Topology, cohomology and sheaf theory

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June 16, 2010

1 Lecture 1

1.1 Manifolds

Definition 1.1 (Locally Euclidean)**.** *A topological space is locally Euclidean if every $p \in M$ has a neighborhood U and a homeomorphism $\phi : U \to V$, where V is an open subset of $\mathbb{R}^n$. We call the pair (U, ϕ) a chart.*

Definition 1.2 (C^∞ Compatible)**.** *Two charts are C^∞ compatible if $\phi \circ \psi^{-1}$ and $\psi \circ \phi^{-1}$ are C^∞ functions.*

Definition 1.3 (Manifold)**.** *A manifold is a locally Euclidean, Hausdorff, second countable topological space on which there existsa covering by C^∞ compatible charts.*

Example 1.4 (The Circle)**.** *The circle is given by $x^2 + y^2 = 1$. Taking the upper and lower semicircles, along with the left and right semicircles, provides the desired cover.*

1.2 Tangent Space

Let r^i be coordinates on an open set in $\mathbb{R}^n$ and x^i be $r^i \circ \phi$.

Definition 1.5 (C^∞ function)**.** *$f : M \to \mathbb{R}$ is C^∞ at p if there exists (U, ϕ) with $p \in U$ such that $f \circ \phi^{-1}$ is C^∞ at $\phi(p) \in \mathbb{R}^n$.*

Definition 1.6 (Partial Derivatives)**.** *If $f \in C^\infty(M)$ and $(U, x^1, \ldots, x^n)$ is a chart containing p, then we define $\frac{\partial}{\partial x^i}|_p f = \frac{\partial}{\partial r^i}|_{\phi(p)} f \circ \phi^{-1}$*

Definition 1.7 (Tangent Space)**.** *T_pM, the tangent space at p, is the vector space with basis $\frac{\partial}{\partial x^i}$ at $p \in (U, \phi)$.*

Theorem 1.8. *Let x^i and y^j be different sets of coordinates at a point p. Then $\frac{\partial}{\partial y^j} = \sum a^i_j \frac{\partial}{\partial x^i}$, where $a^i_j = \frac{\partial x^i}{\partial y^j}$.*

The proof is a direct computation.

Definition 1.9 (Vector Field)**.** *A vector field is a function $X : M \to \coprod_{p \in M} T_pM$ such that $X_p \in T_pM$. We can write it at p as $X = \sum a^i \frac{\partial}{\partial x^i}$ and we call X a C^∞ vector field if the a^i are all C^∞.*

1.3 Differential Forms

Definition 1.10 (Cotangent Space)**.** *The cotangent space is* $T_p^*M = \hom(T_pM, \mathbb{R})$*. It has basis* $(dx^i)_p$ *the duals of* $\frac{\partial}{\partial x^i}$*'s at* p*.*

Definition 1.11 (Differential 1-Form)**.** *A 1-form is a function* $\omega : M \to \coprod_{p\in M} T_p^*M$ *such that* $\omega_p \in T_p^*M$*. We can write them near* p *as* $\omega = \sum a^i (dx^i)_p$ *and call them* C^∞ *if the* a^i *are.*

Definition 1.12 (Wedge Product)**.** *Let* V *be a real vector space, and let* $\alpha, \beta \in V^*$*, then* $(\alpha \wedge \beta)(v, w) = \alpha(v)\beta(w) - \alpha(w)\beta(v)$*.*

It then follows that $\beta \wedge \alpha = -\alpha \wedge \beta$ and $\alpha \wedge \alpha = 0$.

Definition 1.13 (Wedge Product)**.** *Let* $A_k(V)$ *be the set of alternating functions on* V *with* k *inputs. If* $\alpha \in A_k$ *and* $\beta \in A_\ell$ *then* $\alpha \wedge \beta \in A_{k+\ell}$ *and* $\beta \wedge \alpha = (-1)^{k\ell} \alpha \wedge \beta$

We define $A_0(V) = \mathbb{R}$.

Definition 1.14 (k-form)**.** *A* k*-form on* M *is a map* $\omega : M \to \coprod_{p\in M} A_k(T_p^*M)$ *such that* $\omega_p \in A_k(T_p^*M)$*. It is called* C^∞ *if the coefficient functions are.*

We denote $\mathcal{A}^i(M)$ to be the C^∞ i-forms on M. We have that $\mathcal{A}^0(M) = C^\infty(M)$, and $\mathcal{A}^k(M) = 0$ for k greater than the dimension of M.

1.4 Exterior Derivative

Definition 1.15 (Derivative)**.** *If* $f \in C^\infty(M)$*, define a 1-form* $df \in \mathcal{A}^1(M)$ *on a chart* (U, x^i) *by* $df = \sum \frac{\partial f}{\partial x^i} dx^i$*. More generally, let* $\omega \in \mathcal{A}^k(M)$ *by* $\omega = \sum a_I dx^I$*, then we define* $d\omega = \sum d(a_I) \wedge dx^I$*.*

Proposition 1.16. *The exterior derivative* $d : \mathcal{A}^k(M) \to \mathcal{A}^{k+1}(M)$ *satisfies*

1. *Antiderivation:* $d(\omega \wedge \tau) = (d\omega) \wedge \tau + (-1)^{\deg \tau} \omega \wedge d\tau$
2. $d^2 = 0$.

1.5 DeRham Cohomology

We define the kernel of d to be the closed k-forms $Z^k(M)$ and the image to be the exact k-forms, $B^k(M)$.

Definition 1.17 (DeRham Cohomology)**.** $H^k(M) = Z^k(M)/B^k(M)$*.*

Example 1.18 ($H^0(M)$)**.** $H^0(M)$ *is just* $\ker d : C^\infty(M) \to \mathcal{A}^1(M)$*, because there is no* $\mathcal{A}^{-1}(M)$*. So we end up with the set of locally constant functions on* M*, and, because* M *is a manifold,* $H^0(M)$ *is the set of all functions constant on the connected components.*

Theorem 1.19. *$H^0(M) = \mathbb{R}^r$ where r is the number of connected component of M.*

Example 1.20 ($H^k(\mathbb{R})$)**.** *An element of $\mathcal{A}^1(\mathbb{R})$ is just $f(x)dx$ where $f \in C^\infty(\mathbb{R})$, and element of $d\mathcal{A}^0(\mathbb{R})$ is $dg = g'(x)dx$. So is every C^∞ function equal to $g'(x)$ for some $g \in C^\infty(\mathbb{R})$? Yes, by the fundamental theorem of calculus, so $H^k(\mathbb{R})$ is $\mathbb{R}$ for $k = 0$ and is 0 else.*

2 Lecture 2 - Computation of deRham Cohomology

2.1 Pullback of Forms

If $f : M \to N$ is a C^∞ map, then there is a pullback map $f^* : \mathcal{A}^k(N) \to \mathcal{A}^k(M)$.

For $k = 0$, then $f^*(h) = h \circ k$. In generall,y locally $\omega \in \mathcal{A}^k(N)$ can be written $\omega = \sum a_I dx^I = \sum a_I dx^{i_1} \wedge \ldots \wedge dx^{i_k}$ and we define $f^*\omega = \sum (f^* a_I) d(f^* x^{i_1}) \wedge \ldots \wedge d(f^* x^{i_k})$.

Proposition 2.1.

1. $1_M^* = 1_{\mathcal{A}^k(M)}$
2. $(f \circ g)^* = g^* \circ f^*$
3. $f^* d = d f^*$

By 3, we know that f^* of a closed form is closed. If $d\omega = 0$, then $d(f^*\omega) = f^* d\omega = 0$. Now, $f^*(d\tau) = d(f^*\tau)$, and so exact forms pullback to exact forms. Thus, f^* induces a map (also denoted by f^* on cohomology $H^k(N) \to H^k(M)$.

If $f : M \to N$ is a diffeomorphism, then there exists $g : N \to M$ such that $f \circ g = 1_N$ and $g \circ f = 1_M$, then $g^* \circ f^* = 1_{H^*(N)}$ and $f^* \circ g^* = 1_{H^*(M)}$.

Theorem 2.2. *If $f : M \to N$ is a diffeomorphism, then $f^* : H^*(N) \to H^*(M)$ is an isomorphism.*

2.2 Homological Algebra

A cochain complex $\mathcal{C}$ is $0 \to C^0 \xrightarrow{d} C^1 \to \ldots$ such that $d^2 = 0$. So $H^k(\mathcal{C}) = \ker d_k / \mathrm{im} d_{k-1}$ is defined.

A sequence of vector spaces $A \xrightarrow{i} B \xrightarrow{j} C$ is exact at B if $\mathrm{im}(i) = \ker j$.

A linear map $f : (\mathrm{A}, d) \to (\mathrm{B}, d)$ of cochain complexes is a cochain map if $f \circ d = d \circ f$ (this is the data of a map in each degree).

A cochain map induces $f^* : H^k(\mathcal{A}) \to H^k(\mathcal{B})$.

Theorem 2.3 (Zig-Zag Lemma)**.** *A short exact sequence of cochain complexes $0 \to \mathcal{A} \xrightarrow{i} \mathcal{B} \xrightarrow{j} \mathcal{C} \to 0$ induces a long exact sequence on cohomology $\ldots \to H^k(\mathcal{A}) \to H^k(\mathcal{B}) \to H^k(\mathcal{C}) \xrightarrow{d^*} H^{k+1}(\mathcal{A}) \to \ldots$.*

2.3 Mayer-Vietoris Sequence

Suppose that M is covered by open sets U, V so that $M = U \cup V$. Then we have $\mathcal{A}^k(M) \xrightarrow{i} \mathcal{A}^k(U) \oplus \mathcal{A}^k(V) \xrightarrow{j} \mathcal{A}^k(U \cap V)$ by the restriction maps followed by the difference of restrictions.

Theorem 2.4 (Mayer-Vietoris Sequence)**.** *The sequence above is exact.*

2.4 $H^*(S^1)$

We cover S^1 by the union of two copies of $\mathbb{R}$. Then $H^*(U) = H^*(V) = H^*(\mathbb{R})$ and $H^*(U \cap V)$ is two copies of $\mathbb{R}$, and so $H^*(U \cap V) = H^*(\mathbb{R}) \oplus H^*(\mathbb{R})$.

Then the Mayer-Vietoris sequence becomes $0 \to \mathbb{R} \to \mathbb{R}^2 \to \mathbb{R}^2 \to H^1(S^1) \to 0 \to 0 \to \ldots$ So by exactness, $H^1(S^1) = \mathrm{im}(d^*) \cong \mathbb{R}^2/\ker d^* = \mathbb{R}^2/\mathrm{im}(j^*) \cong \mathbb{R}$, because the image of j^* is the diagonal.

2.5 Smooth Homotopy

Two C^∞ maps $f_0, f_1 : M \to N$ are smoothly homotopic if there existsa C^∞ map $F : M \times [0,1] \to N$ such that $F(x,0) = f_0(x)$ and $f(x,1) = f_1(x)$ (where we say that F is C^∞ if it can be extended to a C^∞ function in a neighborhood of $M \times [0,1]$ in $M \times \mathbb{R}$). We write $f_0 \sim f_1$.

Definition 2.5 (Homotopy Inverse)**.** *$f : M \to N$ has a homotopy inverse if there exists $g : N \to M$ such that $f \circ g \sim 1_N$ and $g \circ f \sim 1_M$*

Then we say that M and N have the same homotopy type and f is a homotopy equivalence.

Homotopy Axiom:Homotopic maps $f, g : M \to N$ induce the same map on cohomology $f^* = g^* : H^*(N) \to H^*(M)$.

Example 2.6. *$\mathbb{R}^n$ has the homotopy type of a point $\{0\}$. We have i the inclusion of the origin and π the unique map to the point. Then $\pi \circ i$ is the identity map on the point, and $i \circ \pi : \mathbb{R}^n \to \mathbb{R}^n$ sends every point to 0. We claim this is homotopic to the identity. Define $F : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ by $F(x,t) = (1-t)x$ to be the homotopy.*

Corollary 2.7. *If $f : M \to N$ is a homotopy equivalence, then $f^* : H^*(N) \to H^*(M)$ is an isomorphism.*

Corollary 2.8 (Poincare Lemma)**.** *$H^*(\mathbb{R}^n) = H^*(pt)$ is $\mathbb{R}$ in degree 0 and 0 else.*

3 Lecture 3 - Presheaves and Cech Cohomology

Definition 3.1 (Presheaf)**.** *A presheaf on a topological space X is a function that assigns to each open $U \subset X$ an abelian group $\mathscr{F}(U)$ and to every inclusion $i_U^V : U \to V$ a group homomorphism $\mathscr{F}(i_U^V) = \rho_V^U : \mathscr{F}(U) \to \mathscr{F}(V)$ such that $\rho_U^U = 1_{\mathscr{F}(U)}$, and $\rho_W^U = \rho_W^V \circ \rho_V^U$.*

Example 3.2. *$\mathcal{A}^k(U)$ is the C^∞ k-forms on U. This is a presheaf on a manifold M.*

Example 3.3. *If G is an abelian group, for every open $U \subset X$, define $\underline{G}(U)$ to be the locally constant functions $f : U \to G$. Then $\underline{G}$ is a presheaf.*

3.1 Cech Cohomology of an Open Cover

Let $\{U_\alpha\}$ be an open cover of a topological space indexed by a totally ordered set. We'll denote intersections by putting the subscripts together.

When $\alpha = 0, 1$ gives the cover, we have Mayer-Vietoris, which says $0 \to \mathcal{A}^k(M) \to \prod \mathcal{A}^k(U_i) \to \mathcal{A}^k(U_{01}) \to 0$.

Now, let $\mathscr{F}$ be a presheaf on a topological space X. We then have a sequence

$$0 \to \mathscr{F}(X) \to \prod_\alpha \mathscr{F}(U_\alpha) \to \prod_{\alpha<\beta} \mathscr{F}(U_{\alpha\beta}) \to \dots$$

Define $C^p(\mathcal{U}, \mathscr{F})$ to be the term involving p open sets. Then we define $\delta : C^p(\mathcal{U}, \mathscr{F}) \to C^{p+1}(\mathcal{U}, \mathscr{F})$ by $(\delta\omega)_{\alpha_0,\dots,\alpha_{p+1}} = \sum_{i=0}^{p+1}(-1)^i \omega_{\alpha_0,\dots,\hat{\alpha}_i,\dots,\alpha_{p+1}}$.

It turns out that $\delta^2 = 0$, and so we define the cohomology of this complex to be the Cech cohomology $\check{H}^k(\mathcal{U}, \mathscr{F})$.

3.2 Direct Limits

Definition 3.4 (Directed Set)**.** *A directed set is a set I with a binary relation $<$ that is reflexive, transitive and such that any two elements have a common upper bound.*

Example 3.5. *Fix $p \in X$. Let I be the set of neighborhoods of p in X and say that $U < V$ iff $U \subset V$.*

Fix a topological space X. Then I be the set of all open covers of X. An open cover $\mathcal{V}$ refines $\mathcal{U}$ if every $V \in \mathcal{V}$ is contained in some $U \in \mathcal{U}$. Refinement gives a directed set structure to the set of covers. A refinement $\mathcal{V}$ of $\mathcal{U}$ can be given by a refinement map on the index sets stating which U each V is contained in.

A directed system of groups is a collection $\{G_i\}_{i\in I}$ of groups indexed by a directed set U such that for all $a < b$ we have a homomorphism $f_{ab} : G_a \to G_b$ satisfying that $f_{aa} = 1$ and $f_{ac} = f_{bc} \circ f_{ab}$.

Example 3.6. *Let I be the neighborhoods of $p \in X$. Then set $G_U = C^\infty(U)$ and say that (U, f) and (V, g) are equivalent iff there exists $W \subset U \cap V$ such that $f|_W = g|_W$. We call these the germs of functions at p.*

In $\coprod_{i\in I} G_i$, let $g_a \in G_a$ and $g_b \in G_b$. Then we say $g_a \sim g_b$ if there exists $c > a, b$ such that $f_{ac}(g_a) = f_{bc}(g_b)$ and define $\varinjlim_{i\in I} G_i = \coprod G_i/\sim$.

3.3 Cech Cohomology of a Topological Space

For each open cover, we have $\check{H}^k(\mathcal{U}, \mathscr{F})$, and we have restrictions making it into a directed system of abelian groups. So we define $\check{H}^k(X, \mathscr{F})$ to be the limit of this system.

3.4 C^∞ partitions of unity

Definition 3.7 (Partition of Unity)**.** *A C^∞ partition of unity on a manifold M is a collection of C^∞ functions ρ_α with $\sum \rho_\alpha = 1$ and $\rho_\alpha : M \to [0, 1]$.*

We define the support of a function to be the set where it is nonzero.

A collection of subsets $\{A_\alpha\}$ in X is locally finite if every $p \in X$ has a neighborhood that meets only finitely many of the A_α.

Theorem 3.8. *Given any open cover of a manifold, there exists a C^∞ partition of unity with each element's support contained in one of the open sets of the cover.*

4 Lecture 4 - Sheaves and the Cech-deRham Isomorphism

Version 6 of lecture notes on Summer School website.

Exercise 4.1. *Compute H^* of S^n, $\mathbb{R}^n \setminus \{0\}$, $\mathbb{CP}^1$, $\mathbb{CP}^2$ for problem session today.*

Problem Session: Does $H^*(\mathcal{U}, \mathscr{F})$ depend on the order on the index set?

4.1 Sheaves

Definition 4.2 (Sheaf)**.** *A sheaf on a topological space X is a presheaf such that for any open set U and any open cover U_i of U, we have*

1. *Uniqueness: If $s \in \mathscr{F}(U)$ such that $s|_{U_i} = 0$ for all i, then $s = 0$.*
2. *Gluing: If $s_i \in \mathscr{F}(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j, then there exists $s \in \mathscr{F}(U)$ such that $s|_{U_i} = s_i$.*

Example 4.3.

1. *$\mathcal{A}^k$ is a sheaf for any manifold.*
2. *Z^k, the closed C^∞ k-forms on a manifold is a sheaf.*
3. *$\mathcal{A}^{p,q}$ is a sheaf for any complex manifold.*
4. *$\underline{\mathbb{R}}$, the locally constant functions to $\mathbb{R}$, is a sheaf.*

However, the presheaf of constant functions is not a sheaf.

4.2 Cohomology in Degree 0

Let $\mathscr{F}$ be a sheaf on a topological space X. We want to know $\check{H}^0(\mathcal{U}, \mathscr{F})$. Let $\mathcal{U} = \{U_i\}$ be an open cover of X. Then as $\mathscr{F}$ is a sheaf, we have the Cech sequence.

So then $\check{H}^0(\mathcal{U}, \mathscr{F}) = \ker \delta$ as $C^{-1} = 0$ which is just $\mathrm{im}(r)$ where r is the restriction map $\mathscr{F}(X) \to \prod \mathscr{F}(U_i)$. But this is precisely $\mathscr{F}(X)$.

Now, let $\mathcal{V}$ be a refinement of $\mathcal{U}$. Both Cech groups are isomorphic to $\mathscr{F}(X)$, and so they are isomorphic.

Theorem 4.4. *If $\mathscr{F}$ is a sheaf on a topological space X, then $\check{H}^0(X, \mathscr{F}) = \mathscr{F}(X)$.*

4.3 Further Computations

For all q and all $k > 0$, $\check{H}^k(M, \mathcal{A}^q) = 0$. But how do we prove this? The standard method for proving that cohomology is zero is finding a map $K : C^p \to C^{p-1}$ such that $1 - 0 = \delta K + K\delta$. Applying $(\delta K + K\delta)$ to a cocycle just gives δK. Therefore, this induces the zero map on cohomology, and so $1^* = 0$, which can only happen when $H^q = 0$. We call K a chain homotopy. There is a theorem guaranteeing that such a K exists when the cohomology is zero.

Let $\mathcal{U} = \{U_\alpha\}$ be an open cover of M, and $\{\rho_\alpha\}$ be a C^∞ partition of unity subordinate to $\mathcal{U}$. For $k \geq 1$, then $K : C^p(\mathcal{U}, \mathscr{F}) \to C^{p-1}(\mathcal{U}, \mathscr{F})$ by $(K\omega)_{\alpha_0,\ldots,\alpha_{p-1}} = \sum_\alpha \rho_\alpha \omega_{\alpha,\alpha_0,\ldots,\alpha_{p-1}}$.

We must check that $\delta K + K\delta = 1$ for $k \geq 1$, and then we have the vanishing of the Cech cohomology, when we take $\mathscr{F} = \mathcal{A}^q$. We in fact have $H^k(M, \mathcal{A}^{p,q}) = 0$ for $k \geq 1$ on a complex manifold.

But $H^k(M, \Omega^q)$ is not necessarily zero!

4.4 Sheaf Morphism

A morphism of sheaves is a collection of maps, one for each open set, compatible with the restriction maps

We define the stalk of a presheaf $\mathscr{F}_p$ at $p \in X$ to be $\varinjlim_{p \in U} \mathscr{F}(U)$. So a presheaf homomorphism $\phi_p : \mathscr{F}_p \to \mathscr{G}_p$ by $(U, s) \mapsto (U, \phi(s))$.

4.5 Exact Sequences of Sheaves

Theorem 4.5. *A short exact sequence of sheaves $0 \to \mathscr{E} \to \mathscr{F} \to \mathscr{G} \to 0$ induces a long exact sequence $\ldots \to H^k(X, \mathscr{E}) \to H^k(X, \mathscr{F}) \to H^k(X, \mathscr{G}) \to H^{k+1}(X, \mathscr{E}) \to \ldots$.*

Theorem 4.6 (Poincaré Lemma)**.** *The sequence of sheaves $0 \to \mathbb{R} \to \mathcal{A}^0 \to \ldots$ is exact.*

Proof. Look at $0 \to \mathbb{R} \to \mathcal{A}^0 \to \mathrm{im}(d_0) \to 0$, and similarly, we can make $0 \to Z^{k-1} \to \mathcal{A}^{k-1} \to Z^k \to 0$ for all k.

So we have $0 \to H^{k-1}(M, Z^1) \to H^k(M, \underline{\mathbb{R}}) \to H^k(M, \mathcal{A}^0) = 0$, and we can finish the proof by basic homological manipulations. $\square$