subset \mathbb{C}^n$, $f_i : U \to \mathbb{C}$ holomorphic for $i \in I$. Then we know $(\mathbb{C}^n, \mathscr{O}_{\mathbb{C}^n})$ is a locally ringed space, look at $(U, \mathscr{O}_{\mathbb{C}^n}|_U)$. Set $X = \{f_i = 0, i \in I\}$ and then $(f_i)\mathscr{O}_U = \mathscr{I}$ is a sheaf of ideals.

So, we can now distinguish between $(0, \mathbb{C})$ and $(0, \mathbb{C}\{x\}/x^2)$, the first is just a point, the second is a double point, and can be viewed as the intersection of a parabola and a line tangent to its vertex.

2.3 Affine Schemes

Let A be a ring (commutative with identity). Then Spec$A$ is the set of prime ideals of A, and the closed sets are given by tkaing an ideal I in A and setting $V(I)$ to be the set of prime ideals containing I. The open sets are their complements.

We define on Spec$A$ the sheaf $\mathscr{O}_A$, whose stalks at P is A_P, and for any open set U, we define $\mathscr{O}_A(U)$ by s is a section if $s : U \to \coprod_{P \in U} A_P$ with for all P there exists an open neighborhood V and $a, b \in A$ such that for all $Q \in V$ we have $b \notin Q$ and $s(Q) = a/b$.

The set of morphisms $(\text{Spec}A, \mathscr{O}_A) \to (\text{Spec}B, \mathscr{O}_B)$ is the same as the set of homomorphisms $B \to A$.

2.4 Schemes

A locally ringed space $(X, \mathscr{O}_X)$ is a scheme if for every x there exists a U such that $(U, \mathscr{O}_X|_U)$ is isomorphic to an affine scheme.

If R is a graded ring $\oplus_{g \geq 0} R_d$, then we define a scheme $(\text{Proj}R, \mathscr{O}_R)$ by Proj$R$ is the set of homogeneous prime ideals in R. We want to set up $\mathscr{O}_{R,P}$ to be $R_{(P)}$, which is the set of elements of degree zero in $T^{-1}R$, where T is the set of homogeneous elements which are not in P. We set $s \in \mathscr{O}_R(U)$ if $s : U \to \coprod_{P \in U} R_{(P)}$ such that for all $P \in U$ there exists V and a, b homogeneous of the same degree such that for all $Q \in V$, $b \notin Q$ and $s(Q) = a/b$.

3 Lecture 3 - Projective Schemes

Let R be a graded ring, look at $(\text{Proj}R, \mathscr{O}_R)$. Let $f \in \oplus_{d \geq 0} R_d = R_+$, then we define $D_+(f)$ to be the homogeneous primes not containing f. $(D_+(f), \mathscr{O}_R|_{D_+(f)})$ is isomorphic to $(\text{Spec}R_{(f)}, \mathscr{O}_{R_{(f)}})$.

For any ring $R = A[x_0, \dots, x_n]$, we have $(\text{Proj}R, \mathscr{O}_R)$ and we'll call it $\mathbb{P}^n_A$.

3.1 Gluing Schemes

Let $(X_1, \mathscr{O}_1)$ and $(X_2, \mathscr{O}_2)$ be two schemes wuch that $(U_1, \mathscr{O}_1|_{U_1}) \to (U_2, \mathscr{O}_2|_{U_2})$ is an isomorphism. Then we can construct a new scheme by identifying them along this map.

3.2 Schemes over a scheme

Let $\phi : (X, \mathscr{O}_X) \to (S, \mathscr{O}_S)$. We call this an S-scheme, and often abuse notation by calling X an S-scheme.

In particular, we will look at schemes over $(\text{Spec}k, k)$.

3.3 Varieties and Schemes

Any variety is covered by a finite number of affine algebraic varieties.

This means that we can take any variety V over k, and make a scheme over $\text{Spec}k$ out of it, by just taking affine varieties to $\text{Spec}A(E)$.

Now, we say that a scene is connected if X is, irreducible if X is, it is reduced if the rings are all reduced (have no nilpotents) and integral similarly.

A scheme is integral if and only if it is reduced and irreducible.

A scheme is locally noetherian if it has a covering by spectra of noetherian rings.

Now, let $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ be a morphisms of schemes.

We say f is locally of finite type if $(Y, \mathscr{O}_Y)$ is covered by $(\text{Spec}A_i, \mathscr{O}_{A_i})$ such that $f^{-1}(\text{Spec}A_i) = \text{Spec}B_{ij}$ with the B_{ij} being A_i-algebras of finite type, that is, are finitely generated as algebras. We say that it is of finite type if the B_{ij} are finite as modules.

An example is the map from a parabola to a line, which induces $k[x] \to k[x, y]/(y - x^2)$.

We can construct fiber products, take $\phi : X \to S$ and $\psi : Y \to S$, we can get $X \times_S Y$, it's the unique scheme such that for all maps $Z \to X$ and $Z \to Y$ that are equal after composing with the maps to S, we get a unique map $Z \to X \times_S Y$.

This allows us to define base change: If we have a map $X \to Y$ and another $Y' \to Y$, we can define $X' = X \times_Y Y'$ and we have $X' \to Y'$, the base change of the morphism.

For any $X \to Y$, we have a map $X \to X \times_Y X$ by using the same map to create the fiber product. If this morphism is closed, then we say that f is separated.

We call a morphism proper if it is separated, of finite type and universally closed, and we say that a scheme which is proper over $\text{Spec}k$ is complete.

3.4 Projective Morphisms

Let $(Y, \mathscr{O}_Y)$ be a scheme. We define $\mathbb{P}^n_Y \to Y$ to be projective space over Y, where $\mathbb{P}^n_Y = Y \times_{\mathrm{Spec}\mathbb{Z}} \mathbb{P}^n_{\mathbb{Z}}$.

We say that a morphism is projective if it factors through $\mathbb{P}^n_Y \to Y$ and the map $X \to \mathbb{P}^n_Y$ is a closed immersion.

Projective morphisms of Noetherian schemes are proper, and quasi-projective morphisms are separated and of finite type.

So a variety over k turns out to just be a scheme over k which is integral and of finite type.

4 Lecture 4 - Sheaves of Modules

Let $(X, \mathscr{O}_X)$ be a ringed space and $\mathscr{M}$ a sheaf on X such that for all U, $\mathscr{M}(U)$ is an $\mathscr{O}_X(U)$-module and these structures are compatible with the restriction maps.

4.1 Locally Free Modules

$\mathscr{M}$ is free if for all U, $\mathscr{M}(U)$ is a free $\mathscr{O}_X(U)$-module.

On a scheme $(X, \mathscr{O}_X)$, say $(\mathrm{Spec}\, A, \mathscr{O}_A)$, then an $\mathscr{O}_A$-module on $\mathrm{Spec}\, A$ is given by an A-module M in the following way. Define $\tilde{M}(U)$ by the sections over U are $s : U \to \prod_{P \in U} M_P$ such that for all $P \in U$, there exists V, m, a such that for all $Q \in V$, we have $s(Q) = \frac{m}{a}$. Then $\tilde{M}_P = M_P$. These are the prototypes of "good" sheaves.

Definition 4.1 (Quasi-Coherent)**.** *A sheaf of $\mathscr{O}_X$-modules is quasi-coherent if there exists covering by open affines such that it's restrictions are of the form $\tilde{M}_i$.*

Definition 4.2 (Coherent)**.** *$\mathscr{F}$ is coherent if the M_i are finite A_i-modules.*

4.2 Differential Forms

Let A be a ring, B an A-algebra and M a B-module. Then an A-derivation of B into M is a map $d : B \to M$ such that d is additive, for all $b, b' \in B$ we have $d(bb') = bd(b') + b'd(b)$ and for all $a \in A$, we have $d(a1) = 0$.

There exists a universal object, a module $\Omega_{B/A}$ and derivation $\delta : B \to \Omega_{B/A}$ such that any other derivation factors through it. We can construct it by looking at $\Delta : B \otimes_A B \to B$ given by $\Delta(b \otimes b') = bb'$, setting $I = \ker \Delta$ and then $\Omega_{B/A} = I/I^2$, with $\delta : B \to I/I^2$ given by $\delta(b) = 1 \otimes b - b \otimes 1$ modulo I^2.

Now, letting $X \to Y$ be a morphism of schemes, we can take $\Delta : X \to X \times_Y X$ and and let $\mathscr{I}$ be the ideal sheaf of the image of Δ. Then $\Omega_{X/Y} = \mathscr{I}/\mathscr{I}^2$.

Let $A \xrightarrow{h} B \xrightarrow{k} C$. Then we have exact sequences $\Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{B/A} \to 0$ and if I is an ideal of B and $C = B/I$, then we have $I/I^2 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to 0$.

The sheaf is well-behaved with respect to base change: let $f : X \to Y$ a morphism, and base change along $g : Y' \to Y$, then $\Omega_{X'/Y'} = (g')^*(\Omega_{X/Y})$.

More generally, let $f : X \to Y$ and $g : Y \to Z$ morphisms of schemes. Then $f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$ is exact, and, if $Z \subset X$ is a closed subscheme with ideal $\mathscr{I}$, we have $\mathscr{I}/\mathscr{I}^2 \to \Omega_{X/Y} \otimes_{\mathscr{O}_X} \mathscr{O}_Z \to \Omega_{X/Z} \to 0$.

4.3 Nonsingular Varieties

Let $Y \subset k^n$ affine, with $I(Y) = (f_1, \dots, f_k)$. Then we say that $\dim Y$ is the Krull dimension of $A(Y)$, and we'll denote it by r. Then a point x is a nonsingular point of Y if and only if rank $\left(\frac{\partial f_i}{\partial x_j}\right) = n - r$ at x. We say that Y is nonsingular if it is nonsingular at every point.

It is a theorem that $y \in Y$ is nonsingular if and only if $\mathscr{O}_{Y,y}$ is a regular local ring, so we take this as the definition, on schemes.

What does it mean to be regular? For a Noetherian local ring of dimension r, the following are equivalent and taken to define regularity:

1. $\mathfrak{m}$ is generated by r elements.
2. The associated graded ring with respect to $\mathfrak{m}$ is a polynomial ring in r variables.
3. $\dim(\mathfrak{m}/\mathfrak{m}^2) = r$.

Now, let X be a locally ringed space isomorphic to an analytic space. This is locally a local analytic space, and it is Hausdorff, and there exists an analytification functor which takes varieties to analytic spaces.