

# 1 Lecture 1

Topics:

1. Chern-Weil Theory
2. K-Theory and Index Theorem
3. Cobordism Theory
4. Surgery Theory

## 1.1 Chern-Weil Theory

This theorem inspired Chern to define Chern classes and is a special case of the index theorem

**Theorem 1.1** (Gauss-Bonnet). *Let  $X$  be a compact  $C^\infty$  2-manifold and  $\kappa$  its Gaussian curvature. Then  $\int_X \kappa = 2\pi\chi(X)$ .*

**Theorem 1.2** (Lichnerowicz's Theorem). *If  $M$  is a  $4k$  dimensional Spin manifold and  $\hat{A}(M) \neq 0$ , then  $M$  has no metric of positive scalar curvature.*

Let  $E^k \rightarrow X$  be a complex smooth vector bundle over a  $C^\infty$  manifold.

**Definition 1.1** (Connection). *A connection  $\nabla : C^\infty(X, E) \rightarrow C^\infty(X, E \otimes T_X^*)$  is a  $\mathbb{C}$ -linear map satisfying  $\nabla(sf) = \nabla(s)f + s \otimes df$ .*

*If  $\xi \in C^\infty(X, TX)$ , then  $\nabla_\xi(s) \in C^\infty(X, E)$ , and  $\nabla_{f\xi}(s) = f\nabla(s)$ .*

Note that  $\nabla$  can be restricted to  $C^\infty(U, E) \rightarrow C^\infty(U, E \otimes T^*X)$  when  $U \subset X$  open.

In fact,  $\nabla$  extends to  $\Omega^i(X; E) \rightarrow \Omega^{i+1}(X; E)$ , the forms on  $X$  with values in  $E$ .

Let  $\Omega = \nabla^2 : \Omega^i(X; E) \rightarrow \Omega^{i+2}(X; E)$ .  $\Omega$  is a tensor.  $\nabla^2(sf) = \nabla^2(s)f - \nabla(s)df + \nabla(s)df - sd^2f = \nabla^2(s)f$ . So  $\Omega \in \Omega^2(X, \text{End}(E))$ .

$$\Omega(\xi_1, \xi_2)s = \nabla_{\xi_1}\nabla_{\xi_2}s - \nabla_{\xi_2}\nabla_{\xi_1}s - \nabla_{[\xi_1, \xi_2]}s.$$

Now, if  $\nabla^1$  and  $\nabla^2$  are two connections on  $E$ , then  $(\nabla^1 - \nabla^2)(sf) = \nabla^1(sf) - \nabla^2(sf) = (\nabla^1 - \nabla^2)(s)f$ . So two connections differ by a tensor in  $\Omega^1(X, \text{End}(E))$ .

Cartan's Moving Frames (reperé mobile)

Let  $U$  be a contractible open set in  $X$ . Let  $\nabla : C^\infty(U, E) \rightarrow \Omega^1(U; E)$ . We trivialize  $E|_U$  by choosing a frame  $s_1, \dots, s_k$ . Then  $\nabla s_i = \omega_i^j \otimes s_j$ . We will use Einstein's convention on repeated indices.

Now let  $s'_1, \dots, s'_k$  be a second frame. Write  $s'_i = s_j g_i^j$ , with  $g_i^j \in C^\infty(U)$ . That is,  $g \in C^\infty(U, GL_k(\mathbb{C}))$ . Let  $\omega'$  be the connection matrix of  $s'_i$ .  $\nabla(s'_i) = s'_j \omega_i^j$ . So then  $s'_j \omega_i^j \nabla(s'_i) = \nabla(s_j g_i^j) = s_k \omega_j^k g_i^j + s_j dg_i^j = s_k (\omega_j^k g_i^j + dg_i^k)$ . Therefore  $g_i^k \omega_j^j = (\omega_j^k g_i^j + dg_i^k)$ .

$$\text{So } \omega_i^j = (g^{-1})_i^k \omega_k^j + (g^{-1})_i^k dg_k^j, \text{ and so } \omega' = g^{-1}\omega g + g^{-1}dg.$$

So now we calculate  $\Omega$  in terms of  $\omega$ .  $\nabla(\nabla s) = \nabla(s \otimes w) = \nabla(s)\omega + sd\omega = s(\omega \wedge \omega + d\omega)$ , and so  $\Omega = d\omega + \omega \wedge \omega$ .

Claim:  $\Omega' = g^{-1}\Omega g$ .

**Lemma 1.3** (Bianchi).  $d\Omega = [\Omega, \omega] = \Omega \wedge \omega - \omega \wedge \Omega$ .

*Proof.*  $\Omega = d\omega + \omega \wedge \omega$ .  $d\Omega = d\omega \wedge \omega - \omega \wedge d\omega$ . □

Let  $P : M_k(\mathbb{C})^n \rightarrow \mathbb{C}$  be a multilinear symmetric function. Such a function is equivalent to a polynomial function on  $M_n(\mathbb{C})$  with  $P \mapsto f(X) = P(X, X, \dots, X)$ . One goes back by the polarization identity.

$P$  is called invariant if  $P(g^{-1}X_1g, \dots, g^{-1}X_ng) = P(X_1, \dots, X_n)$ .

**Lemma 1.4.** *Invariant holds iff  $\forall Y, X_1, \dots, X_k \in \mathfrak{gl}_k(\mathbb{C})$ , we have  $P([Y, X_1], X_2, \dots, X_n) + \dots + P(X_1, \dots, [Y, X_n]) = 0$ .*

Take  $P(\Omega, \dots, \Omega)$ .

Facts:

1. If  $\Omega'$  is the curvature in a different frame, then  $\Omega' = g^{-1}\Omega g$ , and so  $P(\Omega, \dots, \Omega)$  is independent of frame. Thus, we have a global  $n$ -form.
2.  $P(\Omega, \dots, \Omega)$  is closed.
3. The cohomology class represented by  $P(\Omega, \dots, \Omega)$  is in  $H^{2n}(X)$  and is independent of the connection.

*Proof.* Let  $\nabla^1$  and  $\nabla^2$  be two connections. Then  $\nabla^2 = \nabla^1 + \omega$  where  $\omega \in \Omega^1(X, \text{End}(E))$ .  $\Omega^2 = \nabla^2 \nabla^2 = (\nabla^1 + \omega)(\nabla^1 + \omega) = \nabla^1 \nabla^1 + \nabla^1 \circ \omega + \omega \circ \nabla^1 + \omega^2 = \Omega^1 + \nabla^1(\omega) + \omega^2$ . where  $\nabla^1(\omega) = \nabla^1 \circ \omega + \omega \circ \nabla^1$ . □

## 2 Lecture 2

Let  $X$  be a manifold and  $\pi : E^k \rightarrow X$  a vector bundle. Let  $\nabla : C^\infty(X; E) \rightarrow C^\infty(X; E \otimes T^*X)$  be a connection. We can extend to  $\Omega^i(X; E) \rightarrow \Omega^{i+1}(X; E)$ .  $\nabla^2 = \Omega \in \Omega^2(X, \text{End}(E))$

Locally, in a frame,  $\nabla s = s \otimes \omega$  for  $\omega \in \Omega^1(U, M_k(\mathbb{C}))$  the connection form. Locally, we have  $\Omega = d\omega + \omega \wedge \omega$ .

$\Omega' = g^{-1}\Omega g$ .  $d\Omega = [\Omega, \omega]$  is called the Bianchi identity.

Convention: All brackets will be graded, that is,  $[a, b] = ab - (-1)^{|a||b|}ba$ , and  $\mathfrak{g} = \mathfrak{gl}_n$ .

Let  $\Phi$  be a polynomial on  $\mathfrak{g}$  of degree  $k$ , which is the same thing as  $\Phi : \mathfrak{g} \times \dots \times \mathfrak{g} \rightarrow \mathbb{R}$  or  $\mathbb{C}$  with  $\Phi(X_{\sigma_1}, \dots, X_{\sigma_k}) = \Phi(X_1, \dots, X_k)$  for  $\sigma \in S_k$ . Then  $\Phi(X) = \Phi(X, X, \dots, X)$ . We denote this collection by  $S\mathfrak{g}^*$ .

Now suppose  $\Phi \in S^k \mathfrak{g}^*$  which satisfies  $\Phi(ad_g X_1, \dots, ad_g X_k) = \Phi(X_1, \dots, X_k)$ .

Then we can form: take  $\nabla$  in a frame define  $\Phi(\Omega, \dots, \Omega) \in \Omega^{2k}(U)$ . This form doesn't depend on  $s$ , and indeed, defines a global form on  $X$ .

**Proposition 2.1.** 1.  $d\Phi(\Omega) = 0$

2. The class of  $\Phi(\Omega)$  is independent of the connection.

*Proof.*  $d\Phi(\Omega) = d\Phi(\Omega, \dots, \Omega) = \Phi(d\Omega, \Omega, \dots, \Omega) + \Phi(\Omega, d\Omega, \dots, \Omega) + \dots = -(\Phi([\omega, \Omega], \Omega, \dots, \Omega) + \dots = 0$  by invariance.

Let  $\nabla^0, \nabla^1$  be two connections. Then  $\mathcal{O} = \nabla^1 - \nabla^0 \in \Omega^1(X; \text{End}(E))$ . Then  $\nabla^t = \nabla^0 + t\mathcal{O}$ . We will use the following theorem.  $\square$

**Theorem 2.2.**  $\Phi(\Omega^1, \dots, \Omega^1) - \Phi(\Omega^0, \dots, \Omega^0) = k \int_0^1 \Phi(\mathcal{O}, \Omega^t, \dots, \Omega^t) dt$

In the local frame, let  $\omega^0$  be the connection form for  $\Omega^0$ . Then  $\omega^0 + t\mathcal{O}$  is the connection form for  $\nabla^t$ .

So  $\Omega^t = d\omega^t + \omega^t \wedge \omega^t = d(\omega^0 + t\mathcal{O}) + (\omega^0 + t\mathcal{O}) \wedge (\omega^0 + t\mathcal{O}) = d\omega^0 + \omega^0 \wedge \omega^0 + t(d\mathcal{O} + \mathcal{O} \wedge \omega^0 + \omega^0 \wedge \mathcal{O}) + t^2 \mathcal{O} \wedge \mathcal{O}$ .

This is then  $\Omega^0 + t(d\mathcal{O} + [\mathcal{O}, \omega^0]) + \frac{t^2}{2}[\mathcal{O}, \mathcal{O}]$ .

So now we compute the derivative with respect to  $t$  of  $\Phi(\Omega^t, \dots, \Omega^t)$  and apply to symmetry to get eventually  $k\Phi(d\mathcal{O} + [\mathcal{O}, \omega^0] + t[\mathcal{O}, \mathcal{O}], \Omega^t, \dots, \Omega^t)$ .

Now we calculate  $d\Phi(\mathcal{O}, \Omega^t, \dots, \Omega^t) = \Phi(d\mathcal{O}, \Omega^t, \dots, \Omega^t) - (k-1)\Phi(\mathcal{O}, d\Omega^t, \Omega^t, \dots, \Omega^t) = \Phi(d\mathcal{O}, \Omega^t, \dots, \Omega^t) - (k-1)\Phi(\mathcal{O}, [\Omega^t, \omega^t], \Omega^t, \dots, \Omega^t)$

By invariance, we have that  $\Phi([\mathcal{O}, \omega_t], \Omega^t, \dots, \Omega^t) - (k-1)\Phi(\mathcal{O}, [\Omega^t, \omega^t], \Omega^t, \dots, \Omega^t) = 0$ .

And so we have  $\sum_{i=1}^k \Phi(X_1, \dots, [Y, X_i], \dots, X_k) = 0$ .

And so, with more nasty calculations, we obtain  $\frac{\partial}{\partial t} \Phi(\Omega^t) = kd\Phi(\mathcal{O}, \Omega^t, \dots, \Omega^t)$ .

The relative invariants for flat connection:

If  $\nabla^0, \nabla^1$  are flat,  $\Phi(\Omega^0) = \Phi(\Omega^1) = 0$ .

$0 = \Phi(\Omega^1) - \Phi(\Omega^0) = dk \int_0^1 \Phi(\mathcal{O}, \Omega^t, \dots, \Omega^t) dt$ . The form differentiated is then closed and defines a cohomology class.

**Example 2.1.** Define  $c_i$  by  $\deg(\lambda - X/(2\pi i))$  with  $X \in \mathfrak{gl}_n \mathbb{C}$ . It is invariant, and this determinant is then  $\lambda^n + c_1(X)\lambda^{n-1} + \dots + c_n(X)$ , and these are the chern classes.

**Example 2.2.** Let  $E^n \rightarrow X$  be a real vector bundle. look at  $\det(\lambda I - X/2\pi)$ . The coefficients are then  $d_i(X)$ , with  $d_{2i+1} = 0$  and  $p_i = d_{2i}$ .

**Example 2.3.** Let  $\mathfrak{g} = \mathfrak{so}(2n)$ . Then define the Pfaffian of a block diagonal matrix with blocks of the form  $\begin{pmatrix} 0 & \lambda_i \\ \lambda_i & 0 \end{pmatrix}$  to be  $\lambda_1 \dots \lambda_n$ .

We have a map  $Ch - W : S\mathfrak{g}^* \rightarrow H^*(BG, \mathbb{R})$ .

## 2.1 Cobordism

Suppose that  $(M, \partial M)$  is a manifold with boundary and  $T\partial M \rightarrow TM|_{\partial M}$  is an inclusion.

If  $TM$  is oriented, then  $\partial M$  becomes oriented by  $(\xi_1, \dots, \xi_{n-1})$  a frame at  $x \in \partial M$  is positively oriented iff  $(p, \xi_1, \dots, \xi_{n-1})$  is in  $T_x M$  where  $p$  is an outward pointing vector.

**Definition 2.1** (Oriented Cobordant). *Two compact orientable manifolds are oriented cobordant if  $\exists W$  a manifold with boundary such that  $\partial W = M_1 \amalg -M_2$ .*

Let  $\Omega_n$  = the oriented cobordism classes of manifolds. This is not just a set, but in fact an abelian group with operation  $M_1 + M_2 = M_1 \amalg M_2$  and  $\emptyset + M = 0$ . Also,  $M + -M = 0$ , because  $M \amalg -M = \partial M \times I$ .

Moreover,  $\Omega_* = \oplus \Omega_n$  is a graded commutative ring, via the multiplication  $M_1 \cdot M_2 = M_1 \times M_2$ , and with multiplicative identity the manifold which consists of a single point.

$\Omega_0 = \mathbb{Z}$ , because for zero dimensional manifolds, the only invariant is the sum of the signs given to the points.  $\Omega_1 = 0$ , because the only one is the circle which is the boundary of the disc.  $\Omega_2 = 0$ , as surfaces can be embedded as the boundary of open subsets of  $\mathbb{R}^3$ ,  $\Omega_3 = 0$  and then  $\Omega_4 \neq 0$ .

### 3 Lecture 3

So  $\Omega_n$  is the oriented cobordism. The pontrjagin class of  $E^k/\mathbb{R}$  is  $p_i(E) = (-1)^i c_{2i}(E \otimes_{\mathbb{R}} \mathbb{C})$ .  $c_k(\bar{E}/\mathbb{C}) = (-1)^k c_k(E)$ , so if  $E$  is a real vector bundle, then  $\overline{E \otimes_{\mathbb{R}} \mathbb{C}} \cong E \otimes_{\mathbb{R}} \mathbb{C}$ . Thus, odd chern classes are 2-torsion for  $E \otimes_{\mathbb{R}} \mathbb{C}$ .

$p(T\mathbb{C}P^n)$ :  $c(T\mathbb{C}P^n) = (1+x)^{n+1}$  when  $x \in H^2(\mathbb{C}P^n)$  is a generator.

**Remark 3.1.** *If  $E$  is a complex vector bundle, then  $E \otimes_{\mathbb{R}} \mathbb{C} = E \oplus \bar{E}$ .*

Thus,  $c(T\mathbb{C}P^n \otimes_{\mathbb{R}} \mathbb{C}) = c(T\mathbb{C}P^n \oplus \overline{T\mathbb{C}P^n}) = c(T\mathbb{C}P^n)c(\overline{T\mathbb{C}P^n}) = (1+x)^{n+1}(1-x)^{n+1} = (1-x^2)^{n+1} = \sum_{i=0}^{n+1} \binom{n+1}{i} (-1)^i x^{2i}$ . And so,  $p_i(T\mathbb{C}P^n) = \binom{n+1}{i} x^{2i}$ .

Let  $I$  be a partition of  $n$ ,  $\{(i_1, \dots, i_\ell) | i_1 + \dots + i_\ell = n\}$ .

Given  $I$  and a complex manifold  $X^n$ , then  $\langle c_{i_1}(TX)c_{i_2}(TX) \dots c_{i_\ell}(TX), [X] \rangle = c_I(X)$ .

Given an oriented manifold  $M^{4n}$ , we can form  $p_I(M) = \langle p_{i_1}(TM) \dots p_{i_\ell}(TM), [M] \rangle$  where  $i_1 + \dots + i_\ell = 4n$ . Then  $p_I(-M) = -p_I(M)$ .

**Corollary 3.1.** *If  $M^{4n}$  has some nonzero pontrjagin number, then it has no orientation reversing diffeomorphisms.*

**Theorem 3.2.** *If  $M = \partial W$  and  $W$  a compact oriented manifold with boundary, then all  $p_I(M) = 0$ .*

*Proof.* Let  $[W] \in H_{n+1}(W, M)$ . We have  $\partial : H_{n+1}(W, M) \rightarrow H_n(M)$  and  $\partial[W] = [M]$  and  $i : M \rightarrow W$  the inclusion.

Look at  $H^n(W) \xrightarrow{i^*} H^n(M) \xrightarrow{\delta} H^{n+1}(W, M)$  by  $(a, \partial b) = (\delta a, b)$ .

$TW|_M = TM \oplus 1$ . So  $p_k(TW|_M) = pk(TM \oplus 1) = p_k(TM)$ , and so  $p_I(M) = \langle p_{i_1}(TM) \dots p_{i_\ell}(TM), [M] \rangle$ , and this is  $\langle p_{i_1}(TW)|_M \dots p_{i_\ell}(TW)|_M, \partial[W] \rangle$ , which is  $\langle i^*(p_{i_1}(TW) \dots p_{i_\ell}(TW)), \partial[W] \rangle = \langle \delta i^*(p_{i_1}(TW) \dots p_{i_\ell}(TW)), [W] \rangle = 0$ .  $\square$

**Corollary 3.3.**  $\mathbb{C}P^{2n}$  are not boundaries of oriented manifold.

**Exercise 3.1.** Show  $\mathbb{C}P^{2n+1}$  is an oriented boundary.

For each partition  $I$  of  $n$ ,  $p_I : \Omega_{4n} \rightarrow \mathbb{Z}$

**Theorem 3.4** (Pontrjagin's Theorem).  $(\mathbb{C}\mathbb{P}^{2i_1} \times \dots \times \mathbb{C}\mathbb{P}^{2i_\ell})$  generate  $\Omega_{4n} \otimes \mathbb{Q}$ .

### 3.1 Thom Spaces

Let  $E^k/\mathbb{R} \rightarrow X$  be a real vector bundle. Put a metric on  $E$ . Let  $B(E) = \{e \mid \|e\| \leq 1\}$  and  $S(E) = \{e \mid \|e\| = 1\}$ . Then define  $T(E) = B(E)/S(E)$ .

If  $X$  is compact, this is isomorphic to  $E^+$ .

The image of  $S(E)$  is a point in  $T(E)$ , which we'll call  $+$ .

**Proposition 3.5.** *If  $E$  is oriented, then  $H^\ell(X) \cong H^{\ell+k}(T(E), +)$ .*

*Proof.* By excision,  $H^*(T(E), +) \cong H^*(B(E), S(E)) \cong H^{*+k}(X)$  by the Thom isomorphism.  $\square$

If  $X$  is a CW-complex, then  $T(E)$  has one cell of dimension  $n+k$  for each cell of dimension  $n$  in  $X$ , plus the 0-cell  $+$ .

$\pi : E \rightarrow X$ , then  $\pi^{-1}(e_\alpha^n) \cong e_\alpha^n \times \mathbb{R}^k$ , and so inside the Thom space, we have  $E_\alpha^n \times D^k$ , an  $n+k$ -cell.

The Thom space is NOT a manifold, but it is away from  $+$  (if  $X$  is).  $X \xrightarrow{i} T(E)$  as the zero section.

Let  $\gamma^k \rightarrow Gr_k^+(\mathbb{R}^\infty) = BGL^+(k, \mathbb{R})$ . Given  $M, TM$ , with  $M$  oriented, we get  $f : M \rightarrow Gr_k^+(\mathbb{R}^\infty)$ .

$f^*\gamma^k \cong TM$ . Form  $T(\gamma^k) = MU_k$ .

**Theorem 3.6** (Thom).  $\Omega_n \cong \lim_{k \rightarrow \infty} \pi_{n+k}(T(\gamma^k))$ .

*Proof.* Suppose that  $f_0 \simeq f_1 : M \rightarrow W \supset N$ . Let  $H$  be homotopy between  $f_0, f_1$ ,  $N$  a submanifold of codimension  $k$ . Then  $H : M \times I \rightarrow W$  by a small deformation, we can make the images transverse to  $N$ . If  $f_0, f_1$  transverse to  $N$ , then  $H$  can be made transverse to  $N$  with affecting  $H|_{\partial(M \times I)} = f_0 \amalg f_1$ . Then  $f_0^{-1}(N)$  and  $f_1^{-1}(N)$  are manifolds of codimension  $k$ .  $H^{-1}(N)$  is the a cobordism between  $f_0^{-1}(N)$  and  $f_1^{-1}(N)$ .  $\square$

## 4 Lecture 4

Let  $E^k/\mathbb{R} \xrightarrow{\pi} X$  be an oriented real vector bundle. We form  $T(E) = B(E)/S(E)$  with one  $r+k$  cell for each  $r$ -cell of  $X$  and an additional 0-cell  $+$ . If  $f : Y \rightarrow X$  and we have  $f^*E$  the pullback of  $E$  to  $Y$ , we get a map  $T(f) : (T(f^*E), +) \rightarrow (T(E), +)$ .

The Thom space is a manifold away from  $+$ .  $T(E \oplus 1_1) \cong \sum T(E)$ , the reduced suspension.

Do this to  $\gamma^k \rightarrow BSO(n) \rightarrow Gr_k(\mathbb{R}^\infty)$ , then  $\gamma_k \oplus 1_1$  is classified by a map  $i_k : BSO(k) \rightarrow BSO(k+1)$  with  $i_k^*\gamma^{k+1} = \gamma^k \oplus 1_1$ .

So we get a map  $T(i_k) : T(\gamma^k \oplus 1) \rightarrow T(\gamma^{k+1})$  which is  $\sum T(\gamma^k) \rightarrow T(\gamma^{k+1})$ .

If  $f : S^{n+k} \rightarrow T(\gamma^k)$ , then  $\sum(f) = \sum S^{n+k} \rightarrow \sum T(\gamma^k)$  then we can form  $T(i_k) \sum(f) : S^{n+k+1} \rightarrow T(\gamma^{k+1})$ .

Thus, we get a map  $\pi_{n+k}(T(\gamma^k)) \rightarrow \pi_{n+k+1}(T(\gamma^{k+1}))$

**Theorem 4.1** (Pontrjagin-Thom).  $\Omega_n \cong \lim_{k \rightarrow \infty} \pi_{n+k}(T(\gamma^k))$ .

A sequence of spaces  $M_k$  with maps  $\sum M_k \rightarrow M_{k+1}$  is called a prespectrum. Similarly  $M_k \rightarrow \Omega M_{k+1}$ .

**Definition 4.1** (Spectrum). If  $M_k \rightarrow \Omega M_{k+1}$  is a homeomorphism, then the collection is a homeomorphism.

A morphism between spectra  $M_n, N_n$ , are maps  $f_n : M_n \rightarrow N_n$  such that the square

commutes.

Spectra give extraordinary (co)homology theories.

Given a (pre)spectrum  $M_n$ , we define  $M^n(X) = \lim_{k \rightarrow \infty} g_{k \rightarrow \infty}[\sim^k X, M_{n+k}]$  and  $M_n(X) = \lim_{k \rightarrow \infty} \pi_{n+k}(X \wedge M_k)$ .

Geometric constructs of oriented cobordism as a homology: Set  $\Omega_n(X) = \{f : M^n \rightarrow X\}$ . Then  $f_1 : M_1 \rightarrow X$  and  $f_2 : M_2 \rightarrow X$ ,  $\partial W = M_1 \amalg -M_2$  and  $F|_{M_1} = f_1$  and  $F|_{M_2} = f_2$ .

Now let  $M^n$  be a closed, oriented  $n$ -manifold, and  $i : M^n \rightarrow \mathbb{R}^{n+k}$  an embedding.  $TM, \nu = \nu(i)$  gets an orientation by requiring  $TM \oplus \nu \cong T\mathbb{R}^{n+k}$ .

$\nu$  is classified by maps  $c_\nu : M \rightarrow BSO(k)$ . Let's identify  $B(\nu)$  with a tautological neighborhood of  $M, \Pi$ .

Define  $p(m, i) : S^{n+k} \rightarrow T(\gamma^k)$ .  $S^{n+k} = (\mathbb{R}^{n+k})^+ \rightarrow \Pi/\partial\Pi = B(\nu)/S(\nu)$ ,  $\xrightarrow{T(c_\nu)} T(\gamma^k)$ .

Let  $Z_k$  be the zero section of  $\gamma^k$  sitting inside the Thom space.

$M = p(M, i)^{-1}(Z_k)$ .

Claim: If you embed  $\mathbb{R}^{n+k} \rightarrow \mathbb{R}^{k+1}$ , then  $S^{n+k} \xrightarrow{p(M, i)} T(\gamma^k)$ ,  $S^{n+k+1} \xrightarrow{p(M, i \times \{x_0\})} T(\gamma^{k+1})$ ,  $p(M, i \times \{0\}) = Sp(M, i)$ .

**Lemma 4.2.** Let  $j : M + \partial W \rightarrow \mathbb{R}^{n+k}$  be an embedding. Then if  $k$  is big enough, then  $p(M + \partial W, j) = p(M, i)$ .

*Proof.* Sketch.

One embeds  $M \times I + W \rightarrow \mathbb{R}^{n+k} \times I \subset \mathbb{R}^{n+k+1}$  in such a way.  $M = \phi^{-1}(\mathbb{R}^{n+k} \times \{0\}) \rightarrow \mathbb{R}^{n+k} \times \{0\}$  is  $i$  and  $M + \partial W = \phi^{-1}(\mathbb{R}^{n+k} \times \{1\}) \rightarrow \mathbb{R}^{n+k} \times \{1\}$  is  $j$ .

Apply  $p$  construction. Define  $p_t : S^{n+k} \rightarrow T(\gamma^k)$ .  $p_t(M \times I \times W, \phi_t)$ , with  $\phi_t = \phi|_{\phi^{-1}(\mathbb{R}^{n+k} \times \{t\})}$ .  $\square$

**Proposition 4.3.**  $p : \Omega_n \rightarrow \lim_{k \rightarrow \infty} \pi_{n+k}(T(\gamma^k), +)$  is well defined.

*Proof.* By the claim, I can take  $k$  as big as a want. Apply the lemma with  $W = \emptyset$ .  $\square$

**Proposition 4.4.**  $p$  is a homomorphism of groups.

**Proposition 4.5.** *p is surjective.*

*Proof.* Let  $f : (S^{n+k}, +) \rightarrow (T(\gamma^k), +)$  represent an element in  $\pi_{n+k}(T(\gamma^k), +)$ .

Homotope  $f$  to  $g$  where  $g$  satisfies  $g : S^{n+k} \rightarrow T(\gamma^k \rightarrow Gr(\mathbb{R}^{k+n}))$ ,  $g$  is differentiable outside  $+$  and  $g$  is transverse to  $Z_k$  in  $T(\gamma^k)$ .

Then set  $M = g^{-1}(Z_k)$ . This is an  $n$ -manifold and running through the PT-construction, we get back  $g$ .  $\square$

Injectivity is less interesting, so we will not prove it.

**Exercise 4.1.** *Let  $X$  be a  $k$ -connected CW-complex. Then  $\pi_i(X) \otimes \mathbb{Q} \cong H_i(X; \mathbb{Q})$  for  $i \leq 2k$ .*

$T(\gamma^k)$  = cells of  $k + \ell$  for each cell of dimension  $\ell$  in  $BSO(k)$ .

$\lim_{k \rightarrow \infty} \pi_{k+n}(T(\gamma^k)) \otimes \mathbb{Q} \cong H_{k+n}(T(\gamma^k); \mathbb{Q})$  for  $k$  big enough. Then  $\lim H_{k+n}(T(\gamma^k)l\mathbb{Q}) \cong \lim H_n(BSO(k), \mathbb{Q})$ , which is  $\lim H^n(BSO(k), \mathbb{Q}) = \mathbb{Q}[p_1, \dots]$  with  $p_i$  the Pontrjagin class.

$\Omega_{4n} \otimes \mathbb{Q}$  is then the number of partitions of  $n$ , and  $\Omega_k \otimes \mathbb{Q}$  is 0 for  $4 \nmid k$ .

$MU_k$  = bordism of manifolds whose tangent bundle has stable almost complex structure.

$MU_* = \mathbb{Z}[c_1, c_2, \dots]$  where  $c_i$  is the chern class in degree  $2i$ .

Then  $MU_*$  has the structure of a 1-dimensional formal group.

## 5 Lecture 5

Today will be rather informal.

Pontrjagin Duality takes  $G$  a locally compact abelian group and defines  $\hat{G} = \text{hom}(G, U(1))$  continuous maps. Then  $\hat{\hat{Z}} = U(1)$  and  $\hat{\mathbb{R}} = \mathbb{R}$ . This gives  $H^k(M, \mathbb{Z})^* \cong H^{n-k}(M, U(1))$ .

### 5.1 K-Theory and the Index Theorem

Poincare duality is teh combination of two theorems:

First is the Duality of Chomology and Homology

Ex. Alexander Duality:  $H^k(X) \cong H_{n+1-k}(S \setminus X)$  use this as definition of a homology theory corresponding to a cohomology theory.

An extraordinary chomology theory is a cofunctor  $E : \text{Pairs} \rightarrow \mathbb{Z}$ -graded abelian groups and satisfies homotopy invariance, long exact sequence of pairs and excision. (If you add the normalization that  $E^0(pt) = \mathbb{Z}$  and  $E^n(pt) = 0$  for all other  $n \in \mathbb{Z}$  on the category of CW-complexes, this singles out  $H^*$ )

We call complexes  $X$  and  $Y$  dual if  $f : X, Y \rightarrow S^n$  for  $n$  large with  $X \cap Y = \emptyset$  then  $Y$  is a deformation retract of  $S^n \setminus X$ .

Define a map  $X \times Y \rightarrow S^n$  by: remove a point of  $S^n$ , WLOG it is  $\infty$ . Then  $\infty \notin X, Y$ , and so define  $\delta(x, y) = \frac{x-y}{|x-y|}$ . This gives a map  $X \rightarrow \text{Maps}(Y, S^n)$ .

By analogy, in rings  $R$ , we define  $M^* = \text{hom}_R(M, R)$  the dual of  $M$ .

**Theorem 5.1** (Atiyah). *If  $M$  is a manifold and  $\nu$  the normal bundle of  $M$  in some embedding, then  $T(\nu)$  is dual to  $M$ . So for any extraordinary cohomology theory  $E$ ,  $E^*T(\nu) \cong E_*(M)$ .*

The Thom isomorphism theorem tells me when  $E^*(T(\nu)) \cong E^*(M)$ . We need a notion of orientable for a bundle relative to  $E^*$ .

I am going to approach the index theorem from the point of view that it is Poincare Duality for  $K$ -theory. There will therefore be two aspects to the proof.

1. Show that  $H^*(T^*X) \rightarrow K_*(X)$  is an isomorphism (operator theoretic)
2. Then isomorphism saying that when  $K^*(TX) \cong K^*(X)$ , the thom isomorphism in  $K$ -theory is Bott Periodicity.

#### Index Theorem

Look at  $M$  a compact manifold and let  $E, F$  be smooth complex vector bundles. Then  $D : C^\infty(M; E) \rightarrow C^\infty(M; F)$  a differential operator. In local coordinates (and trivialization),  $Ds = \sum_\alpha a_\alpha(x) \frac{\partial s}{\partial x^\alpha}$ .

We define  $\sup_{a_\alpha \neq 0} |a_\alpha|$  to be the degree or order of the operator.

#### Space of solutions

Dimension of space of solutions, that is,  $\ker D$ , is not invariant under perturbations. But the index is.  $ind(D) = index(D) \equiv \dim \ker D - \dim \operatorname{coker} D$ . So we need  $D$  to be Fredholm.

How to express of  $D$  in terms of data from  $D$  and from  $M$ .

**Example 5.1.**  $\Delta$ , the Laplacian extended to forms. Gives us a map  $\Omega^k \rightarrow \Omega^k$ , that is,  $C^\infty(M \wedge^k T^*) \rightarrow C^\infty(M; \wedge^k T^*)$ . The index is zero by self-adjointness.

**Example 5.2.**  $D = d + d^*$ , where  $d^* = *d*$  and  $*$  is the Hodge star. Then the index is  $\sum (-1)^i b_i$ , where  $b_i$  are the Betti numbers, and so the Atiyah-Singer Index Theorem becomes the Gauss-Bonnet-Chern Theorem.

**Example 5.3.** If  $X$  is a complex compact manifold and  $E$  is a holomorphic vector bundle. Let  $(\bar{\partial})_E : \Omega^{0,k}(X; E) \rightarrow \Omega^{0,k+1}(X; E)$ . Then  $D = \bar{\partial}_E + \bar{\partial}_E^*$ . This takes  $\Omega^+ \rightarrow \Omega^-$ . The index is then  $\sum (-1)^i \dim H^i(X; E)$ .

**Theorem 5.2** (Index Theorem). *If  $M$  is compact and  $D : C^\infty(M, E) \rightarrow C^\infty(M, F)$  is an elliptic operator, then  $ind(D) = (-1)^n \langle ch(\sigma(D)) + d(TM \otimes \mathbb{C}), [M] \rangle$  where  $\sigma(D)$  is a vector bundle you construct out of  $D$ .*

In the case of the signature theorem  $sign(M) = \langle L(M), [M] \rangle$  where  $L(M) = \prod_{i=1}^n \frac{x_i}{\tanh x_i}$  where  $x_i$  are the chern roots of  $TM \otimes \mathbb{C}$  is a polynomial in the pontrjagin classes

So  $Sign(M)$  is a homotopy invariant of  $M$ .  $p_i(TM) \in H^{4i}(M; \mathbb{Q})$  is NOT homotopy invariant. It is a big theorem of Novikov that  $p_i(TM) \in H^{4i}(M; \mathbb{Q})$  are homeomorphism invariants. It's a surprising fact that  $L(M) = \langle \phi(p_i)(TM), [M] \rangle$  is a homotopy invariant.

Browder and Novikov prove that  $L(M)$  is the only relation among pontrjagin classes which is homotopy invariant.

Let  $M$  be an abelian monoid. Define  $K_0M = \{(m, n) \in M \times M\} / \sim$  by  $(m_1, n_1) \sim (m_2, n_2)$  if and only if there exists  $r \in M$  such that  $m_1 + n_2 + r = m_2 + n_1 + r$ . Then  $K_0M$  is an abelian group, and it is universal in that any map from  $M$  to an abelian group factors through it.

$K_0(\mathbb{N}) = \mathbb{Z}$  and  $K_0(\{0, 1, 2, \text{many}\}) = 0$ .

If  $A$  is a ring, then  $K_0(A) = K_0$  group of the abelian monoid  $\mathcal{P}$ , where  $\mathcal{P}$  is the collection of isomorphism classes of finitely generated projective left  $A$ -modules under direct sum.

Let  $X$  be a compact space, define  $C(X)$  to be the continuous complex valued functions. Then  $K^0(X) \equiv K_0(C(X))$

References:

Atiyah and Singler's original articles.

## 6 Lecture 6

**Theorem 6.1.** *Given  $\pi_i : E_i \rightarrow X$  for  $i = 1, 2$  two vector bundles,  $U_\alpha$  a cover of  $X$  by open sets (if  $X$  is parcompact,  $U_\alpha$  is locally finite cover byh closed subsets) suppose that  $\phi_\alpha : E_1|_{U_\alpha} \rightarrow E_2|_{U_\alpha}$  such that  $\phi_\alpha|_{U_{\alpha\beta}} = \phi_\beta|_{U_{\alpha\beta}}$ . Then there exists a unique  $\phi : E_1 \rightarrow E_2$  a homomorphism such that  $\phi|_{U_\alpha} = \phi_\alpha$ .*

**Theorem 6.2.**  *$X, U_\alpha$  same as above.  $\pi_\alpha : E_\alpha \rightarrow U_\alpha$  vector bundles such that there exist  $g_{\alpha\beta} : E_\beta|_{U_{\alpha\beta}} \rightarrow E_\alpha|_{U_{\alpha\beta}}$  isomorphisms so that  $g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma}$ . Then there exists  $E$  on  $X$  such that  $E|_{U_\alpha} \cong E_\alpha$ .*

**Corollary 6.3.**  $Vect_n(X) \cong \check{H}^1(X, GL_n)$

**Corollary 6.4.**  $Vect_n(\sum X) = [X, GL_n]$ .

**Proposition 6.5.** *Let  $X$  be paracompact and  $\pi : E \rightarrow X$  be a vector bundle. Let  $Y \subseteq X$  closed. Then  $C(X; E) \rightarrow C(Y; E|_Y)$  is surjective. (ie, sections are a soft sheaf).*

**Definition 6.1** (Soft Sheaf). *Like flasque, but with restriction to closed subsets surjective.*

**Proposition 6.6.** *Let  $X$  be paracompact and  $0 \rightarrow E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow 0$  be a short exact sequence of vector bundles. Then  $E_2 \cong E_1 \oplus E_3$ .*

**Proposition 6.7.** *Let  $\pi : E \rightarrow X$  be a vector bundle and  $p \in C(X; \text{End}E)$ ,  $p_x^2 = p_x$  for all  $x \in X$ . Then  $\ker p \subseteq E$  is a vector bundle.*

*Proof.* Question is local, so assume  $E = X \times \mathbb{C}^n$ . Fix  $x_0 \in X$ ,  $p : X \rightarrow M_n(\mathbb{C})$ . Let  $\phi(x) = 1 - p_x - p_{x_0} + 2p_x p_{x_0}$ . Then  $p_x \phi(x) = p_x(1 - p_x - p_{x_0} + 2p_x p_{x_0}) = p_x - p_x - p_x p_{x_0} + 2p_x p_{x_0} = p_x p_{x_0} = \phi(x) p_{x_0}$ .

We have the following commutative diagram:

$$\phi_{x_0} = 1 - p_{x_0} - p_{x_0} + 2p_{x_0} p_{x_0} = 1$$

So there exists a neighborhood  $U$  of  $x_0$  whre  $\phi_x$  is invertible for all  $x \in U$ . So  $\phi$  is giving an isomorphism of  $\ker p$  with the trivial vector bundle over  $U$ .  $\square$

**Theorem 6.8.** *Let  $\pi : E \rightarrow X$  be a vector bundle with  $X$  compact. Then there exists  $F \rightarrow X$  a vb such that  $E \oplus F = X \times \mathbb{C}^N$ .*

*Proof.* Let  $U_\alpha$  be a finite trivializing cover of  $E$ . Then  $s_\alpha^1, \dots, s_\alpha^n$  be a frame for  $E|_{U_\alpha}$ . Let  $\lambda_\alpha$  be a partition of unity. Then  $\lambda_\alpha s_\alpha^i$  extends to a section of  $E$ . So  $\{\lambda_\alpha s_\alpha^i\}$  span the fiber of  $E$  at every point.

Define  $\phi : X \times \mathbb{C}^N \rightarrow E$  by  $\phi(x, z_1^\alpha, \dots, z_n^\alpha) = \sum_{\alpha, i} z_i^\alpha \lambda_\alpha(x) s_\alpha^i(x)$ . This defines a surjective homomorphism, so we can find  $\psi : E \rightarrow X \times \mathbb{C}^N$  a section and  $p = \psi\phi$  is idempotent. Define  $F = \ker p$ . This is a vector bundle with the desired property.  $\square$

**Theorem 6.9** (Serre-Swan). *Let  $X$  be compact and let  $Vect(X)$  be the category of finite dimensional vector bundles on  $X$ . Let  $P(C(X))$  be the category of finitely generated projective modules over  $C(X)$ . Then  $E \mapsto C(X; E)$  is an equivalence of categories.*

*Proof.* Note that  $C(X; X \times \mathbb{C}^N) \cong C(X)^N$  is a finitely generated free  $C(X)$  module, and so  $C(X; E \oplus F) = C(X; E) \oplus C(X; F)$ , so these are all projective and finitely generated.

Now given  $P$  a finitely generated projective module, we know that there exists  $Q$  such that  $P \oplus Q = C(X)^N$ . Define  $p \in M_N(C(X))$  to be projection onto  $P$ . Then  $C(X)^N = P \oplus Q \rightarrow P \oplus Q \cong C(X)^N$  is  $p$ , and so  $p^2 = p$ . Take  $E$  to be the image of  $p$  which is  $\ker(1 - p)$ . Then  $E$  is a vector bundle.  $\square$

Let  $P_n(\mathbb{C}^N)$  be the space of idempotent  $N \times N$  matrices of rank  $n$ . Then  $Gr_n(\mathbb{C}^N) \subset P_n(\mathbb{C}^N)$ , where the first is the self-adjoint idempotents.

**Theorem 6.10.**  *$Gr_n(\mathbb{C}^N)$  is a deformation retract of  $P_n(\mathbb{C}^N)$ .*

*Proof.* Define  $H : P_n(\mathbb{C}^N) \times I \rightarrow P_n(\mathbb{C}^N)$  by  $H(p, t) = g(t)pg(t)^{-1}$  where  $g(t) = \sqrt{1 + t(2p - 1)^*(2p - 1)}$ . Then  $H(p, 0) = p$ ,  $H(p, t) = p$  for all  $t$  if  $p^* = p$  and  $g(1)^2 p = (1 + (2p - 1)^*(2p - 1))p = 2p^*p$ . Also  $p^*g(1)^2 = 2p^*p$ . So conjugating gives  $p^*$ . Thus,  $H(p, 1) = g(1)^{-1}p^*g(1)$ , and  $H(p, 1)^* = g(1)pg(1)^{-1} = H(p, 1)$ .  $\square$

**Corollary 6.11.**  *$Vect_n(X)$  can be described in the following equivalent ways for compact  $X$ :*

1.  $[X, P_n(\mathbb{C}^N)]$  for  $N$  large
2.  $[X, Gr_n(\mathbb{C}^N)]$  for  $N$  large
3.  $\pi_0(P_n(M_N(C(X))))$
4.  $\pi_0$  of the self-adjoint idempotents of  $M_N(C(X))$  for  $N$  large.

For a space  $X$  compact  $Vect(X)$  is the isomorphism classes of vector bundles. This is a semigroup under  $\oplus$ . Define  $K^0(X)$  the Grothendieck group of  $Vect(X)$  to be  $K^0(C(X))$ . An element of  $K^0(X)$  can be written as  $[E] - [F]$ . Find  $G$  such that  $F \oplus G = 1_N$ , then  $[E] - [F] = [E \oplus G] - [F \oplus G]$  which is  $[E \oplus G] - [1_N]$ .

So every class in  $K^0(X)$  can be written as  $[E'] - [1_N]$ . Also,  $[E] = [F]$  in  $K^0(X)$  iff there exists  $N$  with  $E \oplus 1_N \cong F \oplus 1_N$  that is,  $E, F$  are stably isomorphic.

For a based space, define  $K^0(X, x_0) = \ker K^0(X) \rightarrow K^0(x_0) = \mathbb{Z}$ .  $K^0(X)$  is functorial for continuous maps between compact spaces, so  $K^0(X, A) \cong K^0(X/A, A/A)$ . So  $K^0(X) \cong K^0([X, \coprod_n Gr_n(\mathbb{C}^\infty)]) \cong [X, \mathbb{Z} \times BGL_\infty(\mathbb{C})]$ .

For  $X$  locally compact, define  $K^0(X) = K^0(X^+, +)$ .

## 7 Lecture 7

$K^0(X)$  and  $K^0(X, A) \cong K$ ,  $\tilde{K}^0(X) = \text{coker } K^0(pt) \rightarrow K^0(X)$ .

This is then isomorphic to  $K^0(X, x_0)$  for  $x \in X$ , which is  $\ker K^0(X) \rightarrow K^0(x_0)$ .

$$K^0(X) \cong \tilde{K}^0(X) \oplus \mathbb{Z}.$$

$$K^0(X) \cong [X; \mathbb{Z} | \text{times } BGL_\infty(\mathbb{C})].$$

Another description of  $K^0(X, A)$ :

Let  $X$  be compact and  $A \subset X$  closed. Consider triples  $(E, F, \alpha)$  where  $E, F \rightarrow X$  are vector bundles and  $\alpha : E|_A \rightarrow F|_A$  is an isomorphism. Then take  $\Gamma(X, A)$  the set of triples, and a morphism of triples is  $\phi : E_1 \rightarrow E_2, \psi : F_1 \rightarrow F_2$  asuch that  $\psi \circ \alpha_1 = \alpha_2 \circ \phi$  on  $A$ . An isomorphism is an isomorphism in this category.

A triple  $(E, E, \alpha)$  is called degenerate if  $\alpha \in \text{Aut}(E|_A)$  is homotopic through automorphisms to the identity.

Define addition to be taking direct sums in each component of the triple.

**Definition 7.1.**  $'K^0(X, A) = \Gamma(X, A) / \sim$  where  $e_1 \sim e_2$  if there are degenerate triples  $d_1, d_2$  such that  $e_1 + d_1 \cong e_2 + d_2$ .

**Proposition 7.1.** 1.  $'K^0(X, A)$  is an abelian group.

2. If  $A = \emptyset$ , then  $'K^0(X, A) \cong K^0(X, A)$ , and if  $A = x_0$ , then  $'K^0(X, x_0) \cong \tilde{K}^0(X)$ .

3.  $[(E, F, \alpha)] + [(F, E, \alpha^{-1})] = 0$ .

*Proof.* The sum is  $[(E \oplus F, E \oplus F, \begin{pmatrix} 0 & -\alpha^{-1} \\ \alpha & 0 \end{pmatrix})]$ . It is simple to see that this is homotopic to the identity through automorphisms.  $\square$

**Proposition 7.2.** If  $(E, F, \alpha_i)$  for  $i = 1, 2$  are two triples with  $\alpha_1 \simeq \alpha_2$  through  $\text{Iso}(E|_A, F|_A)$ , then  $[(E, F, \alpha_1)] = [(E, F, \alpha_2)]$ .

**Proposition 7.3.** If  $(E, F, \alpha)$  and  $(F, G, \beta)$  are two triples, then  $[(E, F, \alpha)] + [(F, G, \beta)] = [(F, G, \beta\alpha)]$ .

**Theorem 7.4.**  $'K^0(X, A) \cong K^0(X, A) \cong K^0(X/A, A/A) \cong 'K^0(X/A, A/A)$  via the map  $'K^0(X/A, A/A) \rightarrow 'K^0(X, A)$  by pullback.

*Proof.* We will only show surjectivity. Let  $(E, F, \alpha) \in \Gamma(X, A)$ . Assume, WLOG,  $F = 1_N$ .  $\alpha : E|_A \rightarrow F|_A$  an isomorphism. Let  $\bar{U}$  be a closed neighborhood of  $A$  and extend  $\alpha$  to an automorphism of  $\bar{U}$ .

Glue  $E|_{X \setminus A}$  to the trivial vector bundle  $1$  of rank  $n$  over  $\bar{U} \setminus A$ .  $X \setminus A \cong (X/A) \setminus (A/A)$  and so  $\bar{U} \setminus A \cong \bar{U}/A - A/A$ . So we get a vector bundle  $E'$  on  $X/A$  and  $E' \rightarrow 1_n$  over  $X/A$  which is the gluing induced by  $\alpha$ .  $\square$

Now we want to define  $K^{-1}(X)$ . It should satisfy  $K^{-1}(X) \rightarrow K^{-1}(A) \rightarrow K^0(X, A) \rightarrow K^0(X) \rightarrow K^0(A)$ . etcetera.

So it should parameterize the number of ways that different elements of  $K^0(X, A)$  can go to the same thing in  $K^0(X)$ .

Define  $\Gamma^{-1}(X)$  to be pairs  $(E, \alpha)$  with  $\alpha \in \text{Aut}(E)$  with direct sum. Then a degenerate pair is  $(E, \alpha)$  with  $\alpha$  homotopic to  $1$  in  $\text{Aut}(E)$ . Then  $K^{-1}(X) = \Gamma^{-1}(X)/\sim$  with  $\sim$  as for  $\Gamma(X, A)$  before.

**Theorem 7.5.**  $K^{-1}(X) \cong [X, GL_\infty(\mathbb{C})]$

*Proof.* By the same proof as in  $K^0$  case, if  $\alpha_1 \simeq \alpha_2$ , then  $(E, \alpha_1) \sim (E, \alpha_2)$ . But now because  $(E, \alpha) \sim (E \oplus F, \alpha \oplus 1_F)$ , we can assume that  $E = 1_n$ . Then  $\alpha$  is just an automorphism of the trivial bundle, and so we have  $X \rightarrow GL_n(\mathbb{C})$ . But the addition tells us that we're going to get the same element.  $\square$

**Corollary 7.6.**  $K^{-1}(X) \cong \tilde{K}^0(\sum X)$ .

In general,  $K^{-n} \equiv \tilde{K}^0(\sum^n X)$ .

**Theorem 7.7** (Bott Periodicity).  $K^{-2}(X) \cong K^0(X)$  canonically.

$\Omega(\mathbb{Z} \times BGL_\infty(\mathbb{C})) = GL_\infty(\mathbb{C})$ ,  $\Omega GL_\infty \mathbb{C} \cong \mathbb{Z} \times BGL_\infty \mathbb{C}$ .

In the real case,  $K_{\mathbb{R}}^{-8}(X) \cong K_{\mathbb{R}}^0(X)$ .

## 8 Lecture 8

Let  $G$  be a locally compact group. Then  $H_G^*(X)$ , the equivariant cohomology of a  $G$ -space, is obtained by  $H^*(EG \times X/G)$ . We want that, if  $G$  acts freely and properly, then  $H_G^*(X) = H^*(X/G)$ , and for it to be a module over  $H_G^*(pt) = H^*(BG)$ .

Let  $G$  be compact, then  $K_G^0(X)$  is the Grothendieck group of  $G$ -equivariant vector bundles.

If  $x \in X$  is a fixed point for  $G$ , then  $G : E_x \rightarrow E_x$  is a representation.

### 8.1 Operator Theory

Let  $V_1, V_2$  be Banach spaces.

**Theorem 8.1.** *TFAE*

1.  $A : V_1 \rightarrow V_2$  is cont

2.  $A : V_1 \rightarrow V_2$  is bounded (ie,  $\|Av\| \leq k\|v\|$ )
3.  $\ker A$  is closed.

Denote by  $L(V_1, V_2)$  the bounded linear operators  $V_1 \rightarrow V_2$ .

**Lemma 8.2.** *Let  $A \in L(V_1, V_2)$  be such that the range of  $A$  has finite codimension. Then  $A(V_1)$  is closed in  $V_2$ .*

*Proof.* Why, we can assume  $A$  is injective (otherwise take  $V_1/\ker A$ ) and so as  $A(V_1)$  has finite codimension, there exists  $W \subset V_2$  finite dimensional such that  $A(V_1) \oplus W = V_2$ . Define  $V_1 \oplus W \rightarrow V_2$  by  $(v, w) \mapsto Av + w$ . This is continuous and bijective, and so is an isomorphism by the Open Mapping Theorem.  $\square$

**Definition 8.1** (Fredholm).  $A : V_1 \rightarrow V_2$  is Fredholm if  $\ker A$  and  $\operatorname{coker} A$  are finite dimensional.

This implies that the range of  $A$  is closed by the last lemma.

**Lemma 8.3.** *TFAE*

1.  $\dim \ker A < \infty$  and range  $A$  is closed.
2. Every sequence  $v_i$  with  $Av_i$  convergent is bounded and has a convergent subsequence.

*Proof.* If  $\|v_i\| \leq 1$ ,  $v_i \in \ker A$ , then  $Av_i = 0$  and so is convergent. So assuming 2, we find that there exists a convergent subsequence of  $v_i$ . Thus, the unit ball in  $\ker A$  is compact, so  $\ker A$  is finite dimensional. We will proceed by proving that there exists  $W \subset V_1$  such that  $W \oplus \ker A \cong V_1$  and  $\|v\| \leq k\|Av\|$  for all  $v \in W$ .

We just proved that 2 implies  $\ker A$  is finite dimensional. So there exists  $W$  such that  $V_1 = \ker A \oplus W$ . If the property were not satisfied, then you'd have  $v_i \in W$ ,  $\|v_i\| = 1$  and  $\|Av_i\| < 1/i$ , and so  $Av_i \rightarrow 0$ . Thus by 2, we have  $v_i$  converges to  $w \in W$  with  $\|w\| = 1$ , and  $W$  is in the complement of  $\ker A$ .

For 1 implies 2: exercise.  $\square$

**Theorem 8.4.** *If  $A$  satisfies the condition in the lemma,  $S \in L(V_1, V_2)$  is small enough, then  $A + S$  satisfies the condition, that is,  $\dim \ker(A + S) \leq \dim \ker A$ ,  $A + S$  has closed range, and  $\operatorname{Ind}(A + S) = \operatorname{Ind}(A)$ .*

*Proof.* Assume that  $A$  is bijective. Then  $A + S = A(1 + A^{-1}S)$ .  $1 + A^{-1}S$  is invertible in  $\|A^{-1}S\| < 1$ , and  $\|A^{-1}S\| \leq \|A^{-1}\|\|S\|$  if  $\|S\|$  is small enough. The general is handled by fiddling with  $\ker$  and  $\operatorname{coker}$  to get a bijective operator.  $\square$

**Theorem 8.5.** *Let  $\mathcal{F}(V_1, V_2) \subset L(V_1, V_2)$  be the Fredholm operators. Then  $\mathcal{F}(V_1, V_2)$  is an open subset of  $L(V_1, V_2)$ , and  $\operatorname{Ind} : \mathcal{F}(V_1, V_2) \rightarrow \mathbb{Z}$  is locally constant.*

**Corollary 8.6.**  $\operatorname{Ind}(A_2A_1) = \operatorname{Ind}(A_1) + \operatorname{Ind}(A_2)$ .

*Proof.*  $A_1 : \ker A_2 A_1 \rightarrow \ker A_2$ ,  $\dim \ker A_2 A_1 \leq \dim \ker A_1 + \dim \ker A_2$ .  $\dim \operatorname{coker} A_2 A_1 \leq \dim \operatorname{coker} A_1 + \dim \operatorname{coker} A_2$ . So  $A_2 A_1$  is still Fredholm.

Then some matrix multiplication happened.  $\square$

**Theorem 8.7.** *If  $A \in \mathcal{F}$ ,  $K$  is compact, then  $A + K \in \mathcal{F}$  and  $\operatorname{Ind}(A + K) = \operatorname{Ind}(A)$ .*

*Additionally, we have Atkinson's Theorem, which says that  $A$  is Fredholm if and only if there exists  $B$  such that  $AB = 1 - K_1$  and  $BA = 1 - K_2$ .*

Let  $H = L^2(S^1) \cong \ell^2(\mathbb{Z})$ . Consider  $H_+$ , the  $L^2$  boundary values of holomorphic plane functions on  $\mathbb{D}$ .  $H_+$  is the Hardy space contained in  $L^2(S^1)$ , and is isomorphic to  $\ell^2(\mathbb{N})$ .

For  $f \in C(S^1)$  the complex valued continuous functions, define  $T_f : H_+ \rightarrow H_+$  by  $T_f \xi = P(f\xi)$  where  $P$  is the projection  $H \rightarrow H_+$ . Call this a Töplitz Operator.

**Theorem 8.8.** *For  $f \in C(S^1)$ ,  $T_f$  is Fredholm iff  $f$  is invertible. Then  $\operatorname{Ind}(T_f) = -w(f)$  where  $w(f)$  is the winding number of  $f$ .  $f$  is then called the symbol of  $T_f$ .*

Let  $\mathcal{T} \subset L(H^+, H^+)$  generated by  $T_{z^n}$  for all  $n$  and compact operators. Then there exists a sequence  $0 \rightarrow K(H_+) \rightarrow \mathcal{T} \xrightarrow{\sigma} C(S^1) \rightarrow 0$ .

From this sequence, it's clear that  $T \in \mathcal{T}$  is Fredholm if and only if  $\sigma T$  is invertible.

The fact about the index follows from every  $f \in C(S^1)^\times$  is homotopic to  $z^n$  for some  $n$ , and  $\operatorname{Ind}$  is locally constant.

## 9 Lecture 9

Let  $V_1, V_2$  be Banach Spaces and  $L(V_1, V_2)$  bounded linear operators. Then  $\mathcal{F}(V_1, V_2)$  the Fredholm Operators.

Now let  $H$  be a separable Hilbert Space. Define  $L(H) = L(H, H)$  and  $\mathcal{F} = \mathcal{F}(H) = \mathcal{F}(H, H)$ . Then  $\mathcal{F} \times \mathcal{F} \rightarrow \mathcal{F}$  by composition, and  $\operatorname{Ind} : \mathcal{F} \rightarrow \mathbb{Z}$  satisfies  $\operatorname{Ind}(AB) = \operatorname{Ind}(A) + \operatorname{Ind}(B)$ .

Define  $[X, \mathcal{F}]$  to be the homotopy classes of maps. It is an abelian semigroup.

Then  $\operatorname{ind} : [X, \mathcal{F}] \rightarrow \check{H}^0(X, \mathbb{Z})$ .

**Theorem 9.1.** *For any compact  $X$ , there exists a homomorphism  $\operatorname{Ind} : [X, \mathcal{F}] \rightarrow K^0(X)$  which is an isomorphism.*

**Lemma 9.2.** *Let  $T \in \mathcal{F}$  and  $V \subseteq H$  a closed subspace of finite codimension such that  $\ker T \cap V = 0$ . Then there exists a neighborhood  $U$  of  $T$  in  $L(H)$  such that for all  $S \in U$*

1.  $V \cap \ker S = 0$
2.  $\cup_{S \in U} H/S(V)$  topologized as a quotient of  $U \times H$  is a trivial vector bundle.

*Proof.* Let  $W = T(V)^\perp$ . Since  $T \in \mathcal{F}$ ,  $W$  is finite dimensional and  $\dim H/V < \infty$ . For  $S \in L(H)$ , define  $\phi_S : V \oplus W \rightarrow H$  by  $\phi_S(v, w) = S(v) + w$ . Then  $S \mapsto \phi_S$  is a continuous map  $L(H) \rightarrow L(\text{oplus } W, H)$  and  $\phi_T(v, w) = T(v) + w$  is an isomorphism.

So there exists a neighborhood  $U$  of  $T$  with  $\phi_S$  being an iso for all  $S \in U$ . This gives the first part.

$\cup_{S \in U} H/S(V) \rightarrow U \times W$  by  $h + S(V) \mapsto (S, p_W \phi_S^{-1}(h))$  is a trivialization.  $\square$

**Proposition 9.3.**  *$X$  compact. Let  $T : X \rightarrow \mathcal{F}$  be a continuous map. Then there exists  $V \subseteq H$  closed and finite codimension such that*

1.  $V \cap T_x = 0$  for all  $x$
2.  $\cup_{x \in X} H/T_x(V) \leftarrow X \times H$  is surjective and a vector bundle (infinite dimensional and trivial)

*Proof.* For each  $x \in X$ , let  $V_x = (\ker T_x)^\perp$ . Let  $U_x$  be a neighborhood in  $\mathcal{F}$  of  $T_x$  guaranteed by the previous lemma. Consider  $T^{-1}(U_x)$ . This forms an open cover of  $X$ . Choose a finite subcover  $T^{-1}(U_{x_i})$ . Let  $V = \cap V_{x_i}$ . Then  $V \cap \ker T_x = 0$  for all  $x \in X$ .

$\cup_{y \in T^{-1}(U_{x_i})} H/T_y(V) = \cup_{s \in U} H/S(V_x)|_{T^{-1}(U_{x_i})}$ .  $\square$

Denote  $\cup_{x \in X} H/T(V_x) \equiv H/T(V)$  the vector bundle.

Now we define  $Ind : [X, \mathcal{F}] \rightarrow K^0(X)$ . Set  $Ind(T) \equiv [X \times (H/V)] - [H/T(V)]$  where  $V$  is the finite codimensional subspace generated by the proposition.

**Theorem 9.4.** *This is well defined.*

*Proof.* Just need to check independence of  $V$ . Let  $W$  be another such space. Then  $W \cap V \subset W$  is a third such space, so we can assume that  $W \subset V$ .

Now we have an exact sequence of vector bundles  $0 \rightarrow X \times V/W \rightarrow X \times H/W \rightarrow X \times H/V \rightarrow 0$  and  $0 \rightarrow X \times V/W \rightarrow H/T(W) \rightarrow H/T(V) \rightarrow 0$ . Thus,  $[X \times H/V] - [X \times H/W] = [X \times V/W]$  and  $[H/T(V)] - [H/T(W)] = [X \times V/W]$ . Thus,  $[X \times H/V] - [H/T(V)] = [X \times H/W] - [H/T(W)]$ .  $\square$

Fact:  $Ind$  is functorial.  $f : Y \rightarrow X$ ,  $T : X \rightarrow \mathcal{F}$ , then a choice  $V$  for  $T$  also works for  $T \circ f$ .

If  $T_0 \simeq T_1$ ,  $T : X \times I \rightarrow \mathcal{F}$ , then  $Ind(T) \in K^0(X \times I) \rightarrow H^0(X \times \{i\})$  is an isomorphism, and so  $Ind(T_0) = Ind(T_1)$ .

Fact:  $Ind : [X, \mathcal{F}] \rightarrow K^0(X)$  is a homomorphism of semigroups.

**Proposition 9.5.** *There exists an exact sequence of semigroups:*

$[X, GL(H)] \rightarrow [X, \mathcal{F}] \rightarrow K^0(X) \rightarrow 0$  with the last nonzero map  $Ind$ .

*Proof.* If  $T : X \rightarrow GL(H)$ , then  $Ind(T) = 0$ , since I can choose  $V$  to be  $H$ .

Let  $T : X \rightarrow \mathcal{F}$  such that  $IndT = 0$ . Then  $[X \times H/V] - [H/T(V)] = 0$ , and so there exists  $n$  such that  $(X \times H/V) \oplus 1_n \cong H/T(V) \oplus 1_n$ .

This means that we can replace  $V$  be a subspace of codimension  $n$ . Let  $W$  be such a subspace. Then  $H/T(W) \cong H/T(W) \oplus 1_n \cong X \times H/W$ , and so we get a section of the map  $X \times H \rightarrow H/T(W)$ .

So we get a bundle map  $X \times H/W \rightarrow X \times H/T(W) \rightarrow X \times H$ . This gives a map  $\phi : X \rightarrow L(H/W, H)$ . Define  $S : X \rightarrow L(H)$  by  $S(x) = T(x) + \phi(x)$ .  $\phi(x)$  is a compact operator, since it factors through  $H/W$  finite dimensional. So  $S(x)$  is Fredholm.  $S_t = T(x) + t\phi(x)$  gives a homotopy between  $S(x)$  which is invertible.  $S_t(x)$  is Fredholm for all  $t$  since  $\phi$  is compact.

To prove surjectivity of  $Int$ , let  $E$  be a vector bundle on  $X$  with  $E \oplus F \cong 1_n \cong X \times V$  where  $V$  is finite dimensional. Let  $p_x : V \rightarrow E_x$  be the projection.

Recall that  $T_f$  are Toplitz Operators,  $H = H^2 = \{f \in L^2(S^1) | \hat{f}(n) = 0, n < 0\}$ ,  $f \in C(S^1)$ ,  $IntT_z = -n$ ,  $T_f\xi(z) = P_f(\xi(z)f(z)) \in L^2(S^1)$ .

Define  $T : X \rightarrow \mathcal{F}(H \otimes V)$ . Then  $T(x) = T_z \otimes p_x + T_1 \otimes (1 - p_x)$ . Then  $\ker T = 0$ . So  $H \otimes V/T(H \otimes V) \cong E$ . So  $IntT = [X \times H \otimes V/H \otimes V] - [E]$ . And so I've constructed  $-[E]$  in  $K^0(X)$  for every vector bundle  $E$ .  $\square$

$$[1_n] = IntS_n \text{ where } S_n(x) = T_{z^{-n}}.$$

Alternately,  $T^* : X \rightarrow \mathcal{F}$  by  $T^*(x) = T(x)^*$ . Then  $IntT^* = [E]$ .

Check that  $\ker A^* \cong (\ker A)^\perp \cong \text{coker } A$ , if  $IndA$  is closed.

**Corollary 9.6.**  $Ind : [X, \mathcal{F}] \rightarrow K^0(X)$

Proof:  $GL(H)$  is contractible. (Kuiper)

**Definition 9.1** (Fredholm Complex). A diagram  $0 \rightarrow H_0 \rightarrow H_1 \rightarrow \dots \rightarrow H_N \rightarrow 0$  with  $A_i : H_i \rightarrow H_{i+1}$  the maps such that the  $H_i$  are Hilbert Spaces the  $A_i$  are bounded linear maps,  $A_i \circ A_{i-1} = 0$ , and  $ImA_i$  is closed in  $H_{i+1}$  and of finite codimension in  $\ker A_{i+1}$ , is called a Fredholm complex.

$H^*(H, A) \equiv \ker A_i / ImA_{i-1}$  is finite dimensional.

$$\chi(H, A) = \sum (-1)^i \dim H^i(H, A).$$

Define  $\Delta_i : H_i \rightarrow H_i$  to be  $\Delta_i = A_i^*A_i + A_{i-1}A_{i-1}^*$ .

**Proposition 9.7.**  $(H, A)$  is a Fredholm complex iff  $\Delta$  is a Fredholm Operator.

*Proof.*  $x \in \ker \Delta_i$  iff  $x \in \ker A_i \cap \ker A_{i-1}^*$ . This is because if  $x \in \ker \Delta_i$ , then  $0 = \langle \Delta x, x \rangle = \langle A_i^*A_i x, x \rangle + \langle A_{i-1}A_{i-1}^* x, x \rangle = \langle A_i x, A_i x \rangle + \langle A_{i-1}^* x, A_{i-1}^* x \rangle$  implies that  $A_i x = A_{i-1}^* x = 0$ .  $\square$

Define  $\mathcal{H}_i = \ker \Delta_i = \ker A_i \cap \ker A_{i-1}^*$  to be the harmonic elements.

**Proposition 9.8** (Hodge Decomposition).  $H_i = \mathcal{H}^i \oplus A_{i-1}(H_{i-1}) \oplus A_i^*(H_{i+1})$ .

**Proposition 9.9.** The following are equal:

1.  $\chi(H, A)$
2.  $Ind(\oplus_i A_{2i} + A_{2i}^* : H_{even} \rightarrow H_{odd})$
3.  $\dim \ker \Delta_{ev} - \dim \ker \Delta_{odd}$

## 10 Lecture 10

If  $X$  is compact, then  $[X, \mathcal{F}] \rightarrow K^0(X)$  by index is an isomorphism.

If  $T : X \rightarrow \mathcal{F}$ , then  $IndT = \ker T - \text{coker } T$  (kind of).

So then  $K^0(X) \rightarrow K^{-2}(X) = K^{-1}(\Sigma X)$  by the Bott Map.

We described relative  $K$ -theory of  $(X, A)$ .  $u : E \rightarrow F$  an iso on  $A$ .

Nice generalization:  $K^0(X, A)$  can be described by complexes of vector bundles  $E^0 \rightarrow E^1 \rightarrow \dots \rightarrow E^N$  exact on  $A$ .

If locally compact,  $K^0(X) = K^0(X^+, +)$ , which can be described by pairs (respectively complexes) of vector bundles  $E \rightarrow F$  where  $u$  is an isomorphism iff a compact set (respectively exact)

Let  $V$  be an  $n$ -dimensional complex vector space. Define the complex bigwedge  $\bigwedge^0 V \rightarrow \bigwedge^1 V \rightarrow \dots \rightarrow \bigwedge^n V \rightarrow 0$ . This is a complex of trivial vector bundles on  $V$  itself.

If  $E \rightarrow X$  is a vector bundle, then  $\pi^*E \rightarrow E$  is a vector bundle. Form the exterior algebra complex. This is exact away from the zero section.

We get  $\lambda_E \in K^0(E)$ .

Multiplication on  $K^0$ : If  $E \rightarrow X$  and  $F \rightarrow Y$  are vector bundles, then  $E \boxtimes F \rightarrow X \times Y$  is a vector bundle. On  $X, Y$  compact, we have  $([E_1] - [E_2]) - ([F_1] - [F_2]) = [E_1 \boxtimes F_1] + [E_2 \boxtimes F_2] - [E_1 \boxtimes F_2] - [E_2 \boxtimes F_1]$ .

If  $u : E_1 \rightarrow E_2$  and  $v : F_1 \rightarrow F_2$ , then we get a map (nasty matrix):  $E_1 \boxtimes F_1 \oplus E_2 \boxtimes F_2 \rightarrow E_1 \boxtimes F_2 \oplus E_2 \boxtimes F_1$ .

So we get a multiplication  $K^0(X) \times K^0(Y) \rightarrow K^0(X \times Y)$  for  $X, Y$  locally compact spaces.

$K^{-1}(X) = K^0(\Sigma X) \cong K^0([0, 1] \times X / \sim, pt) \cong K^0(\mathbb{R} \times X)$ , and so  $K^{-j}(X) = K^0(\mathbb{R}^j \times X)$ . So we get a product  $K^{-i}(X) \otimes K^{-j}(X) \rightarrow K^{-i-j}(X \times X) \rightarrow K^{-i-j}(X)$  by the pullback along the diagonal.

Thom Isomorphism in  $K$ -theory

If  $E \rightarrow X$  is a complex vector bundle, then  $\lambda_E \in K^0(E)$ . Define  $\lambda : K^0(X) \rightarrow K^0(E)$  by  $\lambda([F \rightarrow X]) = \pi^*E \cup \lambda_E \in K^0(E)$ . This is an isomorphism.

**Theorem 10.1** (Bott Periodicity). *If  $E = X \times \mathbb{C} \rightarrow X$ , then  $\beta = \lambda : K^0(X) \rightarrow K^0(X \times \mathbb{C}) \cong K^0(X \times \mathbb{R}^2) = K^{-2}(X)$  is an isomorphism.*

In  $K^0(\mathbb{C})$ , we have  $\mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C} \times \mathbb{C}$ .  $u(z, \lambda) = (z, z\lambda)$ . So we get  $\beta : K^0(X) \rightarrow K^{-2}(X)$ , and  $\beta(x) = x \cdot b \in K^0(X, \times \mathbb{C})$ .

Thus,  $K^{-2}(X) = K^{-1}(\Sigma X) = [\Sigma X, GL_\infty(\mathbb{C})] = [X, \Omega GL_\infty \mathbb{C}]$ . Let  $f : X \rightarrow \Omega GL_\infty(\mathbb{C})$ ,  $X$  compact. Then  $f : X \rightarrow \Omega GL_N(\mathbb{C})$  for some big  $N$ .  $f(x) : S^1 \rightarrow GL_N(\mathbb{C})$ . Given  $\phi : S^1 \rightarrow GL_N(\mathbb{C})$ , define  $T_\phi : H_+ \otimes \mathbb{C}^n \rightarrow H_+ \otimes \mathbb{C}^N$ .

So give  $f : X \rightarrow \Omega GL_N(\mathbb{C})$  the form  $T_f(x) \equiv T_{f(x)}$ , and so  $IndT_f \in K^0(X)$ .

Suppose that we have a map  $\alpha_X : K^{-2}(X) \rightarrow K^0(X)$  defined on compact  $X$ , satisfying:

1.  $\alpha$  is functorial, that is,  $f : Y \rightarrow X$ , then we have  $\alpha_X$  and  $\alpha_Y$  commuting with pullbacks.

2.  $\alpha_X$  is a module homomorphism over  $K^0(X)$ .

3.  $\alpha_{pt}(b) = 1$ .

**Remark 10.1.**  $\alpha(\beta(x)) = \alpha(bx)$ , by A2, this is  $\alpha(b)x$ , and by A3, this is  $1x = x$ .

**Lemma 10.2.** Given  $\alpha$  satisfying A1, A2, and A3, then  $\alpha$  can be extended to locally compact spaces, and  $K^{-(i+\ell)}(X) \rightarrow K^{-i}(X)$ , and is a module for  $K^0(X)$ .

So then  $X \times \mathbb{R}^2 \times \mathbb{T}^2$  by  $\theta : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2 \times \mathbb{R}^2 : (x, y) \mapsto (y, x)$ , look at  $(1 \times \theta)^*$ .

Messy computation.

**Proposition 10.3.** If  $\alpha$  exists satisfying A1-3, then  $\alpha$  is an inverse to the Bott map, and so Bott periodicity is true.

*Proof.*  $\alpha\beta = 1_{K^0(X)}$ .  $\beta(\alpha(x)) = \alpha(x)b = \alpha(xb) = \alpha(b\tilde{x})$ , and so  $\beta\alpha(x) = \tilde{x}$ , and in fact  $(\beta\alpha)^2 = \beta\alpha$ , thus,  $\tilde{x} = x$ .  $\square$

## 11 Lecture 11

Lasttime we proved the following

**Proposition 11.1.** If there exists  $\alpha : K^{-2}(X) \rightarrow K^0(X)$  for all compact  $X$  such that

1.  $\alpha(b) = 1$
2.  $\alpha$  is functorial
3.  $\alpha$  is a  $K^0$ -module homomorphism

then  $\alpha$  is inverse to  $\beta$ , and so Bott Periodicity is true.

Thom Isomorphism:  $\lambda_E \in K^0(E)$  defines an isomorphism  $x \mapsto \pi^*x \cup \lambda_E$  from  $K^0(X) \rightarrow K^0(E)$ .

Let  $M$  be a compact manifold. Then  $K_\infty^0(M) \cong K^0(M)$ , that is, the  $K$ -theory defined with  $C^\infty$ -bundles.

Let  $E, F$  be two  $C^\infty$  vector bundles,  $D : C^\infty(M; E) \rightarrow C^\infty(M; F)$  be a linear differential operator. If  $D$  is elliptic, then  $D$  is fredholm.

Index  $D$  only depends on the highest order term of  $D$ .

Suppose that  $Q$  is a vector bundle on  $M$ . Then we can couple  $Q$  to  $D$ . If  $Q$  is trivial, then  $D_Q : C^\infty(M; E \otimes Q) \rightarrow C^\infty(M; F \otimes Q)$  by  $D \otimes 1_Q$ .

In gneeral, find  $(U_\alpha, \lambda_\alpha)$  where  $U_\alpha$  is a trivializing cover of  $Q$  and  $\lambda_\alpha$  a partition of unity for  $U_\alpha$ . and take  $D_Q = \sum \lambda_\alpha^{1/2} (D|_{U_\alpha} \otimes 1_Q|_{\tilde{U}_\alpha}) \lambda_\alpha^{1/2}$ .

The highest part of  $D_Q$  is independent of  $U_\alpha, \lambda_\alpha$ , and so index  $D_Q$  is independent.

So then, we get a well defined map  $K^0(M) \rightarrow \mathbb{Z}$  by  $Q \mapsto \text{Ind}D_Q$ .

Let  $X$  be a compact space,  $E, F$  be  $C^\infty$  bundles on  $M$  parameterized by  $X$ , and let the inclusions be smooth in the  $M$  direction, cont in the  $X$  direction.

Let  $D$  be a continuous family of elliptic operators  $E \rightarrow F$  parameterized by  $X$ . Locally on  $M \times X$ , on  $U \times V$ , we have  $D = \sum a_\alpha(m, x) \partial_\alpha$

We then get  $x \mapsto D_x$  a map from  $X$  to Fredholm operators.  $D_x : C^\infty(M \times \{x\}, E|_{M \times \{x\}}) \rightarrow C^\infty(M \times \{x\}, F|_{M \times \{x\}})$ . This Index  $D$  is in  $K^0(X)$ .

Given  $Q \in K^0(M \times X)$ , couple to  $D$ , then  $IndD_Q \in K^0(X)$ . Moreover, if  $Q = p_2^* Q_0$ ,  $Q_0 \in K^0(X)$ , and  $p_2^* Q_0|_{M \times \{x\}}$  is trivial, then  $(D_Q)_x = D_x \otimes Q_{0,x}$ . So  $\ker D_Q = \ker D \otimes Q_0$ .

$\text{coker } D_Q = \text{coker } D \otimes Q_0$ . Thus,  $Index_D : K^0(M \times X) \rightarrow K^0(X)$  is a  $K^0(X)$ -module homomorphism. The fact that Index is functorial is trivial.

**Remark 11.1.** *Let  $M$  be a complex manifold. A holomorphic line bundle  $E$  which has transition functions  $g_{\alpha\beta}$  which are holomorphic. To  $E$ , there is associated a  $\bar{\partial}_E$ -operator, locally a section  $s : U \rightarrow \mathbb{C}^n$ ,  $\bar{\partial}(s) = \bar{\partial}$  is such a function.*

*Changing frames,  $s' = g_{\alpha\beta} s$  gives  $\bar{\partial}(s') = \bar{\partial}(g_{\alpha\beta} s) = \bar{\partial}(g_{\alpha\beta}) s + g_{\alpha\beta} \bar{\partial}(s) = g_{\alpha\beta} \bar{\partial}(s)$ .*

*So  $\bar{\partial}_E : A^{0,q}(M, E) \rightarrow A^{0,q+1}(M, E)$  has  $\bar{\partial}^2 = 0$ .*

On  $\mathbb{C}\mathbb{P}^n$ , we have  $\gamma^n$  the tautological line bundle. Then it is holomorphic.  $H^k(M, E) = H^k(A^{0,1}(M, E), \bar{\partial}_E)$ .

Let  $D = \bar{\partial}$  on  $\mathbb{C}\mathbb{P}^1$ . Then  $Ind\bar{\partial}_1 = Ind\bar{\partial} = 1$ ,  $Ind\bar{\partial}_{\gamma^1} = 0$ .

Define  $\alpha : K^{-2} \rightarrow K^0$  as follows.

$K^{-2}(X) = K^0(X \times \mathbb{R}^2) = \ker\{K^0(X \times \mathbb{C}\mathbb{P}^1) \rightarrow K^0(X \times pt)\}$ . Then  $(1, -\gamma) \in K^0(X \times \mathbb{R}^2)$ . And so we can take  $Index_{\bar{\partial}}$  to  $K^0(X)$ , and the index of  $1 - \gamma$  is 1, and that it is functorial is easy.

This DOESN'T COMMUTE

But if we start with  $E$ , take it as a direct sum of line bundles  $L_1 \oplus \dots \oplus L_n$  and define  $x_i = c_1(L_i)$ , then  $ch(E) = \sum_i e^{x_i}$ .

$\lambda_E(a) = \pi^* a \cup \lambda_E$  for  $a \in K^0(X)$ , and  $\Phi_E(x) = \pi^* x \cup \Phi_E$ .

Let  $t = \frac{i^* ch(\lambda_E)}{i^* \Phi_E}$  where  $i : X \rightarrow E$ . Then  $t$  is the multiplier that mates the diagram commute.  $i^* ch(\lambda_E) = ch(i^* \lambda_E) = \dots = \sum (-1)^i \wedge V$ . We eventually see that  $t = td(E^*)$ , and to make the diagram above commute, take the bottom map to be  $\Phi_E \cup td(E^*)$ .

Umkehr Maps: wrong way.

If  $M$  is a manifold and  $i : N \rightarrow M$  is an embedding with oriented normal bundle, then there exists a map  $i_1 : H^j(N) \rightarrow H_c^{j+r}(M)$ .

Similarly, if  $N \rightarrow M$  an embedding and the normal bundle has complex structure, then we get  $K^0(N) \xrightarrow{\lambda_N} K^0(\nu) \cong K^0(U) \rightarrow K^0(M)$ .

$[E] - [1_n]$  has a trivialization for a compact set in  $U$ . Use this trivialization to extend  $E$  trivially to  $M$ .

## 12 Lecture 12

We've shown that  $K^0, K^1, \dots$  and  $K^{-1}, K^{-2}, \dots$  form an extraordinary cohomology. But this is only a functor on the category of locally compact spaces with proper maps.

The coefficient ring of  $K$ ,  $K^*(pt) = \mathbb{Z}[b, b^{-1}]$ ,  $|b| = -2$ .

Different topic:

Let  $V$  be a real vector space and let  $(-, -)$  be a symmetric bilinear form of signature  $p, q$ .

For an algebra generated by  $V$  and with the relation  $vw + wv = -2(v, w)1$ , and we want  $v^2 = -(v, v)1$ .

**Proposition 12.1.** *There exists a unique isomorphism class of algebras  $C(V)$  universal for give  $V \rightarrow A$ .*

*Proof.*  $T(V)/I$ , where  $I$  is the two sided ideal generated by  $v \otimes w = w \otimes v + 2(v, w)1$ . □

**Example 12.1.** *If  $(-, -) = 0$ , then  $C(V) = \bigwedge V$ .*

**Example 12.2.** *If  $V = \mathbb{R}$ ,  $(v, v) = 1$ , then  $C(V) = \mathbb{C}$ .*

If  $V = \mathbb{R}^2$  with a positive definite form, we get the quaternions.

In general,  $C(V)$  is not  $\mathbb{Z}$ -graded, but is  $\mathbb{Z}/2$ -graded (which is "super")

$C(V) = C^+(V) \oplus C^-(V)$ .

define spin group

other stuff, get notes from Hilaf?

## 13 Lecture 13

MISSED

## 14 Lecture 14

So now we have  $\text{Spin}_n \xrightarrow{\pi} SO(n)$ . When you have a spin structure, you have a bundle of spinors  $\mathcal{S}^\pm$  from which you can construct a Dirac operator.

We now want to define  $\text{Spin}^c$ . It is  $\text{Spin}_n \times_{\mathbb{Z}/2} U(1)$ . It is obtained by taking the Spinor representation of  $\text{Spin}_n$  (which is  $U(S)$ ) and the natural  $U(1)$  action, and then taking the fibered product.

So we have a map  $\text{Spin}_n^c \rightarrow SO(n) \times U(1)$ .

So now a manifold has  $\text{Spin}^c$ -structure if there exists a principal  $\text{Spin}^c$ -bundle such that the associated  $SO(n)$  is the frame bundle and the associated bundle of  $U(1)$  is an extra structure, ie a line bundle  $L$ .

So now, if a manifold has a  $\text{Spin}^c$  structure then it has an associated spinor bundle  $\mathcal{S}^\pm$ . In order to get a connection on  $\mathcal{S}^\pm$ , you need to choose a connection on the associated line bundle.

Principal Bundle Connections:

$G \rightarrow P \rightarrow M$  and a representation  $\pi$  on a vector space  $V$ , I get a vector bundle  $V \rightarrow P \times_G V \rightarrow M$ .

Sections are  $C^\infty(M; P \times_G V) = \{f : P \rightarrow V \mid C^\infty \text{ and } f(pg) = \pi(g)^{-1}f(p)\}$ .

Let's recall moving frames over a trivializing neighborhood of  $E$ ,  $E|_U \cong U \times \mathbb{R}^n$ . Then  $\nabla s = s \otimes \omega$ .

If we changed frames, we have  $\omega' = g^{-1}dg + g^{-1}\omega g$ .

Principal bundle connections are an attempt to have global data. A connection on  $P$  is the following:  $\omega \in \Omega^1(P; \mathfrak{g})$  satisfying two conditions:

1. It should be invariant under  $G$
2. If  $X \in \mathfrak{g}$  then  $\tilde{X}$  denotes the associated vector field on  $P$ .  $i_{\tilde{X}}\omega = X$ .

Explanation:  $g^{-1}dg \in \Omega^1(G, \mathfrak{g})$ , the Maurer-Cartan Form.

The kernel of  $\omega$  is an invariant distribution  $H \subset TP$  which is complementary to the vertical vectors.  $P \rightarrow M$ ,  $p_* : TP \rightarrow TM$  and so  $p_* : H \rightarrow TM$  is an isomorphism.

Now, suppose we're given  $G \rightarrow P \rightarrow M$  and  $\pi : G \rightarrow GL(V)$ .  $\mathcal{V} = P \times_G V$  and  $\omega$  a principal bundle connection. Then  $\pi$  gives  $\pi : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ .

$\nabla : C^\infty(M; \mathcal{V}) \rightarrow C^\infty(M; T^*M \otimes \mathcal{V})$ . Then  $\nabla f = df + \pi(\omega)f \in \Omega^1(P; V)$ .

**Exercise 14.1.** *If  $G$  is a compact Lie group acting on  $M$  arbitrarily, then  $H^*(\Omega_{G\text{-basic}}^*(M), d) \cong H^*(M/G)$ . Additionally,  $H^*(\Omega^*(M)^G) \cong H^*(M)^G$ .*

**Exercise 14.2.**  $H^*(G)^G \cong H^*(\wedge^* \mathfrak{g}^*, d) = H^*(G)$  if  $G$  is connected.

So now, if  $\omega$  is a connection on  $P$ , its curvature  $\Omega$  is  $d\omega + \frac{1}{2}[\omega, \omega]$ .

Now  $\Omega \in \Omega^2(P; \mathfrak{g})$  and  $\Omega$  is basic, so it descends to  $\Omega^2(M; P \times_{Ad} \mathfrak{g})$ .

A manifold has  $\text{Spin}^c$  structure iff  $w_2 = x \pmod{2}$  where  $x \in H^2(M; \mathbb{Z})$  is a generator.

So complex manifolds have  $\text{Spin}^c$  structures.

Fact:  $C(TM) \otimes \mathbb{C}$  having a spinor bundle (ie fiberwise irreducible) is equivalence to  $M$  having  $\text{Spin}^c$ .

Connection between positive scalar curvature and Dirac Operator (For a Spin Structure)

Let  $M$  be a closed spin manifold.  $D$  the Dirac operator. Then  $D$  is essentially self-adjoint and  $D^2 = \nabla^* \nabla + r/4$  where  $r$  is the scalar curvature function.

If  $Ds = 0$  then  $0 = \langle D^2 s, s \rangle = \langle \nabla^* \nabla s + rs/4, s \rangle = \langle \nabla s, \nabla s \rangle + \langle r/4 s, s \rangle$ . The first part is  $\geq 0$ , the second is  $\int r(x)/4 |s(x)|^2 dx > 0$  whenever  $r > 0$  everywhere. So, if  $M$  is spin and has a metric of positive scalar curvature, there are no solutions to  $Ds = 0$ .

We can split the Dirac operator into  $D_+ : \mathcal{S}^+ \rightarrow \mathcal{S}^-$  and  $D_- : \mathcal{S}^- \rightarrow \mathcal{S}^+$ . The index of  $D_+$  is  $\dim \ker D_+ - \dim \text{coker } D_+ = \dim \ker D_+ - \dim \ker D_-$ . And so if  $M$  is Spin and has positive scalar curvature, then  $\text{Ind } D_+ = \langle \hat{A}(M), [M] \rangle = 0$ .

**Theorem 14.1** (Gromov-Lawson). *If  $M$  has psc and we do surgery in codimension  $\geq 3$ , then the result has psc.*

On simply connected spin manifolds, being spin cobordant is equivalent to differing by surgery in codimension  $\geq 3$ .

**Theorem 14.2** (Gromov-Lawson). *If  $M$  is simply connected and not spin, then it has a metric of psc.*

## 15 Lecture 15

$K$ -homology is the homology dual to  $K$ -theory.

**Remark 15.1** (Supports on Homology and Cohomology). *We can pair homology with compact support with cohomology with arbitrary supports and vice versa.*

*Homology with noncompact support: Borel-Moore Homology, or homology with locally finite support.  $H_*^{lf}(X) = H_*(\sum a_\sigma \sigma : \text{for each compact set } K \text{ the number of } a_\sigma \neq 0 \text{ such that } \sigma(\Delta) \cap K \neq \emptyset \text{ is finite})$*

*Poincaré isomorphism preserves the type of supports. So  $H_*^{lf}(\mathbb{R})$  is 0 for  $* = 0$  and  $\mathbb{Z}$  for  $* = 1$ .*

Anyway...

If  $X$  is compact, let  $C(X)$  be the complex valued continuous functions on  $X$ . A 0  $K$ -cycle on  $X$  is given by the following data: Let  $H = H^+ \oplus H^-$  be a super Hilbert Space with  $C(X)$  acting on it, let  $P : H^\pm \rightarrow H^\mp$  be a bounded operator such that

1.  $P - P^* \in K(H^\pm)$  the compact operators
2.  $[P, f] \in K(H^\pm)$  for all  $f \in C(X)$
3.  $P^2 - 1 \in K(H^\pm)$

Define an equivalence relation  $(H_1^\pm, P_1) \sim (H_2^\pm, P_2)$  if there exists  $(H, P)$  a pair for  $C(X \times I)$  and  $(H, P)$  should restrict to  $(H_i, P_i)$  for  $i = 1, 2$ .

Let  $\alpha = (\alpha_1, \dots, \alpha_n)$  a multiindex, and  $D^\alpha$  is that  $\left(-i \frac{\partial}{\partial x_1}\right)^{\alpha_1} \dots \left(-i \frac{\partial}{\partial x_n}\right)^{\alpha_n}$ , and  $|\alpha| = \alpha_1 + \dots + \alpha_n$ . So  $D^\alpha$  is an order  $|\alpha|$  derivative.

Let  $E, F \rightarrow M$  be vector bundles on  $M$ . Then a differential operator of order  $k$  is a linear operator  $P : C^\infty(M; E) \rightarrow C^\infty(M; F)$  which locally looks like  $\sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha$ , with  $a_\alpha \in C^\infty(M; \text{hom}(E, F))$ .

A differential operator is local in the following sense:  $\text{supp } Pf \subset \text{supp } f$ .

**Theorem 15.1** (Peetre). *A linear operator  $P : C^\infty(M; E) \rightarrow C^\infty(M; F)$  which is local is a differential operator.*

To a differential operator  $P$  of order  $k$ , we associate its principal symbol  $\sigma(P)(x, \xi) = \sum_{|\alpha|=k} a_\alpha(x) \xi^\alpha$ .

Claim:  $\sigma(P) \in C^\infty(T^*M; \text{hom}(\pi^*E, \pi^*F))$  where  $\pi : T^*M \rightarrow M$ .

We want to evaluate  $\sigma(P)(x, \xi)(e)$  where  $x \in M, \xi \in T_x^*M$  and  $e \in E_x$ . Let  $\phi$  be any function such that  $\phi(x) = 0, d\phi(x) = \xi$ , and  $\eta \in C^\infty(M; E)$  with  $\eta(x) = e$ . Look at  $\frac{i^k}{k!} P(\phi^k \eta)|_x$ . You can take  $\phi = \xi_1 x_1 + \dots + \xi_n x_n$ , so that  $(x_1, \dots, x_n)$  is a coordinate about  $x$ .

$\sigma(P)$  is independent of the coordinate systems or trivializations.

$\sigma : \text{Diff}_k(M; E, F) \rightarrow \text{Sym}_k(M; E, F)$  (the differential operators of order  $k$  and the set  $\{p(x, \xi) \in C^\infty(T^*M, \text{hom}(\pi^*E, \pi^*F)) | p(x, \lambda\xi) = \lambda^k p(x, \xi) \text{ for } \lambda > 0\}$ ).

**Proposition 15.2.** *There exists an exact sequence  $0 \rightarrow Diff_{k-1}(M; E, F) \rightarrow Diff_k(M; E, F) \rightarrow Sym_k(M; E, F)$  with the last map  $P \mapsto \sigma_k(P) = 0$ .*

$Diff(M, E, F)$ , the union of the  $Diff_k$ 's, is filtered by the associated graded  $Diff_k/Diff_{k-1} = Sym_k$ , which are polynomials in  $\xi$ .

**Proposition 15.3.** *If  $P \in Diff_k$  and  $Q \in Diff_\ell$ , then  $Q \circ P \in Diff_{k\ell}$  and  $\sigma(Q \circ P) = \sigma(Q) \circ \sigma(P)$ , which is just pointwise composition.*

Let  $M$  be a Riemannian manifold and let  $E, F$  be Hermitian vector bundles, with  $\langle e_1, e_2 \rangle = \int_M \langle e_1(x), e_2(x) \rangle_E dvol(x)$  for  $e_1, e_2 \in C^\infty(M; E)$ .

We say that  $P \in Diff_k(M; E, F)$  is formally adjoint to  $P^* \in Diff_k(M; F, E)$  if  $\langle Pe, f \rangle = \langle e, P^*f \rangle$  for all  $e \in C^\infty(M; E)$  and  $f \in C^\infty(M; F)$ .

**Proposition 15.4.** 1. *There exists exactly one formal adjoint*

$$2. (P + Q)^* = P^* + Q^*$$

$$3. (QP)^* = P^* \circ Q^*$$

$$4. P^{**} = P$$

$$5. \sigma(P^*) = \sigma(P)^t$$

**Definition 15.1** (Elliptic). *An operator  $P \in Diff_k(M; E, F)$  is elliptic if  $\sigma(P)(x, \xi)$  is invertible for all  $(x, \xi) \in T^*M \setminus 0$ .*

For now, we work on  $\mathbb{R}^n$ .

Let  $f \in C_c^\infty(\mathbb{R}^n)$ , and  $\hat{f}(\xi) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) dx$ . Fourier inversion says that  $f(x) = \int_{\mathbb{R}^n} e^{ix \cdot \xi} \hat{f}(\xi) d\xi$ .

One way to differentiate is to take  $f \mapsto \hat{f} \mapsto \xi_j \hat{f} \mapsto \int e^{ix \cdot \xi} \xi_j \hat{f}(\xi) d\xi$ . This gives  $-\frac{\partial f}{\partial x_j}$ .

If on  $\mathbb{R}^n$ , I have  $P = \sum a_\alpha(x) D^\alpha$ . The total symbol  $p(x, \xi) = \sum a_\alpha(x) \xi^\alpha$  is a function on  $\mathbb{R}^n \times \mathbb{R}^n$  which is polynomial in  $\xi$ .

$$Pf(x) = \int_{\mathbb{R}^n} e^{ix \cdot \xi} p(x, \xi) \hat{f}(\xi) d\xi.$$

The idea of pseudodifferential operators is to let  $p(x, \xi)$  be in a much wider class of functions, but now you have to be careful that your function has good properties.

Let  $U \subset \mathbb{R}^n$  an open set in  $\mathbb{R}^n$ . Let's define  $(Pf)(x) = \int_{\mathbb{R}^n} e^{ix \cdot \xi} p(x, \xi) \hat{f}(\xi) d\xi$  for  $f \in C_c^\infty(U)$ .

Here we are going to require some conditions on  $p$ .

$$p \in C^\infty(U \times \mathbb{R}^n). \quad |D_x^\beta D_\xi^\alpha p(x, \xi)| \leq C_{\alpha, \beta} (1 + |\xi|)^{k - |\alpha|}.$$

**Theorem 15.5.** *If  $p(x, \xi)$  satisfies this condition, then  $P$  defines an operator  $C_c^\infty(U) \rightarrow C^\infty(U)$ .*

$$Pf(x) = \frac{1}{(2\pi)^n} \iint e^{i(x-y) \cdot \xi} p(x, \xi) f(y) dy d\xi.$$

We'd like to write  $\int K(x, y) f(y) dy$  where  $K(x, y) = \frac{1}{(2\pi)^n} \int e^{i(x-y) \cdot \xi} p(x, \xi) d\xi$ .

## 16 Lecture 16

Suppose that  $P = P(D)$ , and  $D = (D_1, \dots, D_n)$  with  $D_i = -i \frac{\partial}{\partial x_i}$  a constant coefficient differential operator.  $P(D)u = f$ ,  $f \in C_c^\infty(U)$  and  $U \subset \mathbb{R}^n$ . Then we Fourier transform and get  $P(\xi)\hat{u} = \hat{f}$ .

Formally,  $\hat{u} = \hat{f}/P(\xi)$ , and so  $u(x) = \int e^{ix\xi} \hat{f}(\xi)/P(\xi) d\xi$ . The problem is that  $P(\xi)$  has zeroes.

Write  $P(\xi) = \sigma_m(\xi) + q(\xi)$ , where  $\sigma_m(\xi)$  is homogeneous of degree  $m$  and degree  $q$  is less than  $m$ .

Zeros of  $\sigma_m(\xi)$  form a cone. If  $P(D)$  is elliptic, then  $\sigma_m(\xi)$  has only zero as a zero.

Even more, the zeroes of  $P(\xi) = \sigma_m(\xi) + q(\xi)$  are a compact set in  $\mathbb{R}^n$ , so we let  $\rho$  be a number such that  $B_\rho(0)$  contains the zeros of  $P(\xi)$ . Let  $\chi = 0$  in  $B_\rho(0)$  and 1 outside  $B_{\rho+c}(0)$ .

Now we form  $v(x) = \int e^{ix\xi} \hat{f}\chi(\xi)/P(\xi) d\xi$ . Then  $Pv(x) = \int e^{ix\xi} P(\xi) \hat{f}(\xi) \chi(\xi)/P(\xi) d\xi = \int e^{ix\xi} \hat{f}(\xi) \chi(\xi) d\xi$ .

This is then  $\int e^{ix\xi} \hat{f}(\xi) (1 - (1 - \chi(\xi))) d\xi = f(x) - Rf(x)$ , where  $R$  is the operator  $Rf(x) = \int e^{ix\xi} \hat{f}(\xi) (1 - \chi(\xi)) d\xi$ . This is then  $\frac{1}{(2\pi)^n} \iint e^{i(x-y)\xi} f(y) (1 - \chi(\xi)) dy d\xi$ , but we can exchange the order of integration, and get  $\int k(x, y) f(y) dy$  where  $k(x, y) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} (1 - \chi(\xi)) d\xi$ .

Now, we note that  $\int e^{ix\xi} g(\xi) d\xi \rightarrow 0$  as  $x \rightarrow \infty$ . In fact, it does so very quickly, faster than any polynomial can grow.

So  $v = \int e^{ix\xi} \hat{f}(\xi)/P(\xi) \chi(\xi) d\xi = Qf$ , and  $PQ(f) = F - Rf$ , where  $R$  is a special type of operator, which is called smoothing.

A smoothing operator is an operator  $S : C_c^\infty(U) \rightarrow C^\infty(U)$ .

Now,  $C_c^\infty(U)$  has a topology defined by as a limit of  $C^\infty(K)$  with  $K$  compact, and each of these has a topology defined by seminorms  $\|f\|_n = \sup_K \sum_{|\alpha| < n} |\partial^\alpha f|$ . A distribution is then a continuous linear functional on  $C_c^\infty(\mathbb{R}^n)$ .

Now, all distributions form the dual of  $C_c^\infty(\mathbb{R}^n)$ , and compactly supported ones form the dual of  $C^\infty(\mathbb{R}^n)$ .

A linear operator  $A : C_c^\infty \rightarrow C^\infty$  can be extended to distributions  $A : \mathcal{E}' \rightarrow \mathcal{E}'$ . An operator is called smoothing if  $A : \mathcal{E}' \rightarrow C^\infty \subset \mathcal{E}'$ .

**Proposition 16.1.** *An operator is smoothing iff there exists  $k \in C^\infty(U \times U)$  such that  $Au(x) = \int_U k(x, y) u(y) dy$  for  $u$  a distribution.*

If  $k(x, y) \in C^\infty(U \times U)$  then  $Au : L^2(U) \rightarrow L^2(U)$  is a compact operator.

Given the original  $P$ , we found  $Q$  such that  $PQ = 1 - R$  and  $R$  smoothing. Thus, compact.

**Proposition 16.2.** *Let  $P$  be any differential operator on  $U$ ,  $Q$  a parametrix (ie,  $PQ = 1 - R$ ,  $R$  smoothing), then for any  $f$  and  $x$  in  $U$ , there exists  $x \in V \subset U$  and  $u$  with  $Pu = f$  on  $V$ .*

This is because  $PQ = 1 - R$ , and if we are willing to shrink  $U$ , we have  $Rf = \int_V k(x, y) f(y) dy$ , and  $\|R\| \leq c\mu(V)$ , and so if  $V$  is small enough such that  $\|R\| < 1$ , then  $1 - R$  is invertible by the geometric series.

Now consider the torus  $T^n = \mathbb{R}^n/\mathbb{Z}^n$ . Then  $L^2(T^n)$  is the completion of  $C^\infty(T^n)$  with respect to  $\langle u, v \rangle = \int u(x)\bar{v}(x)dx$ .

We define a new inner product  $\langle u, v \rangle'_S = \sum_{|\alpha| \leq S} \langle D^\alpha u, D^\alpha v \rangle$ .

The completion of  $C^\infty(T^n)$  with respect to  $\langle, \rangle'_S$  is called  $H^s$ , the  $s$ th Sobolev Space.

**Theorem 16.3** (Sobolev Embedding Theorem). *If  $s > n/2 + k$  then  $H^2 \subseteq C^k(T^n)$ . Additionally,  $\cap_s H^s = C^\infty(T^n)$  and  $\cup_s H^s = \mathcal{E}(T^n)$ .*

**Theorem 16.4** (Rellich's Lemma).  *$H^2 \rightarrow H^t$  for  $s > t$  is a compact operator.*

**Theorem 16.5.** *If  $D$  is a differential of order  $m$ , then  $D$  extends to a bounded operator  $H^2 \rightarrow H^{s-m}$  for any  $s$ .*

## 17 Lecture 17

Look at  $T^n = (S^1)^n$ . Then look at  $\hat{f}(\lambda) = \frac{1}{(2\pi)^n} \int_{T^n} f(x)e^{-i\langle x, \lambda \rangle} dx$  for  $\lambda \in \mathbb{Z}^n$ .

Then  $f(x) = \sum_\lambda e^{i\langle \lambda, x \rangle} \hat{f}(\lambda)$ .

Now we define  $\langle f_1, f_2 \rangle_t = \sum_\lambda \hat{f}_1(\lambda)\bar{\hat{f}}_2(\lambda)(1 + |\lambda|^2)^t$  for  $t \in \mathbb{R}$ .

This defines an inner product on  $C^\infty(T^n)$ . The completion is  $H^t$ , the  $t$ th Sobolev space.

Since  $D_i f = -i \frac{\partial}{\partial x_i} f$ , it has transform  $\xi_i \hat{f}$ .

Thus, the Sobolev inner product for  $t \in \mathbb{N}$  is equivalent to  $\sum_{|\alpha| \leq t} \langle D^\alpha f_1, D^\alpha f_2 \rangle$ .

**Proposition 17.1.**  $C^k(T^n) \subset H^k$

*Proof.*  $\widehat{D^\alpha f} = \xi^\alpha \hat{f}$ . So  $|\hat{f}(\lambda)| \leq |\alpha|^{-\alpha}$ , so  $|\widehat{D^\alpha f}| \leq C|\lambda|^{-\alpha}$ . Thus we have  $|\hat{f}(\lambda)||\lambda|^\alpha \leq C$ , and so  $f \in H^k$ .  $\square$

**Theorem 17.2** (Sobolev Embedding Theorem). *If  $p > n/2$  then  $H^{k+p} \subset C^k$*

*Proof.* See Analysis last semester  $\square$

**Lemma 17.3** (Rellich's Lemma). *If  $s > t$  then  $H^s \rightarrow H^t$  is a compact linear map.*

*Proof.* Let  $B$  be the unit ball in  $H^s$  and let  $\epsilon > 0$  be given. Then let  $S_N = \{f \in H^s \mid \hat{f}(\lambda) = 0 \text{ such that } |\lambda| < N\}$ .

There exists  $N$  such that  $f \in B \cap S_N$  implies that  $\|f\|_t < \epsilon$ .

$\|f\|_s^2 = \sum_\lambda |\hat{f}(\lambda)|^2 (1 + |\lambda|^2)^s < 1$ . If  $f \in S_N$ , then  $\|f\|_t^2 = \sum_\lambda |\hat{f}(\lambda)|^2 (1 + |\lambda|^2)^t < \|f\|_s^2 \sum_{|\lambda| \geq N} (1 + |\lambda|^2)^{t-s}$ .

Now  $S_N$  has finite codimension in  $H^s$ , so  $H^s/S_N$  is finite dimensional and  $B/S_N \subset H^s/S_N$  is precompact, that is, there exists a finite cover of balls with radius  $< \epsilon$  that cover  $B/S_N$ . But  $B \cap S_N$  has radius  $< \epsilon$  in  $H^t$ , and so  $B$  can be covered by a finite number of balls of radius  $2\epsilon$  in  $H^t$ .  $\square$

Now, if  $E$  is a complex vector bundle on  $X$  a compact manifold, let's define  $H^s(X, E)$  as follows: choose coordinate/trivializing charts  $\phi : U_\alpha \rightarrow T^n$  such that  $E|_{U_\alpha} \cong U_\alpha \times \mathbb{C}^n$ . Choose a partition of unity  $p_\alpha$ .

Define  $\langle s_1, s_2 \rangle_t = \sum_\alpha \langle p_\alpha s_1(\phi_\alpha^{-1}x), p_\alpha(\phi_\alpha^{-1}x) \rangle_t$ .

Claim: All the statements about Sobolev spaces on  $T^n$  hold for  $H^t(X; E)$ .

Prototype:  $Pu(x) = \int_{\mathbb{R}^n} p(x, \xi) e^{i\langle x, \xi \rangle} \hat{u}(\xi) d\xi$ . Now  $|D_x^\beta D_\xi^\alpha p(x, \xi)| \leq C_{\alpha, \beta} (1 + |\xi|)^{k - |\alpha|}$  where  $P$  is defined for  $C_c^\infty(U)$ .

**Theorem 17.4.**  $P : C_c^\infty(U) \rightarrow C^\infty(U)$

If  $P$  is a differential operator on  $\mathbb{R}^n$ , then  $p(x, \xi) = e^{-i\langle x, \xi \rangle} P(e^{i\langle x, \xi \rangle})$ .

For pseudodifferential operators, we can't apply  $P$  to  $e^{i\langle x, \xi \rangle}$ , you want to consider those operators such that for all  $\xi \in C_c^\infty(U)$ , then  $p_\chi(x, \xi) = e^{-i\langle x, \xi \rangle} P(\chi(x) e^{i\langle x, \xi \rangle})$ .  $p_\chi$  should satisfy the estimates  $|D_x^\beta D_\xi^\alpha p_\chi| \leq C_{\alpha, \beta, \chi} (1 + |\xi|)^{k - |\alpha|}$ .

But now,  $p_\chi(x, \xi)$  depends on  $x$ , even if  $\chi = 1$  in a neighborhood of  $x$ . Let's consider those  $p(x, \xi)$  which satisfy  $\lim_{\lambda \rightarrow \infty} \frac{p(x, \lambda\xi)}{\lambda^k}$  exists as  $\sigma_k(x, \xi)$ .

Let  $X$  be a manifold and  $E, F$  complex vector bundles.

A  $\psi$ do of order  $k$  is an operator  $P : C^\infty(X; E) \rightarrow C^\infty(X; F)$  such that there exists a cover  $U_\alpha$  of  $X$  trivializing  $E$  and  $F$  such that the local operators  $P_{\alpha, \chi} : C_c^\infty(U_\alpha, E) \rightarrow C^\infty(U, F)$  with  $\chi$  supported in a compact subset of  $U_\alpha$ , this should be a  $\psi$ do on  $U_\alpha$ .

That is,  $p_\chi(x, \xi) = e^{-i\langle x, \xi \rangle} P(\chi(x) e^{i\langle x, \xi \rangle})$  should satisfy estimates and also  $\sigma_k(x, \xi) = \lim_{\lambda \rightarrow \infty} \frac{p_\chi(x, \lambda\xi)}{\lambda^k}$  is independent of  $\chi$  if  $\chi = 1$  in a neighborhood of  $x$ .

So the  $\sigma_k(x, \xi)$  piece together to give a global element  $\sigma_k(x, \xi) : \pi^*E \rightarrow \pi^*F$  which is homogeneous of degree  $k$ .

**Proposition 17.5.** 1. If  $P$  is a  $\psi$ do of order  $k$  then for all  $s$   $P$  extends to a bounded operator  $H^s \rightarrow H^{s-k}$ .

2. If  $P, Q$  are  $\psi$ dos of order  $k, \ell$  then  $PQ$  is a  $\psi$ do of order  $k + \ell$  and  $\sigma_{k+\ell}(PQ) = \sigma_k(P)\sigma_\ell(Q)$ .

3. If  $P$  is a  $\psi$ do of order  $k$  and  $\sigma_k(P) = 0$ , then  $P$  is a  $\psi$ do of order  $k - 1$ .

4. Given a symbol  $\sigma \in \text{Sym}^k(X; E, F) = \{\sigma : \pi^*E \rightarrow \pi^*F, \sigma(x, \lambda\xi) = \lambda^k \sigma(x, \xi)\}$  then there exists a  $\psi$ do  $P$  with  $\sigma_k(P) = \sigma$ .

Let  $P$  be a  $\psi$ do which is order  $k$  and elliptic.  $\sigma_k(x, \xi)$  is invertible for all  $(x, \xi) \neq 0$ .

We define  $q(x, \xi) = \sigma_k(x, \xi)^{-1}$ . Then  $q$  is order  $-k$ , and let  $Q$  have  $\sigma_{-k}(Q) = q$ . Then  $\sigma_0(Q \circ P) = \sigma_{-k}(Q)\sigma_k(P) = 1$ , and so  $\sigma_0(1 - Q \circ P) = 0$ . So  $1 - Q \circ P$  is an operator of order  $-1$ . If  $T$  is  $\psi$ do of order  $-1$ , then  $T : H^0(X, E) \rightarrow H^1(X, E) \rightarrow H^0(X, E)$  and the last map is compact by Rellich, and so  $1 - QP$  is compact. So  $P : H^k(X, E) \rightarrow H^0(X, F)$  is Fredholm.

So now  $1 - QP$  is  $\Psi^{-1}(X, E, E)$ .

If  $P$  is elliptic, then any solution to  $Pu = f \in C^\infty$  must be  $C^\infty$ .

## 18 Lecture 18

Today we'll be talking about Rational Homotopy Theory.

We will assume that all spaces are simply connected.

We say that two spaces are rationally homotopy equivalent if there exists  $f : X \rightarrow Y$  such that  $f \otimes \mathbb{Q} : \pi_i(X) \otimes \mathbb{Q} \rightarrow \pi_i(Y) \otimes \mathbb{Q}$  is an isomorphism for each  $i$ .

**Theorem 18.1.** *There exists a process sending spaces  $X \rightarrow X_{(0)}$  where  $X \rightarrow X_{(0)}$  is a rational homotopy equivalence and  $X_{(0)}$  is a rational space, that is,  $\pi_i(X_{(0)}) = \pi_i(X) \otimes \mathbb{Q}$  for all  $i$ .*

**Proposition 18.2.** *TFAE*

1.  $X \rightarrow Y$  is a rational homotopy equivalence.
2.  $H_i(X; \mathbb{Q}) \rightarrow H_i(Y; \mathbb{Q})$  is an isomorphism for all  $i$ .

Look at Bausfield/Kan

The problem of rational homotopy theory is to give an algebraic model for rational homotopy type

Look at Sullivan "Infinitesimal Calculations in Topology"

We will now look at (graded) commutative differential graded algebras (cdga)

We define  $f : A \rightarrow B$  to be a quasi-isomorphism if it is an algebra homomorphism that induces isomorphisms on cohomology.

**Definition 18.1** (Free CGDA). *A cdga is called free if it is  $(\bigwedge^* V, d)$  for a graded  $\mathbb{Q}$  vector space  $V$ .*

**Definition 18.2** (Minimal). *A cdga is called minimal if it is free,  $A^0 = \mathbb{Q}$ ,  $A^1 = 0$  and  $dA \subset A^+A^+$ , where  $A^+ = \bigoplus_{i \geq 1} A^i$ .*

A minimal model for a cdga  $A$  is  $(M, \rho)$  where  $\rho : M \rightarrow A$  is a quasi-isomorphism and  $M$  is minimal.

**Theorem 18.3** (Sullivan). *Every connected, simply connected cdga has a minimal model which is unique up to isomorphism.*

This is basically a version of deRham's Theorem for singular cohomology (Block "deRham's Theorem is not a Theorem about manifolds")

Comment about homotopy groups forming a lie algebra over  $\mathbb{Z}$ .

## 19 Lecture 19

Let  $U \subseteq \mathbb{R}^n$ , and  $S^k(U) = \{p(x, \xi) : U \times \mathbb{R}^n \rightarrow \mathbb{C} \mid \|D_x^\beta D_\xi^\alpha p(x, \xi)\| \leq C(1 + |\xi|)^{k-|\alpha|}\}$ , and  $P_u(x) = \frac{1}{(2\pi)^n} \int p(x, \xi) e^{i\langle x, \xi \rangle} \hat{u}(\xi) d\xi$ .

Define  $\tilde{\Psi}^k = P : C_c^\infty(U) \rightarrow C^\infty(U)$  such that  $\forall f \in C_c^\infty(U), \exists p_f(x, \xi)$  such that  $p(f_u) = p_f(x, \xi)u$ .

Or, equivalently,  $P : C_c^\infty(U) \rightarrow C^\infty(U)$  and for all  $f \in C_c^\infty(U)$ , we have  $p_f(x, \xi) = e^{-i\langle x, \xi \rangle} P(f e^{i\langle x, \xi \rangle}) \in S^k(U)$ .

Define  $\Psi^h = \{P \in \Psi^h | \forall f, \lim_{\lambda \rightarrow \infty} p_f(x, \lambda \xi) / \lambda^n \text{ exists}\}$ .

So for  $P \in \Psi^h(U)$ , set  $\sigma(P)(x, \xi) = \sigma(p_f)(x, \xi)$  where  $f \in C_c^\infty(U)$ , and  $f = 1$  near  $x$ . Then  $\sigma(P)$  is independent of  $f$  and  $\tilde{\Psi}^h$  and  $\Psi^h$  are invariant under diffeos.

Thus,  $\Psi^k(X; E, F)$  for  $X$  a  $C^\infty$  manifold and  $E, F$  vector bundles can be defined locally in coordinate charts on  $X$  and frames on  $E, F$ .

So  $\sigma(P)(x, \xi)$  is a globally defined element of  $Sym^k(X; E, F) = \{\sigma : \pi^*(E) \rightarrow \pi^*(F) | C^\infty \text{ homogeneous in } \xi \text{ and of order } k\}$  where  $\pi$  is the map  $T^*X \setminus X \rightarrow X$

So now we let  $X$  be a compact oriented manifold

**Proposition 19.1.** 1.  $\sigma_{k+\ell}(P \circ Q) = \sigma_k(P) \circ \sigma_\ell(Q)$  for  $P \in \Psi^k$  and  $Q \in \Psi^\ell$ .

2. If  $P \in \Psi^k(X; E, F)$  and its formal adjoint  $P^* \in \Psi^k(X; F^*, E^*)$  then  $\sigma(P^*) = \sigma(P)^*$

3. There is a short exact sequence  $0 \rightarrow K \rightarrow \Psi^k(X; E, F) \rightarrow Sym^k(X; E, F) \rightarrow 0$ .

Take  $H^s(X; E)$  to be the  $s$ th Sobolev space. Chose an inner product on  $E$  and a connection on  $E$ , and define  $\Delta = 1 + \nabla^* \nabla$  a differential operator on  $E$ . Define  $\langle u, v \rangle_0 = \int_X \langle u, v \rangle \text{ vol}$  and  $\langle u, v \rangle_s = \int_X \langle \Delta^s u, v \rangle \text{ vol}$ .

**Definition 19.1.** We define  $Op_k^s(X; E, F)$  to be the Banach Space  $\{T : H^s(X; E) \rightarrow H^{s-k}(X; F) \text{ with } T \text{ continuous}\}$ .

**Proposition 19.2.** For all  $s$ ,  $\Psi^k(X; E, F) \rightarrow Op_k^s(X; E, F)$  by  $P \mapsto P_s$

We also have  $\Psi^k(X; E, F) \xrightarrow{\sigma} Sym^k(X; E, F)$ , and so  $\sigma$  extends to  $\overline{\Psi^k(X; E, F)} \subset Op_k^s(X; E, F)$  to  $\overline{Sym^k(X; E, F)}$ , the continuous symbols.

So now, we define  $P \in \Psi^k(X; E, F)$  to be elliptic if  $\sigma(P)$  is invertible.

If  $P \in \Psi^k(X; E, F)$  is elliptic, then  $\sigma_k(P)$  is invertible, so we can find  $Q \in \Psi^{-k}(X; F, E)$  such that  $\sigma_{-k}(Q) = 1/\sigma(P)$ , so  $\sigma(PQ) = \sigma(QP) = 1$ .

So now,  $\sigma_0(1 - PQ) = \sigma_0(1 - QP) = 0$ , so  $1 - PQ, 1 - QP \in \Psi^{-1}$ .

Now,  $\Psi^{-1}(X; E, F)$  gives  $H^s(X; E) \rightarrow H^{s+1}(X; F)$  and the maps are compact.

That is,  $\Psi^{-1}(X; E, F)$  gives compact operators on  $H^s(X; E, F)$  for all  $s$ .

$(1 - PQ) : H^s(X; E) \rightarrow H^s(X; F)$  is compact, and so  $P$  is Fredholm. Thus,  $\ker P$  and  $\text{coker } P$  are finite dimensional.

**Theorem 19.3** (Elliptic Estimate). If  $P$  is elliptic in  $\Psi^k$ , then  $\forall s, \exists C$  such that  $\|Pu\|_s + \|u\|_0 \geq C\|u\|_{s+k}$ .

**Theorem 19.4** (Elliptic Regularity). If  $Pu = v \in H^s$ ,  $u \in H^{s+k}$ . If  $Pu \in C^\infty$  then  $u \in C^\infty$ , and so  $Pu = 0 \Rightarrow u \in C^\infty$ . Thus,  $\text{Ind } P \equiv \dim \ker P_s - \dim \text{coker } P_s$  is independent of  $s$ ,  $P_s = \text{Im } P \in Op_k^s$ .

**Proposition 19.5.** *If  $P \in \Psi^k(X; E, F)$  is elliptic, its index depends only on  $\sigma(P)$ .*

*Proof.* If  $\sigma_k(P) = \sigma_k(P')$ , then  $\sigma_k(P - P') = 0$  so  $P - P' \in \Psi^{k-1}$ . Thus, though  $P, P' : H^s(X; E) \rightarrow H^{s-k}(X, F)$ , we have  $P - P' : H^s(X, E) \rightarrow H^{s-(k-1)}(X; F)$ , which then maps compactly into  $H^{s-k}(X, F)$ .

So  $\text{Ind}P_s = \text{Ind}(P' + (P - P')) = \text{Ind}(P')$  as  $P - P'$  is a compact operator, and adding a compact operator to a Fredholm operator doesn't change the index.  $\square$

**Proposition 19.6.**  *$\text{Ind}P$  only depends on the homotopy class of  $\sigma(P)$  inside  $\text{Sym}^k(X; E, F)^*$  (for invertible). If  $\sigma_t$  is a family, we can find  $P_t$  a continuous family with  $\sigma_k(P_t) = \sigma_t$  and  $\text{Ind}(P_1) = \text{Ind}(P_0)$ .*

**Proposition 19.7.** *If  $\sigma_1 \in \text{Sym}^k(X; E, F)$  and  $\sigma_2 \in \text{Sym}^\ell(X; E, F)$  are both elliptic and they agree on  $ST^*X$ , then  $\text{Ind}P_1 = \text{Ind}P_2$  for  $\sigma(P_1) = \sigma_1$  and  $\sigma(P_2) = \sigma_2$ .*

*Proof.*  $\sigma_1/\sigma_2 = 1$  On the sphere bundle, and we can find a self-adjoint operator  $Q$  with  $\sigma_1(P_1) = \sigma_2(P_2)Q$ . So  $\text{Ind}(P_1) = \text{Ind}(P_2Q) = \text{Ind}(P_2) + \text{Ind}(Q)$ , but  $Q$  is self-adjoint, so  $\ker Q \cong \text{coker } Q$ , so  $\text{Ind}(Q) = 0$ .  $\square$

For  $X$  a manifold, we're going to define two homomorphisms  $K^0(T^*X) \rightarrow \mathbb{Z}$  called  $i_X$  and  $a_X$ .

We define  $a_X$  as follows: take an element of  $K^0(T^*X)$  represented by a bundle homomorphism  $p : \pi^*E \rightarrow \pi^*F$ , where  $p$  is an isomorphism off of the zero section. This  $p$  can be thought of as a symbol for a  $\Psi$ do. Take  $a_X([p]) = \text{Ind}P$ , where  $\Sigma(P) = p$ .

We define  $i_X$  topologically by embedding  $X$  into  $\mathbb{R}^n$  and then  $T^*X$  into  $\mathbb{R}^{2n}$ . So then  $i_X : K^0(T^*X) \xrightarrow{\text{Thom}} K^0(U) \xrightarrow{\text{extension}} K^0(\mathbb{R}^{2n}) \xrightarrow{\text{Bott}} \mathbb{Z}$

## 20 Lecture 20

Let  $X$  be a compact manifold  $i : X \rightarrow Y$  an inclusion into another manifold. We want to define  $i_! : K^0(TX) \rightarrow K^0(TY)$ , by the following:

$X$  has a tubular neighborhood  $X \subset N \subset Y$  which we can identify with  $\nu$  of  $X$  in  $Y$ . Then  $TN$  will be a tubular neighborhood of  $TX$  in  $TY$ , and this can be identified with the normal bundle of  $TX$  in  $TY$ . Then  $TN \cong \pi^*(\nu \otimes \mathbb{C})$  for  $\pi : TX \rightarrow X$ .

We use the Thom isomorphism theorem  $\lambda_{\pi^*(\nu \otimes \mathbb{C})} : K(TX) \rightarrow K(TN)$ , so there exists a map  $K(TN) \rightarrow K(TY)$  since the first is compactly supported, and composition gives a map  $i_! : K(TX) \rightarrow K(TY)$ .

If  $X \rightarrow Y \rightarrow Z$  by  $i$  and  $j$ , then  $j_!i_! = (j \circ i)_!$ .

Now let  $i : X \rightarrow \mathbb{R}^n$  the Whitney embedding, and  $j : * \rightarrow \mathbb{R}^n$  be the inclusion, then define  $t = j_!^{-1} \circ i_! = K(TX) \rightarrow K(T\mathbb{R}^n) \leftarrow K(T\{0\}) = \mathbb{Z}$ .

**Proposition 20.1.**  *$t$  is independent of the embedding.*

$i : X \rightarrow \mathbb{R}^n$  and  $i' : X \rightarrow \mathbb{R}^{n'}$  be two embeddings, let  $k(x) = (i(x), i'(x)) : X \rightarrow \mathbb{R}^{n+n'}$ .

We now define a homotopy  $k_s(x) = (i(x), si'(x))$ . Then  $k_1 = k$  and  $k_0 = i \times 0$ , and  $k_{0!} = k_{1!}$ , so we only need to show that  $i_! = (i \times 0)_!$ .

We have  $X \rightarrow \mathbb{R}^n \rightarrow \mathbb{R}^{n+n'}$  by  $i$  and then the standard inclusion, and the composition is  $i \times \{0\}$ . So  $\nu(i \times 0) = \nu(i) \oplus 1_{n'}$ , and so it works out.

**Definition 20.1.** An index map is an assignment for each compact differentiable manifold  $X$ ,  $\phi_X : K^0(TX) \rightarrow \mathbb{Z}$  satisfying

1. If  $f : X \rightarrow Y$  is a diffeo, then  $K^0(TY) \rightarrow K^0(TX)$ . the pull back map  $f^*$ , is a map over  $\mathbb{Z}$ .
2.  $\phi_{pt} : K^0(Tpt) \rightarrow \mathbb{Z}$  is the identity
3. If  $i : X \rightarrow Y$  is an embedding of compact manifolds, then  $K^0(TX) \xrightarrow{i_!} K^0(TY)$  is a map over  $\mathbb{Z}$ .

**Proposition 20.2.** If  $\phi$  is an index map, then  $\phi = t$ .

*Proof.* Embed  $X \rightarrow \mathbb{R}^n$ , and extend it to a map  $X \rightarrow S^n$  called  $i^+$ , and define  $j : \{0\} \rightarrow \mathbb{R}^n$ . We get the following diagram:

$$\begin{array}{ccccc}
 & & K^0(T\mathbb{R}^n) & & \\
 & \nearrow i_! & \downarrow & \nwarrow j_! & \\
 K^0(TX) & \xrightarrow{i_!^+} & K^0(TS^n) & \xleftarrow{j_!^+} & K^0(T0) \\
 & \searrow \phi_X & \downarrow \phi_{S^n} & \swarrow \phi_{pt} & \\
 & & \mathbb{Z} & & 
 \end{array}$$

By the axioms  $\phi_{pt} = 1$ , and the top triangles commute by construction, and the bottom does by axiom 3.

We now identify  $TX = T^*X$ , and  $\alpha : K^0(TX) \rightarrow \mathbb{Z}$  given by: let  $x \in K^0(TX)$  be represented by  $\sigma : E \rightarrow F$  where  $\sigma$  is an isomorphism of zero section, take an elliptic  $\psi$  of order  $k$ ,  $P$ , with symbol  $\sigma$ , so  $\sigma_k(P) = \sigma$ .

Now set  $\alpha(x) = \text{Ind} P_s$ , where  $P_s : H^s(X; E) \rightarrow H^{s-k}(X; F)$ .

Last time we saw that index is independent of  $s$  and  $k$ , and within operators of order  $k$ , it only depends on  $\sigma$ . So we have a well defined map  $\alpha_X : K^0(TX) \rightarrow \mathbb{Z}$ , and it clearly satisfies axioms 1 and 2. We need to show that it satisfies axiom 3, and we will do this later.  $\square$

Recall: given a complex vector bundle  $E \rightarrow X$ , we define  $c_k(E)$  and had the splitting principle  $F(E) \rightarrow X$  such that  $H^*(X) \rightarrow H^*(F(E))$  is an inclusion and  $\pi^*E = L_1 \oplus \dots \oplus L_k$  with  $L_i$  line bundles.

Define  $c_1(L_i) = x_i$  the chern root.

Any symmetric expression in Chern roots descends to  $H^*(X)$ .

$ch(E) = \sum_{i=1}^k e^{x_i} \in H^{ev}(X, \mathbb{Q}) \cong \prod_k H^{2k}(X; \mathbb{Q})$ , then  $ch_0(E) = rank(E)$ ,  $ch(E \oplus F) = ch(E) + ch(F)$  and  $ch(E \otimes F) = ch(E)ch(F)$ .

So  $ch : K^0(X) \rightarrow H^{ev}(X; \mathbb{Q})$  is a homomorphism of rings, and  $K^1(X) = K^0(\sum X) \xrightarrow{ch} H^{ev}(\sum X; \mathbb{Q}) \cong H^{odd}(X; \mathbb{Q})$ , so we get  $ch : K^*(X) \rightarrow H^*(X; \mathbb{Q})$ .

If  $X$  is locally compact, then  $K^*(X) \cong \tilde{K}^*(X^+) \rightarrow \tilde{H}^*(X^+; \mathbb{Q}) \cong H_c^*(X; \mathbb{Q})$ .

**Proposition 20.3.**  $ch \otimes \mathbb{Q} : K^*(X) \otimes \mathbb{Q} \rightarrow H_c^*(X; \mathbb{Q})$  is an isomorphism of  $\mathbb{Z}/2$  graded algebras (if  $X^+$  is a finite CW-complex)

*Proof.* For  $U$  a ball in  $\mathbb{R}^n$ ,  $K^*(U) \cong \tilde{K}^*(U^+ = S^n) \cong \begin{cases} \mathbb{Z} & * = n \\ 0 & * \neq n \end{cases}$

Check that Chern character commutes with Meyer-Vietoris sequence,  $ch : K^n(U) \otimes \mathbb{Q} \rightarrow H_c^n(U; \mathbb{Q})$  for  $n = 0, 1 \pmod 2$  is an isomorphism.  $\square$

So  $Td(E) \equiv \prod_{i=1}^k \frac{x_i}{1-e^{-x_i}}$ . Clearly  $Td(E \oplus F) = Td(E)Td(F)$ .

$$\begin{array}{ccc} K^0(X) & & K^0(E) \\ \downarrow ch & \searrow \lambda_E & \downarrow cd \\ H^{ev}(X; \mathbb{Q}) & \xrightarrow{\Phi_E} & H_c^{ev}(E; \mathbb{Q}) \end{array}$$

Doesn't commute. Instead  $ch(\lambda_E)(x) = (-1)^k \Phi(Td(\bar{E}))^{-1} ch(x)$ .

**Theorem 20.4.**  $X$  a compact orientable manifold. If  $P$  is an elliptic  $\psi$ do, with symbol  $\sigma$ , then  $IndP = \langle (-1)^n Td(TX \otimes \mathbb{C}) ch(\sigma), [X] \rangle$

**Theorem 20.5.**  $IndP = t(\sigma)$ .

If  $G$  is a compact group acting on  $X$  and  $P$  is an elliptic operator for which  $gP = Pg$ , then  $\ker P$  and  $\text{coker } P$  are representations of  $G$ . So  $IndP \in R(G)$  the representation ring of  $G$ .

$R(G)$  is the direct sum of  $\mathbb{Z}$ 's indexed by irreducible representations of  $G$  with products given by tensor product of representations and then decomposition.

Let  $T$  be a compact torus, then  $R(T) = \mathbb{Z}[t_1^{\pm 1}, \dots, t_k^{pm1}]$ , and  $R(G) = R(T)^W$  where  $W$  is the Weyl group.

If  $X \times Y$  and  $E, F$  on  $X \times Y$ , and  $P_y$  a family of elliptic  $\psi$ dos on  $E|_{X \times \{y\}}$ . Then there exists an index  $Y \rightarrow Fred$  such that  $IndP \in K^0(Y)$ , the proof of index extends to this.

## 21 Lecture 21

We define  $Td(E) = \prod_{i=1}^k \frac{x_i}{1-e^{-x_i}}$  the Todd genus, it satisfies  $Td(E \oplus F) = Td(E)Td(F)$ .

$$\Phi_E^{-1}(ch \lambda_E(x)) = (-1)^k Td(\bar{E})^{-1} ch(x).$$

So now we have  $t = j_i^{-1} \circ i_!$ , where  $i_! : K^0(TX) \xrightarrow{\lambda_{T\nu}} K^0(T\nu) \xrightarrow{k_*} K^0(T\mathbb{R}^k) \xleftarrow{j_i} T^0(Tpt)$ .

This gives us a map  $H_c^{ev}(TX) \xrightarrow{\Phi_{T\nu}} H_c^{ev}(T\nu)H_c^{ev}(T\mathbb{R}^k) \xleftarrow{\Phi_j} H_c^{ev}(Tpt)$  by taking the Chern character.

Now,  $Td(T\mathbb{R}^k) = 1$  since it is trivial. So then as  $\bar{T\nu} \cong \pi^*\nu \otimes \mathbb{C}$ , we have  $Td(\bar{T\nu}) = Td(T\nu)$ , and  $T(TX) \oplus T\nu \cong T\mathbb{R}^k|_{TX}$ , so we get that  $TTX \cong \pi^*TX \otimes \mathbb{C}$

Thus,  $Td(TTX \oplus T\nu) = Td(T\mathbb{R}^k|_X) = 1$ , but it is also  $Td(TTX)Td(T\nu)$ , and so  $Td(T\nu) = Td(\pi^*TX \otimes \mathbb{C})^{-1}$ .

So we have  $\Phi_{\pi^*\nu}^{-1} \circ ch \circ \lambda_{\pi^*\nu}(x) = (-1)^{k-n}Td(\bar{T\nu})^{-1}ch(x)$ , which is the same as  $(-1)^{k-n}Td(\pi^*TX \otimes \mathbb{C})ch(x)$ .

We also have  $\Phi_j^{-1}ch\lambda_{\mathbb{R}^k}(x) = (-1)^kTd(T\mathbb{R}^k)ch(x) = (-1)^kch(x)$ .

So  $ch(i_!x) = ch(k_* \circ \lambda_{\pi^*\nu}(x)) = ch(j_!t(x)) = (-1)^k\Phi_j(t(x)) = (-1)^kt(x)\Phi_j(1)$ .

So  $t(x) = \langle t(x)\Phi_j(1), [T\mathbb{R}^k] \rangle = (-1)^k \langle ch k_* \lambda_{\pi^*\nu}(x), [T\mathbb{R}^k] \rangle = (-1)^k \langle ch k_* \lambda_{\pi^*\nu}(x), [T\nu] \rangle = (-1)^n \langle \Phi_{\pi^*\nu}^{-1}(Td(\pi^*TX \otimes \mathbb{C}))ch(x), [T\nu] \rangle = (-1)^n \langle \pi^*Td(TX \otimes \mathbb{C})ch(x), [TX] \rangle$ .

If  $X$  is oriented, we can reduce this to  $X$  as follows:

$\Phi_{TX} : H^*(X) \rightarrow H_c^*(TX)$  satisfies  $\langle \Phi_{TX}(a), [TX] \rangle = (-1)^{n(n-1)/2} \langle a, [X] \rangle$ .

**Theorem 21.1.**  $X$  a compact manifold and  $P$  a  $\psi$ do, elliptic, with symbol  $\sigma$ . Then  $IndP = (-1)^n \langle \pi^*Td(TX \otimes \mathbb{C})ch(\sigma), [TX] \rangle$ .

If  $X$  is oriented, it is  $(-1)^{n+n(n-1)/2} \langle Td(TX \otimes \mathbb{C})\Phi_{TX}^{-1}(\sigma), [X] \rangle$ .

Axiom 1:  $a_{pt} : K^0(Tpt) \rightarrow \mathbb{Z}$  is the identity.

Axiom 2: If  $i : X \rightarrow Y$  an inclusion, then  $a_X(x) = a_Y(i_!(x))$ .

Axiom B1: If  $j : U \rightarrow X$  is an open subset and  $j' : U \rightarrow X'$  as an open subset, then  $a_X \circ j_* = a_{X'} \circ j'_*$ .

If we have this axiom, the  $a$  extends to open manifolds, as long as they embed as open subsets of closed manifolds.

We also see that for any vector bundle  $E \rightarrow X$ , that  $a_E : K^0(TE) \rightarrow \mathbb{Z}$  is defined.

Claim:  $a$  satisfies B1.

Let  $j : U \rightarrow X$  be an open inclusion and suppose  $x \in K^0(TU)$  is represented by a homogeneous  $0 \rightarrow \pi^*E \xrightarrow{\alpha} \pi^*F \rightarrow 0$ , with  $\alpha$  homogeneous of order 0.

We can assume that  $\alpha(x, \xi) = 1$  outside of a compact set  $K \subseteq U$ .

Off of  $K$ , we have  $E \rightarrow (U \setminus K \times \mathbb{C}^n)$  an isomorphism and  $F \rightarrow (U \setminus K \times \mathbb{C}^n)$  an iso.

Construct  $j_*(\pi^*E \xrightarrow{\alpha} \pi^*F)$ , that is, extending  $\pi^*E$  and  $\pi^*F$  via trivialization: making them trivial off  $U$ . Then  $\alpha$  extends to be 1 off  $U$ .

Construct  $P$  a  $\psi$ do of order 0 with symbol  $\alpha$  such that off  $K$   $P = \beta^{-1}1\alpha$ .

Let  $P'$  be the same, constructed on  $X$ . If  $u \in C^\infty(X; j_*E)$ , and  $P'u = 0$ , then  $\text{supp } u \subseteq U$ .

Since  $P'u = \beta^{-1}1\alpha u$  on  $X \setminus K$ , so if  $P'u = 0$  then  $u \in \ker P$ , so same for  $P^*$ . So  $IndP = IndP'$ .

## 22 Lecture 22

Let  $G$  be a compact group, then  $K_G^0(X)$  is the Grothendieck group of equivariant vector bundles. Define  $K_G^{-1}(X) \cong K_G^0(\sum X)$ . Bott Periodicity is still true.

If  $V$  is a complex  $G$ -representation, then  $K_G^0(X \times V) \cong K_G^0(X)$ .

$K_G^0(pt) = R(G)$ , and in general  $K_G^0(X)$  is a module over  $R(G)$ .

Facts:

$$K_G^0(G/H) = K_H^0(pt) = R(H).$$

If  $G$  acts freely on  $X$ , then  $K_G^0(X) \cong K^0(X/G)$ .

In general,  $K_G^*(X)$  is a combination of these. That is, there exists a spectral sequence  $E_2^{pq} = H^p(X/G, R^q) \Rightarrow K_G^*(X)$  where  $R^q = 0$  if  $q$  odd and  $R_x^q = R(G_{\bar{x}})$  for  $x \in X/G$  and  $\bar{x}$  maps to  $x$ .

Recall non-Equivariant K-Theory

We want a multiplicative formula for  $a$  and to calculate  $a(\lambda_E)$ . We will do the first.

Suppose that we have a principal  $G$ -bundle  $G \rightarrow P \rightarrow X$  and given a  $G$ -space  $F$ , we have  $Y = P \times_G F \rightarrow X$ . The action of  $G$  on  $F$  gives an action of  $G$  on  $TF$ , and so we can form  $P \times_G TF \rightarrow Y$  over  $X$ . Now, we get  $P \times_G TF \rightarrow TY$  which identifies with vectors tangent to  $F$ .

There exists a noncanonical isomorphism  $\pi^*TX \oplus P \times_G TF \cong TY$  by choosing a metric on  $Y$ .

This gives a product  $K^0(TX) \otimes K^0(P \times_G TF) \rightarrow K^0(TY)$ .

Define  $K^0(TX) \otimes K_G^0(TF) \rightarrow K^0(TX) \otimes K_G^0(P \times TF) = K^0(TX) \otimes K^0(P \times_G TF) \rightarrow K^0(TY)$ .

**Proposition 22.1.** *If  $x \in K^0(TX)$  and  $y \in K_G^0(TF)$  then  $a(x \cdot y) = a(x)a^G(y)$  provided that  $a^G(y)$  is an integer multiple of the trivial representation.*

*Proof.* Let  $x$  be represented by  $0 \rightarrow \pi^*E^0 \xrightarrow{\alpha} \pi^*E^1 \rightarrow 0$ . Let  $A$  be a  $\psi$ do elliptic of symbol  $\alpha$ . Trivialize  $P$ ,  $U_\alpha$  a cover of  $X$  and  $p^{-1}(U_\alpha) \cong U_\alpha \times G$ .

So  $Y$  also trivializes at  $U_\alpha$ .  $q^{-1}(U_\alpha) = U_\alpha \times F$ .

Let  $\phi_\alpha^2$  be a partition of unity.

Define  $A_\alpha = \phi_\alpha A \phi_\alpha$

Lift to  $q^{-1}(U_\alpha) = U_\alpha \times F$ . Define  $\tilde{A} = A \vee_G \left( \sum \tilde{A}_\alpha \right)$  since  $\sigma(\sum A_\alpha) = \sum \phi_\alpha^2 \sigma(A) = \sigma(A)$ , we have  $\sigma(\tilde{A}) = \tilde{\alpha}$ .

For  $y \in K_G^0(TF)$ , choose a  $G$ -invariant  $\psi$ do  $B$ , with symbol  $0 \rightarrow \pi^*F_0 \xrightarrow{\beta} \pi^*F_1 \rightarrow 0$ . Let  $\tilde{B}$  be a lift of  $B$  to  $P \times F$ . Since  $\tilde{B}$  is  $G$ -invariant, it induces an operator on  $P \times_G F$ .

Let's form  $D$  a matrix with first column  $\tilde{A}, \tilde{B}$  and section  $-\tilde{B}^*, \tilde{A}^*$ . Then  $\sigma(D) = \begin{pmatrix} \tilde{\alpha} & -\tilde{\beta}^* \\ \tilde{\beta} & \tilde{\alpha}^* \end{pmatrix}$  which takes  $\pi^*E_0 \oplus \pi^*F_0 \rightarrow \pi^*E_1 \oplus \pi^*F_1$  and  $\pi^*F_1^* \oplus \pi^*E_1^* \rightarrow \pi^*F_1 \oplus \pi^*E_0^*$ .

This represents  $\tilde{\alpha}\tilde{\beta} \in K^0(TY)$ .

We claim that  $\tilde{A}, \tilde{B}$  commute. Now,  $D * D$  is diagonal with  $A_0, B_0$  and  $DD^*$  is diagonal with  $A_1, B_1$ .

Now,  $\ker D = \ker D^*D$ , and  $\ker A_0 \oplus \ker B_0$ .

$\ker D^* = \ker DD^* = \ker A_1 \oplus \ker B_1$ .

So  $\text{Ind}D = \ker A_0 - \ker A_1 + \ker B_0 - \ker B_1$ . Consider  $A_0 = \tilde{A}^*A + \tilde{B}^*B$ . Then  $\langle A_0 u, u \rangle = \langle \tilde{A}u, \tilde{A}u \rangle + \langle \tilde{B}u, \tilde{B}u \rangle$ , so  $\ker A_0 \subseteq \ker \tilde{A} \cap \ker \tilde{B}$ .

Since  $\tilde{B}$  is an extension of  $B$  to the fibers of  $Y$ , we have  $\ker \tilde{B} = C^\infty(X; P \times_G \ker B)$ . We call this bundle  $K_B \rightarrow X$ .

Now  $\tilde{A}$  induces an operator  $C$  on  $K_B$ .  $\sigma(C) = \alpha \otimes 1_{K_B} \in K^0(TX)$  and  $\text{Ind}(C) = \alpha^X(x[K_B])$ . Similarly,  $L_B = P \times_G \text{coker } B$ , and we get  $\dim B_0 - \dim B_1 = \alpha^X(x[L_B])$ , and so  $\text{Ind}D = a(xy) = a(x([K_B] - [L_B])) = a(x)([K_B] - [L_B]) = a(x)a(y)$ .  $\square$

Look at  $K^0(\mathbb{C})$ . Bott element was represented by  $(z, \lambda) \mapsto (z, \lambda z d\bar{z})$ ,  $\mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C} \times \mathbb{C} d\bar{z}$ .

Form  $D = \bar{\partial} + z d\bar{z} : L^2(\mathbb{C}, \wedge^0 T^{0,1}) \rightarrow L^2(\mathbb{C}, \wedge^1 T^{0,1})$ , where  $\bar{\partial} = 1/2(\partial_x + i\partial_y) \wedge d\bar{z}$ .

Then  $\Delta = DD^* + D^*D$ . This works out to be just  $-\frac{\partial^2}{\partial z \partial \bar{z}} + |z|^2$ .

## 23 Lecture 23

## 24 Lecture 24

## 25 Lecture 25