Introduction

There will be no specific text in mind. Sources include Griffiths and Harris as well as Hartshorne. Others include Mumford's "Red Book" and "AG I- Complex Projective Varieties", Shafarevich's Basic AG, among other possibilities.

We will be taught to the test: that is, we will be being prepared to take orals on AG. This means we will de-emphasize proofs and do lots of examples and applications. So the focus will be techniques for the first semester, and the second will be topics from modern research.

1. Techniques of Algebraic Geometry
   (a) Varieties and Sheaves on Varieties
      i. Cohomology, the basic tool for studying these (we will start this early)
      ii. Direct Images (and higher direct images)
      iii. Base Change
      iv. Derived Categories
   (b) Topology of Complex Algebraic Varieties
      i. DeRham Theorem
      ii. Hodge Theorem
      iii. Kähler Package
      iv. Spectral Sequences (Specifically Leray)
      v. Grothendieck-Riemann-Roch
   (c) Deformation Theory and Moduli Spaces
   (d) Toric Geometry
   (e) Curves, Jacobians, Abelian Varieties and Analytic Theory of Theta Functions
   (f) Elliptic Curves and Elliptic fibrations
   (g) Classification of Surfaces
   (h) Singularities, Blow-Up, Resolution of Singularities
2. Problems in AG (mostly second semester)
   (a) Torelli and Schottky Problems: The Torelli question is "Is the map that takes a curve to its Jacobian injective?" and the Schottky Problem is "Which abelian varieties come from curves?"
   (b) Hodge Conjecture: "Given an algebraic variety, describe the subvarieties via cohomology."
   (c) Class Field Theory/Geometric Langlands Program: (Curves, Vector Bundles, moduli, etc) Complicated to state the conjectures.
   (d) Classification in Dim $\geq$ 3/Mori Program (We will not be talking much about this)

(e) Classification and Study of Calabi-Yau Manifolds

(f) Lots of problems on moduli spaces

  i. Moduli of Curves (See Angela Gibney's Course)

  ii. Moduli of Bundles (related to Geometric Langlands)

Assume knowledge of manifolds and cohomology and DeRham theory, cup product, some Sheaf theory, some Kähler manifolds. From algebra, rings and modules, projective modules, etc.

1 Varieties and Sheaves over $\mathbb{C}$

1.1 First Definitions

Definition 1.1 (Affine Variety). *Affine Space $\mathbb{A}^n = \mathbb{C}^n$ as a topological space. An affine variety $Z \subset \mathbb{A}^n$ is the subset where a collection of polynomials vanish.*

Example 1.1. $\mathbb{A}^m \subset \mathbb{A}^n$ *for* $m < n$

Example 1.2 (Quadrics). *A quadric $Q \subset \mathbb{A}^n$, the set of points $\{\xi | q(\xi) = 0\}$ for q a quadratic polynomial.*

If $n = 1$, then the 0-dimensional quadric is either two points or a double point in $\mathbb{A}^1$, these are the cases where $q(x) = x^2 - 1$ and $q(x) = x^2$.

If $n = 2$, we get a conic over $\mathbb{R}$, which are the cases of the ellipse, parabola, hyperbola, two lines, line, point, and the empty set which correspond to $4x^2 + y^2 - 1$, $y - x^2$, $x^2 - y^2 - 1$, $x^2 - y^2$, x^2, $x^2 + y^2$ and $x^2 + y^2 + 1$. Over $\mathbb{C}$, however, the ellipse, hyperbola and the empty set all become the same thing smooth conic, $S^2 \setminus \{2 \text{ points}\}$. The parabola is $S^2 \setminus \{pt\}$. There still remain the singular conic (pair of lines/point) and the double line

Example 1.3 (Affine Plane Curves). *Let $C \subset \mathbb{A}^2$ be the set of points $\{(x, y) \in \mathbb{A}^2 | f(x, y) = 0\}$ where f is polynomial in x, y of some degree. Assume that C is nonsingular, that is, $f = \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} = 0$ have no common solutions. So C is a complex submanifold of $\mathbb{A}^2$.*

In degree 1 there is 1, in degree 2 there are 2, in degree 3 the question is more complicated.

Definition 1.2 (Projective Space). *Projective space $\mathbb{P}^n$: as a set, $\mathbb{P}^n = \mathbb{A}^{n+1} \setminus \{0\}/\sim$, where $(x_0, \ldots, x_n) \sim (\lambda x_0, \ldots, \lambda x_n)$ where $\lambda \in \mathbb{C}^*$. That is, the set of lines through 0 in $\mathbb{A}^{n+1}$.*

Alternate description $\mathbb{P}^n = \coprod_{i=0}^n U_i / \sim$ where $U_i \simeq \mathbb{A}^n$ and then glued together.

We will be working over $\mathbb{C}$. We should think of $\mathbb{A}^n = \mathbb{C}^n$ not as a linear space but as an algebraic manifold.

Definition 1.3 (Affine Algebraic Manifold). *An affine manifold is a subset $X \subset \mathbb{A}^n$ defined by polynomial equations $\{f_i = 0\}$ which is nonsingular. That is, it is a complex analytic submanifold of $\mathbb{A}^n$.*

Definition 1.4 (Projective Algebraic Manifold)**.** *A projective manifold is a complex analytic submanifold realized as a subset of* $\mathbb{P}^n$ *given by* $\{F_i(x) = 0\}$ *where* F_i *are homogeneous polynomials.*

Lemma 1.1. *The union of two disjoint affine submanifolds of* $\mathbb{A}^n$ *is an affine submanifold.*

Proof. The union $X \cup Y$ with X defined by f_i and Y defined by g_i, is defined by $\{f_i g_j\}$. □

Definition 1.5 (Projective Variety)**.** *A projective variety is* $X \subset \mathbb{P}^n$ *for some* n *given by* $F_i = 0$ *for some collection of homogeneous polynomials in* $n+1$ *variables.*

Now that we have objects, we need to ask about maps. In particular, what sort of equivalence do we want? We could use diffeomorphism, equivalence of complex manifolds, or we could attempt to define equivalence of complex varieties algebraically. We will use all of these. The algebraic notion is defined as follows:

Definition 1.6 (Morphism)**.** *A morphism of varieties* $f : X \to Y$ *is a continuous map which is given locally coordinatewise by rational functions.*

An isomorphism of varieties is a morphism with a two-sided inverse.

We will refine this notion a bit later, once we have the notion of a locally ringed space.

Now consider the set of all affine cubics. This is $\mathbb{A}^{10}$, or better $\mathbb{P}^9$, because f and af define the same variety for $a \neq 0$.

Theorem 1.2 (Existence of the j-invariant)**.** *There exists an explicit morphism* $j : \mathbb{P}^9 \to \mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$ *such that* $f = 0$ *(a projective plane cubic) is nonsingular iff* j *is finite and* $f_1 = 0$ *is isomorphic to* $f_2 = 0$ *iff* $j(f_1) = j(f_2)$*.*

If f *is taken to be an affine plane cubic, then there are three such equations satisfying the same conclusion.*

Proof. Theorem IV.4.1 in Hartshorne. □

We will talk more about the j-invariant later. It leads to parameter spaces and moduli spaces. The parameter space is all possible objects (ie, $\mathbb{A}^{10}$), and the moduli space is roughly the set of isomorphism classes which is also an algebraic variety, in this case, $\mathbb{A}^1$.

The biggest difference between varieties and algebraic manifolds is that varieties may be singular:

Definition 1.7 (Singular)**.** *Let* V *be a variety define by polynomials* $f_1, \ldots, f_k$ *(in some affine or projective space). Then* $p \in V$ *is singular iff* $f(p) = \frac{\partial f}{\partial x_i}(p) = 0$ *for all* i*. Otherwise,* p *is nonsingular.*

In fact, we now come naturally to the notion of reducibility versus irreducibility:

Definition 1.8 (Reducible)**.** *A variety V is reducible if $V = V_1 \cup V_2$ for V_1, V_2 both proper subvarieties of V. Otherwise it is irreducible.*

A variety is locally reducible if on some open subset (often in the complex topology) it is reducible.

Note that any reducible variety is locally reducible.

Example 1.4 (Singularity Types)**.**

1. *$xy = 0$, the union of two lines. This is globally reducible.*

2. *$y^2 = x^2(x-1)$, the nodal cubic. This is globally irreducible, but locally reducible.*

3. *$y^2 = x^3$, the cuspidal cubic. Take a map $\mathbb{C}^2 \to \mathbb{P}^1$ by $(x, y) \mapsto y/x$. If we throw away the origin, this maps the cubic to $\mathbb{C} \setminus \{0\}$, and the map extends continuously but not algebraically to the origin to make $C \simeq \mathbb{C}$ as topological spaces.*

As an unfortunate fact of language, there is a notion that something be reduced, which is very different from being reducible.

Definition 1.9 (Reduced)**.** *An ideal $I \subset \mathbb{C}[x_1, \ldots, x_n]$ is said to be reduced (or radical) if $\mathbb{C}[x_1, \ldots, x_n]/I$ has no nilpotent elements.*

Look at the intersection of $y = x^2$ and $y = 0$. This is a nonreduced intersection, because $V(y - x^2) \cap V(y) = V(y - x^2, y) = V(x^2)$.

Roughly speaking, an affine or projective variety is a subset of $\mathbb{A}^n$ or $\mathbb{P}^n$ defined by an ideal which is reduced and irreducible. (Here, irreducibility is a new requirement. We will revise the old definition, without the irreducible requirement, to be an algebraic set, rather than a variety.)

Example 1.5. *In* $\dim = 0$*, a single point is a variety, 2 points are a variety, but a double point is not.*

Example 1.6. *In* $\dim = 1$*, a line is, a smooth cubic is, a nodal cubic is, a cuspidal cubic is, unions of lines intersecting are not.*

Roughly speaking once again, affine varieties correspond to Stein spaces and projective varieties correspond to compact manifolds.

Definition 1.10. *Let X be a topological space. Then we define a category with objects the open sets of X and morphisms $U \to V$ corresponding specifically to $U \subset V$.*

1.2 Sheaves

We need a good way of collecting local data. The next two definitions give us this:

Definition 1.11 (Presheaf)**.** *A presheaf on a topological space is a contravariant functor on the above mentioned category.*

Definition 1.12 (Sheaf). *A sheaf of abelian groups on a topological space is a presheaf $\mathscr{F}$ satisfying an additional two axioms:*

1. *If U is an open set and $\{V_i\}$ is an open cover of U, $s \in \mathscr{F}(U)$ is an element such that $s|_{V_i} = 0$ for all i, then $s = 0$.*
2. *If U is an open set and $\{V_i\}$ is an open cover of U, if $s_i \in \mathscr{F}(V_i)$ for each i with the property such that for each i, j, we have $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$ then there exists $s \in \mathscr{F}(U)$ such that $s|_{V_i} = s_i$ for all i.*

We think of the first of these conditions as uniqueness and the second as existence.

Example 1.7. *Let $\mathscr{F}(U) = 0$ if $U = \emptyset$ and $= \mathbb{C}$ else. This is a presheaf but not a sheaf. The sheafification (this is defined in Hartshorne, II.1.2) of this presheaf is the sheaf of locally constant functions.*

Definition 1.13 (Subsheaf). *$\mathscr{S} \subset \mathscr{F}$ is a subsheaf iff for all U, $\mathscr{S}(U) \subset \mathscr{F}(U)$ is a subobject.*

Definition 1.14 (Quotient). *A quotient sheaf $\mathscr{F} \to \mathscr{Q}$ is a sheaf such that for all U, $\mathscr{Q}(U)$ is a quotient of $\mathscr{F}(U)$.*

Let $i : U \to X$ be an open inclusion of topological spaces, then if $\mathscr{F}$ is a sheaf on X, then we define $i^*\mathscr{F} = \mathscr{F}|_U$, which is a sheaf on U.

In the other direction, we take a sheaf $\mathscr{G}$ on U and $i_*\mathscr{G}(W) = g(U \cap W)$ with $W \subset X$ open. We could also do extension by zero, which is $\mathscr{G}(U \cap W) = \mathscr{G}(W)$ if $W \subset U$ and otherwise is 0. Extension by zero is denoted by $i_!\mathscr{G}$

Example 1.8. *Let $X = \mathbb{A}^1$ and $U = \mathbb{A}^1 \setminus \{0\}$ with $\mathscr{F} = \mathscr{O}_X$ on $\mathbb{A}^1$. Then $i^*\mathscr{F} = \mathscr{F}|_U = \mathscr{O}_U$. $i_*\mathscr{O}_U(W)$ =holomorphic functions on $W \setminus \{0\}$, and allow arbitrary singularities at the origin. However, $i_!\mathscr{O}_U(W) = 0$ if $0 \in W$, and else the holomorphic functions on W.*

Quotients are not necessarily sheaves, and so we need sheafification and cohomology theories.

Definition 1.15 (Pushforward and Pullback). *Let $f : X \to Y$ be a morphism of varieties, $\mathscr{F}$ a sheaf on X and $\mathscr{G}$ a sheaf on Y. Then the pushforward or direct image of a sheaf is $f_*\mathscr{F}(U) = \mathscr{F}(f^{-1}(U))$, and the pullback is $f^*\mathscr{G}(W) = \mathscr{G}(f(W))$.*

For this, we need to make sense of $\mathscr{G}(V)$ where $V \subset Y$ is a non-open subset. Do it by defining $\mathscr{G}(V) = \varinjlim_{U \supset V} \mathscr{G}(U)$.

In particular, $\mathscr{G}_p = \varinjlim_{U \ni p} \mathscr{G}(U)$ is the stalk of $\mathscr{G}$ at p, which consists of germs of elements of $\mathscr{G}$ at p.

Definition 1.16 (Ringed Space). *A ringed space $(X, \mathscr{O})$ is a pair with X a topological space and $\mathscr{O}$ a sheaf of rings.*

A locally ringed space is a ringed space such that the stalks are local rings.

Example 1.9. *So a C^∞ manifold is a locally ringed space locally isomorphic to* $(\mathbb{R}^n, C^\infty)$*.*

Definition 1.17 (Algebraic Variety)**.** *An algebraic variety is a locally ringed space that is locally isomorphic to an affine variety.*

Definition 1.18 (Scheme)**.** *A scheme is a locally ringed space that is locally isomorphic to* $\operatorname{Spec} R$ *for some ring* R*.*

Next we look at divisors and line bundles. To do so, we need the following definition:

Definition 1.19 (Sheaf of Modules)**.** *Let* $(X, \mathscr{O}_X)$ *be a locally ringed space. Then a sheaf of* $\mathscr{O}_X$*-modules* $\mathscr{F}$ *is a sheaf consisting of* $\mathscr{O}_X(U)$*-modules* $\mathscr{F}(U)$ *for each open set, with the restriction maps homomorphisms of modules compatible with the change of rings.*

We say that an $\mathscr{O}_X$*-module is locally free of rank* n *if there exists an open cover* U_i *of* X *such that* $\mathscr{F}|_{U_i} = \mathscr{O}_{U_i}^{\oplus n}$ *for each* i*.*

We can perform all the usual construction involving modules with $\mathscr{O}_X$-modules, by performing them on each open set and then sheafifying.

1.3 Invertible Sheaves and Divisors

We will focus on locally free sheaves of rank 1. These are called invertible, and this is justified as follows:

Proposition 1.3. *Let* $\mathscr{L}$ *be a locally free sheaf of rank one on a locally ringed space* $(X, \mathscr{O}_X)$*. Then there exists* $\mathscr{G}$ *a locally free sheaf of rank one such that* $\mathscr{F} \otimes \mathscr{G} \simeq \mathscr{O}_X$*.*

Proof. Exercise. □

Definition 1.20. *Given* $D \subset X$*, we take* $\mathscr{O}(D)(U) = \{f \in \mathscr{O}(U \setminus D) | \text{poles are of order} \leq 1 \text{ on } D\}$*. This is a sheaf*

Example 1.10. $D = \emptyset$*, then* $\mathscr{O}(D) = \mathscr{O}$*.*

Example 1.11. *If* $X = \mathbb{A}^1$*, and* $D = (0)$*, then* $U = \mathbb{A}^1 \backslash \{0\}$*, and so* $\mathscr{O}(D)(U) = \{f(x)/x^k | f \in \mathscr{O}(U), k \in \mathbb{N}\}$*.*

Now we will define subsets (in fact, with multiplicities) such that $\mathscr{O}(D)$ is particularly well behaved.

Definition 1.21 (Prime Divisor)**.** *A prime divisor is a codimension 1 subvariety.*

These are analogous to primes in number theory, as the prime numbers are the codimension 1 irreducible subsets of $\operatorname{Spec} \mathbb{Z}$.

Definition 1.22 (Weil Divisor)**.** *A Weil divisor on* X *is an element of the free abelian group generated by the prime divisors. We denote this group by* $\operatorname{Div} X$*.*

Definition 1.23 (Cartier Divisor)**.** *A divisor is Cartier if it is locally defined by a single equation.*

On a smooth variety, any Weil divisor is Cartier, on a singular variety, things are more complicated.

If $D = \sum n_i D_i$ is a divisor, then we define $\mathscr{O}(D)$ to be the sheaf of functions with poles of order at worst n_i on D_i.

Proposition 1.4. *Let D be a Weil divisor. Then $\mathscr{O}(D)$ is an invertible sheaf if and only if D is Cartier, and $\mathscr{O}(-D)$ is its inverse.*

Proof. Exercise. □

We have the following corollary:

Corollary 1.5. *The set of isomorphism classes of invertible sheaves on a variety X is a group, which we shall denote by* $\mathrm{Pic}(X)$*.*

We will often call locally free sheaves vector bundles and, in particular, invertible sheaves will be called line bundles.

In the future, we will talk about how to take a line bundle and get a divisor, along with equivalence relations on divisors.

We should now introduce the notation $\Gamma(U, \mathscr{F})$ for $\mathscr{F}(U)$. This is an important piece of notation because it makes it easier to write the global section functor $\Gamma(X, -)$.

1.4 The Rational map associated to a Line Bundle

First we need to know what a rational map is.

Definition 1.24 (Rational Map)**.** *A rational map $f : X 1 Y$ is a morphism $U \to Y$ for some open $U \subset X$*

Now, let X be a variety and $\mathscr{L}$ be an invertible sheaf on X. Then $\Gamma(X, \mathscr{L})$ is a vector space, which, for reasons that will be clear later, we sometimes denote by $H^0(X, \mathscr{L})$. Define $h^0(X, \mathscr{L})$ to be the dimension of this vector space. Then we get a morphism $X \setminus B_{\mathscr{L}} \to \mathbb{P}^{h^0-1}$, that is, a rational map $X \to \mathbb{P}^{h^0-1}$.

The crude way to find this map is to choose a basis $s_1, \ldots, s_{h^0}$ for $H^0(X, \mathscr{L})$ (we are implicitly assuming h^0 is finite, which it is, but we have not proved this) and define $x \mapsto (s_1(x) : \ldots : s_{h^0}(x))$ where defined, by also choosing an isomorphism $\mathbb{C} \simeq \mathscr{L}_x$, to get an $(h^0 - 1)$-tuple of numbers.

As changing this isomorphism results in multiplication by a nonzero scalar, the point of $\mathbb{P}^{h^0-1}$ is independent of the isomorphism. We define $B_{\mathscr{L}} = \{x \in X | s(x) = 0 \text{ for all sections } s\}$, and so we get a map $\varphi_{\mathscr{L}} : X \setminus B_{\mathscr{L}} \to \mathbb{P}^{h^0-1}$.

Example 1.12 (Veronese Map)**.** *Let $X = \mathbb{P}^n$ and $D = dH$ for H a hyperplane. Then the morphism $\varphi_{\mathscr{O}(D)}$ is called the Veronese map, and has $B_{\mathscr{O}(D)} = \emptyset$ for $d \geq 1$.*

Example 1.13 (Segre Map). *Let $X = \mathbb{P}^r \times \mathbb{P}^s$, and $D = H_1 + H_2$ where H_1 is a hyperplane in $\mathbb{P}^r$ and H_2 is a hyperplane in $\mathbb{P}^s$, that is, $\mathscr{O}(D) = \pi^* \mathscr{O}_{\mathbb{P}^{n_1}}(H_1) \otimes \pi_2^* \mathscr{O}_{\mathbb{P}^{n_2}}(H_2)$. Then $\mathscr{B}_{\mathscr{O}(D)} = \emptyset$. The map $\varphi_{\mathscr{O}(D)}$ is called the Segre Map,*

A more natural definition of $\varphi_{\mathscr{L}}$ is to also give it dependency on $V \subset \Gamma(X, \mathscr{L})$ and call it $\varphi_{\mathscr{L},V} : X \setminus B_{\mathscr{L},V} \to \mathbb{P}(V^*)$ by $x \mapsto \{s \in V | s(x) = 0\}$.

The most important sheaves, in practice, have the following nice property:

Definition 1.25 (Coherent Sheaf). *A sheaf $\mathscr{F}$ of $\mathscr{O}_X$-modules is coherent if it is "finitely presented." That is, there is an exact sequence $\mathscr{L}_1 \to \mathscr{L}_0 \to \mathscr{F} \to 0$ where $\mathscr{L}_i$ is a locally free sheaf of finite rank.*

Under nice circumstances which will hold for this course, coherence is equivalent to $\forall x \in X$, there exists $U \ni x$ with $\mathscr{O}_U^{\oplus m} \to \mathscr{O}_U^{\oplus n} \to \mathscr{F}|_U \to 0$ is exact. We will generally use this version.

Example 1.14 (Coherent Sheaves). *1. $\mathscr{O}_X$.*

2. Any locally free sheaf of finite rank

3. If $\mathscr{F}, \mathscr{G}$ are coherent, so is $\mathscr{F} \oplus \mathscr{G}$ and $\mathscr{F} \otimes \mathscr{G}$.

4. If $i : Y \to X$ is an inclusion of a subvariety, and $\mathscr{F}$ is a coherent sheaf on Y, then $i_* \mathscr{F}$ is coherent.

In contrast to coherent sheaves, a sheaf is quasicoherent if it is coherent or if it is "infinitely generated" in some sense (that we will not define at the moment).

It is a fact that direct images only preserve quasi-coherence in general, but that if the map is proper, then it preserves coherence. (Proper means that inverse image of compact is compact)

2 Cohomology

2.1 Čech Cohomology

We want to define $H^i(X, \mathscr{F})$ for any sheaf of abelian groups $\mathscr{F}$. Because we will be making use of it in the future, we introduce the following:

Definition 2.1 (Constant Sheaf). *Let X be a topological space and let A be an abelian group. Then $\underline{A}$ denotes the constant sheaf, which is the sheafification of the constant presheaf. This is the sheaf of locally constant functions $X \to A$.*

We want our definition to gives us that $H^i(X, \underline{A}) = H^i(X, A)$, as defined in algebraic topology.

The standard way of defining a cohomology theory for some object is to associate to the object a chain complex, and then take the cohomology of this complex.

Definition 2.2 (Čech Cohomology of a Cover). *Let X be a topological space and $\mathscr{F}$ a sheaf of abelian groups on X. Let $\mathcal{U}$ be an open cover with ordered index set I.*

Define $C^p(\mathcal{U}, \mathscr{F}) = \prod_{i_1,\ldots,i_p \in I} \mathscr{F}(U_{i_1,\ldots,i_p})$. We define $d : C^p \to C^{p+1}$ by $(d\alpha)_{i_0,\ldots,i_p} = \sum_{k=0}^{p} (-1)^d \alpha_{i_0,\ldots,\hat{i_k},\ldots,i_p}|_{U_{i_0,\ldots,i_p}}$.

This gives the Čech complex, and so we get notions of Čech cocycles and coboundaries, and so we define Čech Cohomology of the sheaf and the cover to be cocycles modulo coboundaries.

Definition 2.3 (Čech Cohomology). *If $\mathcal{V}$ is a refinement of $\mathcal{U}$, we say that $\mathcal{V} > \mathcal{U}$, and this defines a poset. If $\mathcal{V} > \mathcal{U}$ then we get a map $C^*(\mathcal{U}, \mathscr{F}) \to C^*(\mathcal{V}, \mathscr{F})$, which in fact induces a map $H^p(\mathcal{U}, \mathscr{F}) \to H^p(\mathcal{V}, \mathscr{F})$. Taking the direct limit, we define $\check{H}^*(X, \mathscr{F}) = \varinjlim_{\mathcal{U}} H^*(\mathcal{U}, \mathscr{F})$.*

Proposition 2.1. *Let A be an abelian group. Then $\check{H}^*(X, \underline{A}) = H^*(X, A)$.*

Proof. Omitted. □

So when is a cover U good enough? Direct limits are hard to compute, so we would like to know when $H^p(\mathcal{U}, \mathscr{F}) \to \check{H}^p(X, \mathscr{F})$ is an isomorphism.

Proposition 2.2. *If X is a variety and $\mathscr{F}$ is quasi-coherent, then if $U_{i_0,\ldots,i_p}$ is always affine, then $H^p(\mathcal{U}, \mathscr{F}) \to \check{H}^p(X, \mathscr{F})$ is an isomorphism.*

To get a better understanding of this, we need to look at acyclicity in general. Now we pause to note a common element of all cohomology theories, in the form we will be using:

Theorem 2.3 (Mayer-Vietoris Sequence). *Let $0 \to \mathscr{A} \to \mathscr{B} \to \mathscr{C} \to 0$ be a short exact sequence of sheaves. Then there exists a long exact sequence $0 \to H^0(X, \mathscr{A}) \to H^0(X, \mathscr{B}) \to H^0(X, \mathscr{C}) \to H^1(X, \mathscr{A}) \to \ldots$.*

2.2 Derived Functor Cohomology

Čech Cohomology is good for computations, but for theoretical purposes, there is a different cohomology theory that is often better. We will assume familiarity with abelian categories, which are categories that behave like the category of abelian groups.

Definition 2.4 (Left Exact Functor). *A functor $F : \mathcal{C} \to \mathcal{D}$ of abelian categories is left exact if $0 \to A \to B \to C \to 0$ short exact implies that $0 \to FA \to FB \to FC$ is exact.*

Example 2.1 (Global Sections). *The functor $\Gamma(X, -)$ from sheaves on X to abelian groups is a left exact functor.*

So we will define right derived functors, which fix the failure of exactness, though this will take some effort.

Definition 2.5 (Injective Objects). *An object I in an abelian category is injective if* $\hom(-, I)$ *is an exact functor.*

A category $\mathcal{C}$ has enough injectives if for each object A, there exists an injective object I and an exact sequence $0 \to A \to I$.

This implies that every object has an injective resolution, that is:

Lemma 2.4 (Injective Resolutions). *Let A be an object in a category with enough injective. Then there are injective objects I_j and an exact sequence $0 \to A \to I_0 \to I_1 \to I_2 \to \ldots$.*

Definition 2.6 (Right Derived Functor). *Let $F : \mathcal{C} \to \mathcal{D}$ be a left exact functor and $\mathcal{C}$ have enough injectives. Then we define the right derived functors R^iF to be obtained as follows:*

Let A be an object of $\mathcal{C}$. Then take an injective resolution $0 \to A \to I^*$ of A. Then $R^iF(A) = H^i(I^*)$.

The right derived functors satisfy the Mayer-Vietoris Sequence, and so behave as we would hope a cohomology theory should.

Proposition 2.5. *The category of abelian groups and the category of sheaves of abelian groups both have enough injectives.*

Proof. Omitted. □

Definition 2.7 (Sheaf Cohomology). *Let $H^i(X, \mathscr{F})$ to be the ith derived functor of the global section functor.*

And so now we have a second definition of cohomology.

Definition 2.8 (Acyclic). *An object is acyclic if nonzero cohomology all vanishes.*

The following lemma is very helpful in actually using derived functor cohomology:

Lemma 2.6. *Injective objects are acyclic.*

Flasque sheaves are acyclic for $\Gamma(X, -)$. (A sheaf is flasque if the restriction maps are all surjective.)

Example 2.2. *Injective sheaves are flasque.*

Theorem 2.7 (Leray). *If $\mathcal{U}$ is an acyclic cover, then the natural map from $H^*(\mathcal{U}, \mathscr{F})$ to $\check{H}^*(X, \mathscr{F})$ is an isomorphism.*

Example 2.3. *Hartshorne showed that algebraically, affine varieties are acyclic (when looking at quasi-coherent sheaves)*

Example 2.4. *Analytically, we have that Stein manifolds are acyclic.*

Note that there are two different notions of acyclic here, though they are closely related. A space is acyclic if quasicoherent sheaves on it are acyclic as sheaves.

The analytic analogue of flasque is "fine", that is, a sheaf which always has partitions of unity.

Definition 2.9 (Partition of Unity)**.** *A sheaf of abelian groups has partitions of unity if for all open covers of open subsets, any $s \in \Gamma(U)$ can be written as $\sum s_i$ for $s_i \in \Gamma(V_i)$ with $\bar{V}_i \subset U_i$.*

Example 2.5. *For X any manifold, true for sheaf of C^∞ functions but not analytic or algebraic.*

Next we note the following:

Theorem 2.8. *Let M be a topological space, and $\mathcal{U}$ an acyclic cover for $\mathscr{F}$. Then $\check{H}^i(\mathcal{U}, \mathscr{F}) \simeq H^i(M, \mathscr{F})$*

And finally, we will move toward a statement of the Serre Duality Theorem.

On any X a nonsingular variety $\dim X = n$, there is a vector bundle of rank n, the tangent bundle TX. This defines a locally free sheaf of rank n, $\mathscr{T}$. Then $\mathscr{H}om(\mathscr{T}, \mathscr{O}) = \Omega^1$ is the sheaf associated to the cotangent bundle. We define $\Omega^i = \bigwedge^i \Omega^1$.

Definition 2.10 (Canonical Bundle)**.** *Let X be a nonsingular variety of dimension n. Then the canonical bundle $K_X = \bigwedge^n \Omega^1$.*

The canonical bundle is a line bundle.

Theorem 2.9 (Serre Duality Theorem)**.** *Let X be a nonsingular variety of dimension n and $\mathscr{L}$ a line bundle. Then $H^n(X, \mathscr{L}) \simeq H^0(X, \mathscr{L}^{-1} \otimes K_X)^*$.*

2.3 Cohomology through Forms

As we are working over $\mathbb{C}$, we have an additional option for defining the cohomology of our spaces.

Definition 2.11 (deRham Cohomology)**.** *Let X be a real manifold. Then the deRham Complex is $\mathscr{A}^0 \to \mathscr{A}^1 \to \ldots$ the complex of sheaves of C^∞ p-forms with the exterior derivative. The cohomology of this complex (as a complex, not as sheaves) is the deRham Cohomology of X, denoted $H^*_{dR}(X)$.*

The following is extremely important:

Lemma 2.10 (Poincaré)**.** *If $X = \mathbb{R}^n$, then the sequence $\mathscr{A}^0 \to \mathscr{A}^1 \to \ldots$ is quasi-isomorphic to the sequence $\underline{\mathbb{R}} \to 0 \to 0 \to \ldots$.*

All quasi-isomorphism means is that there exists a map of complexes which induces isomorphisms on cohomology groups. This is stronger than merely having identical cohomology groups.

A consequence of Poincaré's lemma, with quite a bit of additional work is

Theorem 2.11 (deRham)**.** *Let X be a C^∞ real manifold, then $H^n(X, \underline{\mathbb{R}}) \simeq H_{dR}(X)$*

More generally, to a complex of sheaves C^* we will assign its hypercohomology $\mathbb{H}^*(X, C^*)$, but not yet. In the meantime, we will state several properties of hypercohomology:

Proposition 2.12.

1. *If C^* is just a single nonzero sheaf, then $\mathbb{H}^*(X, C^*) = H^*(X, \mathscr{F})$. If the C^i are acyclic, then $\mathbb{H}^i(X, C^*) = H^i(\Gamma(C^i))$.*
2. *If $C_1^* \to C_2^*$ is a quasi-isomorphism, then $\mathbb{H}(C_1) = \mathbb{H}(C_2)$.*

Now we look at Dolbeault Cohomology.

Let X be a complex manifold. Now define a sequence with $\mathscr{A}^p$ the holomorphic p-forms.

Lemma 2.13. *The sequence $\mathscr{A}^0 \to \mathscr{A}^1 \to \dots$ is quasi-isomorphic to $\mathscr{O} \to 0 \to 0 \to \dots$.*

More generally, we define $\mathscr{A}^{p,q}$ to be the sheaf of p-holomorphic and q-antiholomorphic forms. Then $\mathscr{A}^{p,0} \to \mathscr{A}^{p,1} \to \dots$ with $\bar{\partial}$ is quasi-isomorphic to Ω^p.

Theorem 2.14 (Dolbeault's Theorem)**.** *Let $H^{p,q}_{\bar{\partial}}(X) = H^q(\Gamma(\mathrm{Dol}^p))$ where Dol^p is the sequence defined above. Then there is an isomorphism $H^q(X, \Omega^p) \to H^{p,q}_{\bar{\partial}}(X)$.*

Contrast this with the Hodge Theorems. Let M be a compact, complex, Hermitian manifold, then the Laplacian is $\Delta = \bar{\partial}\bar{\partial} * + \bar{\partial} * \bar{\partial}$. Define a harmonic (p, q) form to be $\alpha \in \mathscr{A}^{p,q}$ with $\Delta\alpha = 0$.

Theorem 2.15 (Hodge)**.** *Every $\bar{\partial}$ cohomology class contains a unique element of shortest norm (the norm is induced from Tangent Bundle to Cotangent Bundle to forms), and this element is harmonic.*

Also, $\mathrm{Har}^{p,q}(M) \to H^{p,q}_{\bar{\partial}}(M)$ is an isomorphism of finite dimensional vector spaces.

There exists a similar real Hodge Theorem, but we are mostly concerned with the complex case.

Corollary 2.16. *If X is a real Riemannian manifold, then $H^p(X, \mathbb{R})$ can be computed by restricting to harmonic p-forms.*

Similarly, if X is complex.

A more powerful version is

Theorem 2.17. *If X is a compact Kähler manifold (a complex Riemannian manifold with a symplectic form such that all three structures are compatible) then we have the decomposition $H^n(X, \mathbb{C}) = \oplus_{p+q=n} H^{p,q}(X) = H^{p,q}_{\bar{\partial}}(X) = \mathrm{Har}^{p,q}(X)$.*

Additionally, $H^{p,q}(X) = \overline{H^{q,p}}(X)$.

Example 2.6. *If X is a curve of genus g, then $H^0(X,\mathbb{C}) = \mathbb{C} = H^{0,0}$*
$\mathbb{C}^{2g} = H^{1,0} \oplus H^{0,1}$, and $\mathbb{C} = H^2(X,\mathbb{C}) = H^{1,1} = H^1(\Omega^1) = H^1(K)$.
So the interesting things happen in $H^{0,1}$ and $H^{1,0}$.
$H^{1,0} = H^0(\Omega^1) = H^0(K)$ and $H^{0,1} = H^1(\mathscr{O})$ and Serre Duality says that they are dual and so have the same dimension, and so $H^{1,0}$ and $H^{0,1}$ have dimension g.
So this says that if you vary the complex structure on a Riemann surface, the holomorphic and antiholomorphic parts of H^1 change.

On a general Hermitian manifold, $\nexists$ a relation between the $\Delta_{\bar\partial}$ and Δ_d. The assumption you need is that the manifold is Kähler. Then $\Delta_{\bar\partial} = \Delta_\partial = \frac{1}{2}\Delta_d$.

Example 2.7. *A projective manifold is Kähler.*

Corollary 2.18. *$H^n(M,\Omega^n) = H^{n,n} \simeq \mathbb{C}$ where $n = \dim M$.*

We need to note that Δ commutes with $*$ and $\bar\partial$, where $*$ is the Hodge star.

Corollary 2.19. *$H^q(M,\Omega^p) \times H^{n-q}(M,\Omega^{n-p}) \to H^n(M,\Omega^n)$ is a perfect pairing.*

That we have the Künneth Formula also follows from this.

Corollary 2.20. *$b_{2k+1}(M)$ is even.*
$b_{2k}(M) > 0$.

Proof. $\sum_{p=0}^{2k+1} h^{p,2k-p} = 2\sum_{p=0}^{b} h^{p,2k+1-p}$. ☐

To write all of this information, we use the Hodge Diamond.

$$
\begin{array}{ccccc}
 & & h^{0,0} & & \\
 & h^{1,0} & & h^{0,1} & \\
\vdots & & \vdots & & \vdots \\
 & h^{n,n-1} & & h^{n-1,n} & \\
 & & h^{n,n} & &
\end{array}
$$

Not all Hodge diamonds occur, there are constraints.
$H^{p,q}(\mathbb{P}^n)$? Well, $H^{p,p}(\mathbb{P}^n) = \mathbb{C}$ and else 0.
What is $H^{p,q}$ for a Riemann Surface of genus g?

Definition 2.12 (Calabi-Yau Manifold)**.** *A three dimensional complex manifold is Calabi-Yau if $\Omega^n = K = \mathscr{O}$. Sometimes we also assume that $h^{p,0} = 0$ for $0 < p < n$.*

If we combine this with Serre Duality, we find the hodge diamond:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & a & & 0 & \\ 1 & & b & & b & & 1 \\ & 0 & & a & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

Mirror Symmetry says that if there is a Calabi-Yau manifold with a, b, then there is one with b, a.

2.4 Algorithm for the Cohomology of a smooth hypersurface

Fix the topological type of M. Vary the complex structure by a smooth and proper map $M \to B$, with fibers M_b. Then $H^i(M_b, \mathbb{C})$ are locally independent of b.

The family of cohomologies $H^i(M_b, \mathbb{C})$ forms a vector bundle over B called $R^i\pi_*\underline{\mathbb{C}} = V$, the higher direct images of $\underline{\mathbb{C}}$.

This vector bundle has a natural flat connection $\nabla : V \to V \otimes \Omega^1$ called the Gauss-Manin Connection. Eg, $V = V_{\mathbb{Z}} \otimes \mathbb{C}$ where $V_{\mathbb{Z}} = R^i\pi_*\underline{\mathbb{Z}}$.

How do the $H^{p,q}$ vary? $H^{p,0}$ varies holomorphically. But $H^{p,q}$ doesn't if $q > 0$.

Let $F^i = \oplus_{p+q=n, p\geq i} H^{p,q}$. Then we have a filtration of $F^0 = H^n$.

We will find an algorithm to calculate $h^{p,q}$ of a hypersurface, $H^*(\mathbb{P}^r, \mathscr{O}(n))$ and dR, Dolbeault and Čech cohomology

<u>Cohomology of a Hypersurface</u>

We will need a topological result, the Lefschetz hyperplane theorem.

Theorem 2.21 (Lefshetz Hyperplane)**.** *Let $Y \subset \mathbb{P}^N$ a submanifold and $X = Y \cap H$ a hyperplane section which is nonsingular. Then $H^i(Y, \mathbb{Z}) \to H^i(X, \mathbb{Z})$ is an isomorphism for $0 \leq i < n$, and is injective for $i = n$.*

Example 2.8. *Let $X \subset \mathbb{P}^2$ a curve. Then $\mathbb{Z} = H^0(\mathbb{P}^2, \mathbb{Z}) \simeq H^0(X, \mathbb{Z})$ and $0 = H^1(\mathbb{P}^2, \mathbb{Z}) \to H^1(X, \mathbb{Z})$ is injetive, we can do this by taking a Veronese embedding of $\mathbb{P}^2$ into $\mathbb{P}^N$, where N depends on the degree of X.*

Example 2.9. *Let X be a surface in $\mathbb{P}^3$. Then $0 = H^1(\mathbb{P}^3, \mathbb{Z}) \simeq H^1(X, \mathbb{Z})$, and so surfaces in $\mathbb{P}^3$ are simply connected.*

Example 2.10. *If X is a three fold in $\mathbb{P}^4$, then $H^1(X,\mathbb{Z}) = 0$ and $H^2(X,\mathbb{Z}) = \mathbb{Z}$.*

Our goal is to work out the Hodge diamond of a smooth surface $X \subset \mathbb{P}^{n+1}$. It has ones down the middle and zeros elsewhere in the top half and bottom half, the only thing we don't know is the middle (longest) row, $h^{n,0}, h^{n-1,1}, \dots, h^{0,n}$.

The algorithm produces things called the primitive Hodge numbers $h_0^{p,q} = \dim(H^{p,q}(X)/H^{p,q}(Y))$.

The Hodge decomposition $H^n(X,\mathbb{C})$ give topological information independent of the complex structure of X.

Subgroups $H^{p,q}$ vary real analytically, $H^{n,0}$ varies holomorphically, $H^{0,n}$ varies antiholomorphically.

$F^iH^n = \oplus_{p\geq i} H^{p,q}$ determines the Hodge Filtration of H^n.

Note that $H^{p,q} = F^p \cap \bar{F}^{n-p}$.

We have $\pi : \mathscr{X} \to B$ a family, and we look at the sheaf $\Omega^p_{\mathscr{X}/B}$. This is a vector bundle over $\mathscr{X}$. We push it forward via π to B, then taking derived functors of π_*, we get $R^q\pi_*\Omega^p_{\mathscr{X}/B}$ a sheaf of $\mathscr{O}_B$-modules, and in fact, locally free over B whose fiber at $b \in B$ is $H^q(X_b, \Omega^p_{X_b}) = H^{p,q}(X_b)$.

Compare this with $R^n\pi_*\underline{\mathbb{C}} = V$, which is a locally constant sheaf of $\mathbb{C}$-vector spaces on B.

Then V can be thought of as a vector bundle on B with a flat connection. If we forget the connection, we get a vector bundle, or a sheaf of $\mathscr{O}_B$-modules. Formally, $\mathscr{V} = V \otimes_{\underline{\mathbb{C}}} \mathscr{O}_B$.

The issue is that the natural map $\mathscr{H}^{p,q} \to \mathscr{V}$ is no holomorphic.

Define $\mathscr{F}^q = \oplus_{p\geq i}\mathscr{H}^{p,q}$, the image in $\mathscr{V}$ of each $\mathscr{F}^i$ is a locally free sheaf of $\mathscr{O}_B$-modules.

The upshot is that $\mathscr{V} = \mathscr{F}^0 \supset \mathscr{F}^1 \supset \dots$ and $\mathscr{F}^i$is a holomorphic subbundle of $\mathscr{V}$, and $\mathscr{H}^{p,q} \simeq \mathscr{F}^p/\mathscr{F}^{p+1}$.

So things vary holomorphically as quotients, but not as subbundles.

The connection ∇ sends $\mathscr{V} \to \mathscr{V} \otimes \Omega^1_B$ and $\mathscr{F}^i \to \mathscr{F}^{i-1} \otimes \Omega^1_B$, and this is called Griffiths Transversality.

This implies that $\mathscr{H}^{p,q} \to \mathscr{H}^{p-1,q+1} \otimes \Omega^1_B$ which is $\mathscr{O}$-linear.

Equivalently, $T_B \otimes \mathscr{H}^{p,q} \to \mathscr{H}^{p-1,q+1}$, as $T_B = H^1(X_b, T_{X_b})$, we have that $H^1(X,T_X) \otimes H^q(X,\Omega^p) \to H^{q+1}(X,\Omega^{p-1})$. This map is just the cup product.

So let's guess a formula for the $H^{p,q}$.

$H^{n,0} = H^0(X,\Omega^n) = H^0(X,K_X) = H^0(X, K_{\mathbb{P}^{n+1}}(d)|_X) = H^0(X, \mathscr{O}_X(d-n-2)) = H^0(\mathbb{P}^{n+1}, \mathscr{O}_{\mathbb{P}^{n+1}}(d-n-2) = S_{d-n-2}$ where $S = \mathbb{C}[x_0,\dots,x_{n+1}]$ the homogeneous coordinate ring of $\mathbb{P}^{n+1}$

We have $0 \to \mathscr{O}(-n-2) \to \mathscr{O}_{\mathbb{P}}(d-n-2) \to \mathscr{O}_X(d-n-2) \to 0$. The first term is in fact $K_{\mathbb{P}}$, and so we have $0 \to H^0(K_{\mathbb{P}}) \to H^0(\mathscr{O}(d-n-2)) \to H^0(\mathscr{O}_X(d-n-2)) \to H^1(K_{\mathbb{P}}) \to \dots$ from Mayer-Vietoris. $H^0(K_{\mathbb{P}}) = H^{n+1,0}(\mathbb{P})$, etc.

So we now know that $H^{n,0} = S_{d-n-2}$, $H^{n,0} \oplus H^1(T_X) \to H^{n-1,1}$ and $H^{n-1,1} \otimes H^1(T_X) \to H^{n-2,2}$.

What is $H^1(T_X)$? It's related to S_d, in fact, it is a quotient of S_d by those deformations of the equation which do not change the complex structure.

As $GL(n+2)$ acts on S_d, what we want to do is divide out by this action, if $f = f(x_0, \ldots, x_{n+1})$, and we have $T_0 GL(n+2) = gl(n+2)$ the $n \times n$ matrices. Then E_{ij} is a basis elemtn of $gl(n+2)$.

How does it act? By $\exp(tE_{ij})$. Differentiating with respect to t the function $f \circ \exp(tE_{ij})$ gives us $x_i \frac{\partial f}{\partial x_j}$.

So changing $f \mapsto f + \sum_{ij} g_{ij} x_i \frac{\partial f}{\partial x_j} \epsilon$ for infinitesimal ϵ is the result of $GL(n+2)$ action, and hence does not change the complex structure.

We need to check two things:

1. All deformations of the complex structure on X come from varying the coefficients of the defining equation in $\mathbb{P}^{n+1}$
2. All trivial deformations (on the complex structure) in $\mathbb{P}^{n+1}$ come from the action of $GL(n)$.

To check these, just calculate $H^1(X, T_X)$. How? On projective space, we had the Euler sequence and now we restrict to X and get short exact sequences and then the long exact sequence on cohomology should yield these facts.

The easiest answer would be $H^1(X, T_X) = S_d$, and VHS: $H^{n,0} = S_{d-n-2}$, $H^{n-1,1} = S_{2d-n-2}$, etc. This is not quire right, though.

Let $\mathscr{J}$ be the ideal in S generated by the first partials of f. Then $R = S/\mathscr{J}$ is a graded ring, the Jacobian ring. The answer is in fact $H_0^{p,q} = R_{dq+d-n-2}$.

Why? Because $H^1(X, T_X)$ isn't S_d, but is rather $S_d/\mathscr{J}_d = R_d$, because we can't have things that are divisible by the partials.

3 Curves

First we must define the objects of study.

Definition 3.1 (Curve)**.** *A curve C is a nonsingular one dimensional projective complex variety.*

The first fact we need to know about curves is as follows:

Proposition 3.1. *Let X be a curve and $\mathscr{L}$ be a line bundle on X. Then there exists a divisor D such that $\mathscr{O}(D) \simeq \mathscr{L}$.*

Note that this divisor is not unique.

Definition 3.2 (First Chern Class)**.** *Let $\mathscr{L}$ be a line bundle on a curve. Then $\mathscr{L} \simeq \mathscr{O}(D)$ for some $D = \sum_{i=1}^{N} n_i p_i$, where $p_i \in C$ are points. Now we define $c_1(\mathscr{L}) = \deg \mathscr{L} = \sum_{i=1}^{N} n_i$.*

Proposition 3.2. *The first Chern class $c_1(\mathscr{L})$ is well-defined, that is, it does not depend on the divisor chosen to represent $\mathscr{L}$.*

In fact, we can determine this value without directly using divisors:

Proposition 3.3. *Let $U \subset C$ be an open set such that $\mathscr{L}|_U \simeq \mathscr{O}_U$. Then let $s \in \mathscr{L}(U)$ and identify it with its image under the isomorphism, as a rational function. Then $\deg \mathscr{L} = \sum_{p \in X} \deg_p(s)$.*

And now we will state a powerful theorem which we will not prove.

Theorem 3.4 (Riemann-Roch)**.** *Let C be a curve and $\mathscr{L}$ a line bundle on C. Then $h^0(C, \mathscr{L}) - h^1(C, \mathscr{L}) = \deg \mathscr{L} - g + 1$, where g is the genus of C (as a Riemann Surface).*

We take a moment now to define the Euler characteristic of a sheaf $\mathscr{F}$ on a space X to be $\chi(\mathscr{F}) = \sum_{i=0}^{\infty} (-1)^i h^i(X, \mathscr{F})$.

Serre Duality applied to the Riemann-Roch formula gives us that $h^0(C, \mathscr{L}) - h^0(C, \mathscr{L}^{-1} \otimes K_C) = \deg \mathscr{L} - g + 1$.

Assuming this, we now have enough to make some real statements.

Theorem 3.5 (Computation of Cohomology for Curves)**.** *Let C be a curve, $\mathscr{O}_C$ its structure sheaf, T the tangent sheaf and K the canonical sheaf. Then we have*

	deg	h^0	h^1
$\mathscr{O}$	0	1	g
K	$2g-2$	g	1
T	$2-2g$	$0+\epsilon$	$3g-3+\epsilon$

where $\epsilon = 0$ if $g \geq 2$ and $\epsilon = 1$ if $g = 1$, and $\epsilon = 3$ if $g = 0$.

Proof. Apply Riemann-Roch to $\mathscr{O}$, then $h^0(\mathscr{O}) - h^1(\mathscr{O}) = 0 - g + 1$. As any global holomorphic (and thus regular) function on a compact Riemann surface is constant, $h^0(\mathscr{O}) = 1$, so we get $h^1(\mathscr{O}) = g$. Now we apply Serre Duality, and get $h^0(K) = g$.

Apply Riemann-Roch to K, then $h^0(K) - h^1(K) = \deg K - g + 1$ gives $g - 1 = \deg K - g + 1$, so $\deg K = 2g - 2$.

Apply Riemann-Roch to T now, and we get $h^0(T) - h^1(T) = 2 - 2g - g + 1 = 3 - 3g$, as $\deg T = -\deg T^{-1} = \deg K$. If $g \geq 2$, then $\deg T < 0$, so $h^0(T) = 0$, because otherwise you'd have a function with zeroes but no poles on a compact Riemann surface.

Assume that $g \geq 2$, then we get $-h^1(T) = 3 - 3g$, so $h^1(T) = 3g - 3$, so we get for $g \geq 2$ the following table.

We must still check the genus 1 and 0 cases.

If $g = 1$, consider K. $\deg K = 0$ and $h^0(K) = 1 > 0$. This implies that $K \simeq \mathscr{O}$, and therefore $T = \mathscr{O}$.

If $g = 0$, we get $h^0(T) - h^1(T) = 2 - 0 + 1 = 3$, $h^1(T) = h^0(K^2) = 0$, and so $h^0(T) = 3$ as $\deg K < 0$, $h^1(T) = 0$. □

3.1 Rational Curves

The ϵ above is $\dim H^0(T)$, and $H^0(T)$ is the space of holomorphic vector fields on $\mathbb{C}$, which is also the dimension of the automorphism group of C. Though we have not proved it, it is true that $\mathrm{Aut}(C) = \mathrm{PGL}(2, \mathbb{C})$ for $g = 0$ and is of dimension 3, if C has genus 1, then $\mathrm{Aut}(C) = C \times \mathbb{Z}/2$, which has dimension 1, and if $g \geq 2$, then the group of automorphisms is finite and so has dimension 0.

The reason for that is that a vector field is an infinitesimal automorphism, that is, they arise as derivatives of one-parameter families of automorphisms. And $T_{\mathrm{id}}(\mathrm{Aut}\, C) = H^0(C, T_C)$.

Just for simpicity, we make the following definition:

Definition 3.3 (Rational Curve)**.** *A curve of genus zero is said to be a rational curve.*

We can, in fact, prove the genus zero case fairly easily, if we assume that $\mathrm{Aut}\, \mathbb{P}^1 = \mathrm{PGL}(2, \mathbb{C})$, though the use of the following

Proposition 3.6. *Any rational curve is isomorphic to* $\mathbb{P}^1$.

Proof. Let $g = 0$. Let $L = \mathscr{O}(p)$ for some $p \in C$. Then $\deg L = 1$. Riemann-Roch says that $h^0(L) - h^1(L) = 2$. By Serre Duality, $h^1(L) = h^0(K \otimes L^{-1})$, which has degree -3, and so vanishes, so $h^0(L) = 2$, $h^1(L) = 0$. So when C is of genus 0, and $p \in C$, then $h^0(\mathscr{O}(p)) = 2$ and $h^1(\mathscr{O}(p)) = 0$.

What is the base locus? Well, a section of $\mathscr{O}(p)$ is a function with a pole at p of order at most 1. There is a two dimensional space of functions that vanishes at $\deg(\mathscr{O}_C(p)) = 1$ points. Take a basis f, g, each vanishes at a unique point. Sections are independent iff $p_1 \neq p_2$, and so $B_L = \emptyset$. So we get a map $\phi_L : C \to \mathbb{P}^1$ defined everywhere. This has the property that $\forall x \in \mathbb{P}^1$, $\exists s$ a section of L which vanishes precisely on $\phi_L^{-1}(x)$. That is, $\phi_L^{-1}(x) = pt$. So this map is a bijective algebraic map.

We must still check that the differential of this map is everywhere nonzero. That is, ϕ_L is an analytic local isomorphism. In particular, it is a cover but $\mathbb{P}^1$ is simply connected, so $C = \mathbb{P}^1$. □

With the degree of a line bundle of such importance, we make the following definition:

Definition 3.4 (Stratification of $\mathrm{Pic}(C)$)**.** *We define the set* $\mathrm{Pic}^n(C)$ *to be the set of line bundles of degree* n *on* C.

Also we note the following:

Definition 3.5 (Abel-Jacobi Map)**.** *There is a function* $\alpha_C : C \to \mathrm{Pic}^1 C$ *defined by* $\alpha_C(x) = \mathscr{O}(x)$. *We call this the Abel-Jacobi map.*

So what is $\mathrm{Pic}^1(C)$? Is α injective? Surjective?

For rational curves, it cannot be injective, because, in fact, $\forall p, q$, $\alpha(p) = \alpha(q)$. We saw that $\forall q \in C$, there exists a section of $\mathscr{O}(p)$ such that s vanishes precisely on q. It is surjective, because of linear equivalence, which is defined as follows:

Definition 3.6 (Linear Equivalence)**.** *On a curve, two divisors* D *and* D' *are linearly equivalent* $D \sim D'$ *if* $D - D' = (f)$ *for some rational function* f, *where* $(f) = \sum_{p \in C} \mathrm{ord}_p(f) p$.

We can conclude that if $g = 0$, then $C \simeq \mathbb{P}^1$, so $\text{Pic}^1(C) = \{pt\}$ and $\text{Pic}(C) = \mathbb{Z}$ by the degree map.

Let $L = \mathscr{O}_C(dp)$ for $p \in C$ a rational curve. If $d < 0$, then $B_L = C$ and no map. If $d = 0$, then L is trivial and the base locus is empty, and $\phi_L : C \to pt$. If $d > 0$, then $B_L = \emptyset$ and we get $\phi_L : C \to \mathbb{P}^d$. This is always an embedding, and it is called the d-uple embedding, or the Veronese embedding, as defined earlier.

3.2 Elliptic Curves

Definition 3.7 (Elliptic Curve)**.** *An elliptic curve is a curve of genus one together with a marked point,* (C, x).

Over $\mathbb{C}$, any curve of genus 1 has points and any two points are equivalent. Over $\mathbb{Z}$, or $\mathbb{C}[x]$, or function fields this may not be the case, though we won't immediately say what these mean.

We will now look at $\text{Pic}(C)$ for elliptic curves. First we will do something that works for any genus:

Proposition 3.7. $\text{Pic}(C) \simeq H^1(X, \mathscr{O}^*)$.

And so we can do the following:

Corollary 3.8. $\text{Pic}^0(C) \simeq \mathbb{C}^g/\mathbb{Z}^{2g}$*, and* $\text{Pic}(C) = \text{Pic}^0(C) \times \mathbb{Z}$.

Proof. By the proposition, a line bundle on C corresponds to an element in $H^1(C, \mathscr{O}^*)$. We define a map $\mathscr{O} \overset{\exp}{\to} \mathscr{O}^*$ by taking holomorphic functions f on C to $e^{2\pi f}$. The kernel of this map is $\mathbb{Z}$, and as a sheaf map it is surjective, so we have a short exact sequence of sheaves $0 \to \mathbb{Z} \to \mathscr{O} \overset{\exp}{\to} \mathscr{O}^* \to 0$.

This gives us a long exact sequence on cohomology, $0 \to H^0(\mathbb{Z}) \to H^0(\mathscr{O}) \to H^0(\mathscr{O}^*) \to H^1(\mathbb{Z}) \to H^1(\mathscr{O}) \to H^1(\mathscr{O}^*) \to H^2(\mathbb{Z}) \to 0$. We can break this up into $0 \to \mathbb{Z} \to \mathbb{C} \overset{\exp}{\to} \mathbb{C}^* \to 0$ and $0 \to \mathbb{Z}^{2g} \to \mathbb{C}^g \to \text{Pic}(C) \to \mathbb{Z} \to 0$.

This tells us that we have a map $\text{Pic}(C) \to \mathbb{Z}$ which is surjective. This is the degree map or the first Chern class. The kernel is $\text{Pic}^0(C)$, which is $\mathbb{C}^g/\mathbb{Z}^{2g}$.

So $\text{Pic}^0(C) = (S^1)^{2g}$ topologically, and $\text{Pic}(C) = \text{Pic}^0(C) \times \mathbb{Z}$, because the extension splits (not naturally though). □

Moving back to elliptic curves, we let $\mathscr{L} \in \text{Pic}^0(C)$.

Lemma 3.9. *For a line bundle* $\mathscr{L}$ *of degree zero on an elliptic curve, we have*
$$h^0(\mathscr{L}) = \begin{cases} 1 & \mathscr{L} = \mathscr{O} \\ 0 & else \end{cases}.$$

Proof. Assume not, then $s \in \mathscr{L}(C)$ is a section which has no poles, and so if it has degree 0, it must have no zeros. So it has global sections $\mathbb{C}$ and thus must be trivial, which is a contradiction. □

Also, Riemann-Roch tells us that $h^0 - h^1 = \deg \mathscr{L} - g + 1 = 0$, so $h^1 = 0$ also.

Lemma 3.10. *For $d > 0$, if $\mathscr{L} \in \mathrm{Pic}^d(C)$ where C is an elliptic curve, then $h^0(C) = d$.*

Proof. We know that $h^1(\mathscr{L}) = h^0(\mathscr{L}^{-1} \otimes K)$ by Serre Duality. The sheaf $K \otimes \mathscr{L}^{-1}$ has degree $-d$, and so has no global section, and thus $h^1(\mathscr{L}) = 0$.

Applying Riemann-Roch, we get that $h^0(\mathscr{L}) - h^1(\mathscr{L}) = h^0(\mathscr{L}) = \deg \mathscr{L} - g + 1 = d - 1 + 1 = d$. □

What about K_C? By the computation of cohomology, $\deg K_C = 0$, so $h^0(K_c) = g = 1$ and so by Lemma 3.9, we have that $K_C = \mathscr{O}_C$.

For an element $\mathscr{L} \in \mathrm{Pic}^1(C)$, $B_{\mathscr{L}}$ is a point and $\varphi_{\mathscr{L}} : C \setminus \{pt\} \to \{pt\}$.

This suggests that we define a map $\beta : \mathrm{Pic}^1(C) \to C$ by $\mathscr{L} \mapsto B_{\mathscr{L}}$.

Corollary 3.11. *The Abel-Jacobi map for a genus 1 curve C is an isomorphism $C \simeq \mathrm{Pic}^1(C)$*

Proof. α and β are inverse maps, which can be seen by explicit calculation. □

This is at least topologically true.

Definition 3.8 (Jacobian)**.** *The Jacobian of a curve is* $\mathrm{Pic}^0(C)$.

Now we can see that $C \simeq \mathrm{Pic}^1(C) \simeq \mathrm{Jac}(C)$, which is a g dimensional complex manifold (in fact a variety). The corollary in fact says that there is an isomorphism of varieties $C \simeq \mathrm{Jac}(C)$ if $g = 1$.

Lemma 3.12. *Let $\mathscr{L} \in \mathrm{Pic}^2(C)$, then $B_{\mathscr{L}} = \emptyset$.*

Proof. Looking at $\mathscr{L} \in \mathrm{Pic}^2(C)$, we know $h^0(\mathscr{L}) = 2$ and $h^1(\mathscr{L}) = 0$, so any section will vanish on two points. $B_{\mathscr{L}}$ will then be either one point or the empty set. In fact, it will always be empty, because the Abel-Jacobi map is an isomorphism. □

So for any $\mathscr{L} \in \mathrm{Pic}^2(C)$, we get a map $\varphi_{\mathscr{L}} : C \to \mathbb{P}^1$. What is $\varphi_L^{-1}(pt)$? It is two points or else a point with multiplicity two. This tells us that the topological degree is 2. We can also think about this as a branched cover of $\mathbb{P}^1$, cut along curves between 0 and 1 and between ∞ and a point λ.

Definition 3.9 (Branch Point)**.** *Let $C \to C'$ be a map of curves of topological degree d. Then a point $p \in C'$ is a branch point if $f^{-1}(p)$ does not have cardinality d.*

We say that a map has ramified behavior over branch points and unramified behavior away from them (this is provisional, and not a rigorous definition).

So now we must ask how many branch points does $\varphi_{\mathscr{L}}$ have?

Theorem 3.13 (Hurwitz Formula)**.** *Let $f : C \to C'$ be a map of curves of topological degree d. Then $\chi(C) = d\chi(C') - |B|$, where B is the set of branch points of f.*

Thus, we have $\chi(C) = 2\chi(\mathbb{P}^1) - b$, and so $b = 4$. As $\mathbb{P}^1$ has a three dimensional automorphism group, we can take three of these points to be $0, 1, \infty$, and so the fourth branch point λ determines C, up to an action of S_3.

Theorem 3.14. *If $\mathscr{L} \in \mathrm{Pic}^3(C)$, then $B_{\mathscr{L}} = \emptyset$ and $\varphi_{\mathscr{L}} : C \to \mathbb{P}^2$ is an embedding of C as a cubic plane curve.*

Proof. Let $\mathscr{L} \in \mathrm{Pic}^3(C)$, then $h^0(\mathscr{L}) = 3$ and $B_{\mathscr{L}} = \emptyset$, so we get a map $\varphi_{\mathscr{L}} : C \to \mathbb{P}^2$. $\varphi_{\mathscr{L}}(C)$ is a planar cubic curve, because $\mathscr{L}$ has degree three. We must show that if $p \neq q \in C$, then $\varphi_{\mathscr{L}}(p) \neq \varphi_L(q)$ and for all $p \in C$, $d_p\varphi_L \neq 0$, so it is an embedding.

Assume $\varphi_{\mathscr{L}}(p) = \varphi_{\mathscr{L}}(q)$, that is, for all $s \in \Gamma(\mathscr{L})$, $s(p) = 0$ iff $s(q) = 0$. Consider $\mathscr{L}(-p)$ which has degree 2, and has no base locus. In particular, $q \notin B_{\mathscr{L}(-p)}$, and so there exists $s' \in \Gamma(C, \mathscr{L}(-p))$ such that $s'(q) \neq 0$. Note that $\mathscr{L}(-p) \to L$ is an inclusion, so s' determines a section $s \in \mathscr{L}$ with s vanishing at q_1, q_2 not equal to q and s vanishes at q_1, q_2, p but not at q.

Now for $d_p(\varphi_{\mathscr{L}})$. Assume that $d_p\varphi_{\mathscr{L}} = 0$ for some p. Then for all sections $s \in \Gamma(\mathscr{L})$, $p \in Z(s) \iff 2p \in Z(s)$. The same argument arrives at a contradiction here as well. □

Thus, every genus 1 curve is a smooth plane curve of degree 3. We will in the future need the Adjunction formula, which is a bit more general than Hurwitz's Formula. Another special case of the Adjunction Formula is as follows:

Proposition 3.15 (Adjunction for Subvarieties)**.** *Let X be any projective nonsingular variety and $\mathscr{L} \in \mathrm{Pic}(X)$. Then for any $s \in \Gamma(X, \mathscr{L})$ and $Y = \{s = 0\}$ nonsingular, we have $K_Y = (K_X \otimes \mathscr{L})|_Y$.*

This allows us to prove a converse for Theorem 3.14.

Theorem 3.16. *Any smooth plane curve of degree three is an elliptic curve.*

Proof. We will need the fact that $K_{\mathbb{P}^2} = \mathscr{O}(-3)$. We want to apply Adjunction for $X = \mathbb{P}^2$ and $\mathscr{L} = \mathscr{O}(3)$, as the sections of $\mathscr{L}$ will be cubics. Then $K_Y = \mathscr{O}(-3+3) = \mathscr{O}$. And so, the degree of $K_Y = \deg \mathscr{O} = 0 = 2g - 2$, and so $g = 1$. □

To make this fully rigorous, we need to prove the Adjunction formula and to compute $K_{\mathbb{P}^2}$.

3.3 Curves of Genus Two

Let $g = 2$. Then $\deg K = 2$, $h^0(K) = 2$. What is B_K? We have two independent sections, s_1, s_2, say s_i vanishes on D_i. Then $\deg D_i = 2$.

Lemma 3.17. $D_1 \cap D_2 = \emptyset$

Proof. If $p \in B_K$, then $D_1 = p + q$ and $D_2 = p + v$. Consider $K(-p) = K \otimes \mathscr{O}(-p)$. $D_i - p$ give two independent sections, so $h^0(C, K(-p)) \geq 2$. Let $L = K(-p)$, then L had degree 1, and $h^0(L) = 2$, so $\varphi_L : C \to \mathbb{P}^1$ has degree 1, and so it is an isomorphism, contradicting the genus. □

More generally, the following is true:

Theorem 3.18. *Let $g \geq 1$. Then a line bundle with $\deg \mathscr{L} = 1$ cannot have $h^0(\mathscr{L}) > 1$.*

So we now have $\varphi_K : C \to \mathbb{P}^1$, which realizes any curve C of genus 2 as a double cover of $\mathbb{P}^1$.

By Hurwitz's Formula $\chi(C) = d\chi(\mathbb{P}^1) - b$. For $\varphi_{\mathscr{L}}$ a double cover, this gives $b = 6$.

It is a fact that the dimension of the moduli space of curves of genus g is $3g-3+\epsilon$. For $g = 2$, this is 3. Another way to see this is that there are 6 branch points, so six degrees of freedom, less three degrees of freedom from $\operatorname{Aut} \mathbb{P}^1$, and so the moduli of genus 2 curves has dimension 3.

Proposition 3.19. *If C is a curve of genus 2 and $\mathscr{L} \in \operatorname{Pic}^0(C)$, then $h^0(\mathscr{L}) = \begin{cases} 1 & \mathscr{L} = \mathscr{O}_C \\ 0 & else \end{cases}$, and if $\mathscr{L} \in \operatorname{Pic}^1(C)$, then $h^0(\mathscr{L}) = \begin{cases} 1 & \mathscr{L} \in \alpha(C) \\ 0 & else \end{cases}$ where α is the Abel-Jacobi map.*

Proof. Let $p \in C$. Then $1 \geq h^0(\mathscr{O}(p)) \geq h^0(\mathscr{O}) = 1$. Conversely, for $\mathscr{L} \in \operatorname{Pic}^1$, if it has a section, then that section must vanish at some p, and so $\mathscr{L} \simeq \mathscr{O}(p)$.

In general, $\mathscr{L} \in \operatorname{Pic}^1(C)$ then $h^0(\mathscr{L}) = 1$ if and only if $\mathscr{L}$ is in the image of $\alpha(C)$. □

Lemma 3.20. *For $g \geq 1$, the Abel-Jacobi map α is injective.*

Proof. This is a consequence of the fact that if a curve has points $p, q \in C$ such that p is linearly equivalent to q, then C is isomorphic to $\mathbb{P}^1$. See Hartshorne for proof. □

We also have the following fact:

Lemma 3.21. *If $g > 0$ and $d \geq 1$, then $h^0(\mathscr{L}) \leq \deg \mathscr{L}$ for any line bundle $\mathscr{L}$.*

Proof. If $\mathscr{L}$ has degree d and $h^0(\mathscr{L}) \neq 0$, then choose a section s. Let p be a point where $s(p) = 0$. Then we have $0 \to \mathscr{L}(-p) \to \mathscr{L} \to \mathbb{C}_p \to 0$, where $\mathbb{C}_p$ is the sheaf with value $\mathbb{C}$ if $p \in U$ and 0 else. We also have $0 \to H^0(\mathscr{L}(-p)) \to H^0(\mathscr{L}) \to \mathbb{C}$ and so we have $h^0(\mathscr{L}) \leq h^0(\mathscr{L}(-p)) + 1$. The lemma follows by induction. □

Example 3.1. *If $\mathscr{L} = K$, $g = 2$, then we have that $h^0(\mathscr{L}) = 2$ and $\deg \mathscr{L} = 2$. Serre Duality then tells us that $h^1(\mathscr{L}) = h^0(K \otimes \mathscr{L}^{-1})$, and this will then have degree 0, and so $h^1(\mathscr{L}) = 0$ unless $\mathscr{L} = K$, and $h^1(K) = 1$.*

Riemann-Roch says that $h^0(\mathscr{L}) - h^1(\mathscr{L}) = d - g + 1 = 1$, and so $h^0(\mathscr{L})$ is 1 or 2, and it is 2 iff $\mathscr{L} = K$, and otherwise is 1. So $h^0(\mathscr{L}) \neq 0$ for $\mathscr{L} \in \operatorname{Pic}^2(C)$ with C a genus 2 curve.

In particular, for $g = 2$ and $\mathscr{L} \in \mathrm{Pic}^2(C)$, we have $h^0(\mathscr{L}) \in \{1, 2\}$ if $\mathscr{L} \neq K$.

What about if we have $\deg \mathscr{L} = 3$? We have entered the stable range: that is, whenever $d > 2g - 2$. Calling it this is justified by the following:

Proposition 3.22. *For any g, if $d = \deg(\mathscr{L}) > 2g-2$, then $h^0(\mathscr{L}) = d-g+1$.*

Proof. Riemann-Roch says that $h^0(\mathscr{L}) - h^1(\mathscr{L}) = d - g + 1$ and Serre Duality gives $h^1(\mathscr{L}) = h^0(K \otimes \mathscr{L}^{-1})$ which has negative degree and so is zero, and so Riemann-Roch reduces to the desired quantity. □

Thus, for $g = 2$, $d \geq 3$, there is a unique stratum: all of $\mathrm{Pic}^d(C)$.

We want to look at embeddings, and so we must continue on to $d = 4$. Then $h^0(\mathscr{L}) = 3$, and so we get $\varphi_{\mathscr{L}} : C \setminus B_{\mathscr{L}} \to \mathbb{P}^2$. For general $\mathscr{L}$, the base locus is empty.

So we get $\varphi_{\mathscr{L}} : C \to \mathbb{P}^2$. The degree of the image is 4, because $\mathscr{L} \in \mathrm{Pic}^4(C)$. The image cannot be smooth! This is a consequence of the following corollary of the Adjunction formula:

Corollary 3.23. *Let C be a smooth plane curve of degree d. Then $g = \frac{(d-1)(d-2)}{2}$.*

Proof. Let C be a plane curve of degree d, that is, it is the zero set of a section of $\mathscr{O}_{\mathbb{P}^2}(d)$. Then the Adjunction formula gives us that $\deg K_C = \deg(K_{\mathbb{P}^2} \otimes \mathscr{O}(d))$. That is, $2g - 2 = \deg(\mathscr{O}(-3) \otimes \mathscr{O}(d)) = \deg(\mathscr{O}(d-3)|_C)$. That gives $2g - 2 = d(d-3)$, which can be solved to give $g = \frac{(d-1)(d-2)}{2}$. □

This means that if $d = 4$, we must have $g = 3$ for a smooth plane quartic curve. In fact, for $p, q \in C$

Claim: There exist $p, q \in C$ such that generically $p \neq q$ and $\varphi_L(p) = \varphi_L(q)$, but a special case is $p = q$ and $d\varphi_L|_p = 0$.

However, if $\mathscr{L}$ had degree 5, it will embed C into $\mathbb{P}^3$ for $\mathscr{L}$ a generic line bundle.

3.4 Hyperelliptic Curves

We note first that for any $g \geq 2$, $h^0(K) = g$, $\deg(K) = 2g - 2$.

Definition 3.10 (Hyperelliptic)**.** *A curve is hyperelliptic if there exists a degree two map $\pi : C \to \mathbb{P}^1$.*

Theorem 3.24. *Either C is hyperelliptic in which case φ_K is two to one onto a rational curve in $\mathbb{P}^{g-1}$, or else $\varphi_K : C \to \mathbb{P}^{g-1}$ is an embedding.*

To justify this at all, we first need the following lemma.

Lemma 3.25. *Let C be a curve $g(C) \geq 2$ and K its canonical divisor, then* $\deg K = 2g - 2$ *and $h^0(K) = g$. Then $B_K = \emptyset$.*

Proof. Say $p \in B_L$, then that is the same as $H^0(K(-p)) \to H^0(K)$ (which is injective) is an isomorphism. This would imply that $h^0(K(-p)) = h^0(K) = g$. So p is a base point iff $h^0(K(-p)) = g$. But $h^0(K(-p)) = dg - g + 1 + h^1(K(-p)) = 2g - 2 - 1 + g + 1 + h^1 = g - 1$. □

So we now have a canonical map $\varphi_K : C \to \mathbb{P}^{g-1}$. And we claim that either φ_K is an embedding or else it is a 2 to 1 map onto a rational normal curve, which is the Veronese embedding of $\mathbb{P}^1$ under $\varphi_{\mathscr{O}(g-1)}$.

Before we prove the theorem, we will look at the hyperelliptic case in more detail.

In the case of a hyperelliptic curve, we should make the provisional definition that a point mapped to a branch point is called a ramification point, R, the ramification divisor, is their sum and B is the sum of the branch points, the branch divisor. Things are really a bit more complicated, but this much is true in the hyperelliptic case.

In the case of $\pi : C \to \mathbb{P}^1$ a double cover ramified at $2g+2$ points of C, the Adjunction formula gives $K_C = \pi^* K_{\mathbb{P}^1}(R)$.

More generally, even with the better definition of ramification, the following is true:

Theorem 3.26. *Let $\pi : X \to Y$ be finite map between smooth projective curves. Then we have $K_X = \pi^* K_Y(R)$. If we define $\pi^*\mathscr{O}_{\mathbb{P}^1}(1)$ to be H, then $\deg H = 2$ and $h^0(H) = 2$, and so we have $K_C = H^{-2}(R)$. In fact, $K_C = H^{g-1}$. Equivalently, $\mathscr{O}(R) = H^{g+1}$.*

Now we will prove the characterization of canonical maps:

Proof. Assume that $\varphi_K(p) = \varphi_K(q)$ for some $p \neq q$. This means that any differential form will vanish at p iff it vanishes at q. So $H^0(K(-p-q))$ includes into each $H^0(K(-p))$ and $H^0(K(-q))$ and each of those includes into $H^0(K)$, and the maps commute. So $h^0(K(-p-q)) = g-1$ as it is equal to $2g-2-2-g+1+h^0(\mathscr{O}(p+q))$ and $\mathscr{O}(p+q)$ has dimension 2, and we will call this line bundle H.

By definition, $\deg H = 2$ and we know that $h^0(H) = 2$, and so $\varphi_H : C \to \mathbb{P}^1$ is a double cover. If $x \in \mathbb{P}^1$, then $\varphi_H^{-1}(x) = r + s$. Take $\varphi_K(p) = \varphi_K(q)$ to be y, then $\mathscr{O}(y) = \mathscr{O}(x)$ on $\mathbb{P}^1$, and so $\varphi_H^*(\mathscr{O}(y)) = \varphi_H^*(\mathscr{O}(x))$, and so $\mathscr{O}(r+s) = \mathscr{O}(p+q)$. This says that these two sheaves both have two dimensions of global sections.

Now we note that $\varphi_K(p) = \varphi_K(q)$ iff $h^0(\mathscr{O}(p+q)) = 2$, and so $\varphi_K(r) = \varphi_K(s)$, and so the map φ_K factors through $\mathbb{P}^1$ via φ_H. And so the claim holds. □

The arguments so far have been very repetitive, and we can encode this and stop worrying about it using the Geometric form of Riemann Roch:

Theorem 3.27 (Geometric Riemann-Roch)**.** *Given $p_1, \ldots, p_d \in C$, with $g \geq 1$, so $h^0(C, \mathscr{O}(p_1, \ldots, p_d)) = 1 +$ the number of linear relations among the images $\varphi_K(p_i)$.*

If you lift the points to their spans in $\mathbb{C}^{n+1}$, then this number is just $d - \dim \operatorname{span}\{p_1, \ldots, p_d\}$

Example 3.2. *If $d = 1$, then $h^0(\mathscr{O}(p)) = 1$ as there are no relations.*

If $d = 2$, then $h^0(\mathscr{O}(p+q)) = \begin{cases} 1 & \text{if } \varphi_K(p) \neq \varphi_K(q) \\ 2, & \text{if } \varphi_K(p) = \varphi_K(q) \end{cases}$.

If $d = 3$, then $h^0(\mathscr{O}(p+q+r)) = \begin{cases} 1, & \text{generically} \\ 2, & C \text{ is trigonal} \\ 3, & \text{never} \end{cases}$

Definition 3.11 (Trigonal)**.** *A curve is trigonal if it has a map of degree three to $\mathbb{P}^1$.*

Proposition 3.28. *A curve is trigonal iff $\varphi_K(C)$ sits on some ruled surface and intersects each ruling in 3 points.*

We now prove the Geometric Riemann-Roch Theorem:

Proof. Take $h^0(\mathscr{O}(p_1 + \ldots + p_d)) = 1 + d - 1 - \dim \operatorname{span}(\varphi_K(p_i)) = d - (g - 1 - h^0(K(-\sum p_i)))$. By Serre Duality, this is $d - g + 1 + h^1(\mathscr{O}(\sum p_i))$, and so it is equivalent to standard Riemann-Roch with Serre Duality. □

3.5 The Abel-Jacobi Map

Recall that the Abel-Jacobi map is $\alpha : C \to \operatorname{Pic}^1(C)$ defined by $\alpha(p) = \mathscr{O}(p)$. In fact, we can generalize this to get $\alpha_d : C^d \to \operatorname{Pic}^d(C)$ by $(p_1, \ldots, p_d) \mapsto \mathscr{O}(p_1 + \ldots + p_d)$. Even better, this map factors through $\operatorname{Sym}^d(C)$, just because addition of divisors is commutative. So now we ask: what are the fibers of α_d like?

Theorem 3.29 (Abel)**.** *$\alpha_d(p_1 + \ldots + p_d) = \alpha_d(p'_1 + \ldots + p'_d)$ iff $\mathscr{O}(p_1 + \ldots + p_d) \simeq \mathscr{O}(p'_1 + \ldots + p'_d)$, and so the fibers are projective spaces.*

We can interpret this as follows: Choose a basis $\omega_1, \ldots, \omega_g$ of the differentials on C, then $\alpha(p) = \alpha(p_0) \iff \int_{p_0}^{p} \omega_i = 0$ for all i for some path.

We will look at the Abel-Jacobi map $\alpha : C \to \operatorname{Pic}^1(C) = \mathbb{C}^g/\Lambda$ in more detail. First we note that a choice of a base point $p_0 \in C$ determines an isomorphism $\operatorname{Pic}^1(C) \to \operatorname{Pic}^0(C)$ by $L \mapsto L(-p_0)$. One interpretation of α is as a multivalued function $C \to \mathbb{C}^g$. It can be expressed in terms of a basis $\omega_1, \ldots, \omega_g$ of $H^0(C, K_C)$. So the Abel Jacobi map is $p \mapsto (\int_{p_0}^{p} \omega_1, \ldots, \int_{p_0}^{p} \omega_g)$.

With the algebraic definition of α_d, Abel's Theorem is obvious. Analytically, it is significantly less so.

Example 3.3. *If $g = 1$, look at the elliptic integral $\int_{p_0}^{p} \frac{dx}{\sqrt{x^3+ax+b}}$. What Abel's Theorem says here, look at $\sum_{i=1}^{3} \int_{p_0}^{p_i} \frac{dx}{\sqrt{x^3+ax+b}}$, and we can take the plane cubic $y^2 = x^3 + ax + b$, and so the answer of the integral doesn't depend on which three points we take if they are collinear.*

This connects because dx/y is a global 1-form on an Elliptic Curve (because $dx/y = 2dy/(3x^3 + a)$).

We will now prove Abel's Theorem from the analytic definitions.

Proof. Let $D = \sum p_i$ and $D' = \sum p_i'$ with $\mathscr{O}(D) \cong \mathscr{O}(D') = \mathscr{L}$. Then $h^0(\mathscr{L}) \geq 2$. Consider the function on $\mathbb{P}(H^0(C, \mathscr{L}))$ that takes $D \mapsto \int_{dp_0}^{D} \omega$. This is a multivalued holomorphic function on $\mathbb{P}^n$.

This implies that it is a single valued function in the universal cover, which is $\mathbb{P}^n$ itself, and hence is constant. □

Proposition 3.30. *The map $d\alpha : T_pC \to T_{\alpha(p)} \operatorname{Pic}^1(C) \simeq \mathbb{C}^g = H^0(C, K_C)^*$ takes $C \to \mathbb{P}^{g-1}$ and is equal to φ_K.*

Proof. φ_K is the map that takes $p \mapsto (\omega_1(p), \ldots, \omega_g(p))$, and α is given by the integrals of the ω_i, and so differentiating to get $d\alpha$ returns the values of the ω_i. □

We can restate the analytic proof as starting with $C^d \to \operatorname{Sym}^d \overset{\alpha_d}{\to} \operatorname{Pic}^d(C)$, and ω is a 1-form on C. We can think of it as a 1-form on $\operatorname{Pic}^d(C)$.

Formally, $\alpha : C \to \operatorname{Pic}^1(C)$ induces $\alpha_* : H_1(C) \to H_1(\operatorname{Pic}^1(C))$ an isomorphism, and so $\alpha^* : H^1(\operatorname{Pic}^1(C)) \to H^1(C)$ is an isomorphism.

So $\alpha_1^*\omega$ is a holomorphic 1-form on $\operatorname{Sym}^d C$. We've assumed that there exists $\mathbb{P} \subset \operatorname{Sym}^d(C)$ with $\alpha_d^*\omega|_{\mathbb{P}} \equiv 0$.

4 Vector Bundles

4.1 Standard Bundles

First, we look at line bundles.

Lemma 4.1. *The degree map from $\operatorname{Pic}(\mathbb{P}^n) \to \mathbb{Z}$ is an isomorphism.*

Proof. This follows from the fact that if $X \subset \mathbb{P}^n$ is a degree d hypersurface, then the d-uple Veronese embedding on $\mathbb{P}^n$ takes X to a hyperplane section, and so X is linearly equivalent to dH where H is a hyperplane in $\mathbb{P}^n$. □

Example 4.1. *$\mathscr{O}_{\mathbb{P}^n}(1)$ is the hyperplane bundle, and it is a generator of $\operatorname{Pic}(\mathbb{P}^n)$. For $d \geq 0$, the sections of $\mathscr{O}_{\mathbb{P}^n}(d)$ are the homogeneous polynomials of degree d in $n+1$ variables. Thus, $h^0(\mathscr{O}(d)) = \binom{n+d}{n}$.*

Example 4.2. *Let T be the tangent bundle. The sections are $\sum a_{ij} x_i \frac{\partial}{\partial x_j}$. Note that $\mathbb{P}^n = \mathbb{P}(V)$, and we can get an exact sequence $0 \to \mathscr{O} \to \mathscr{O}_{\mathbb{P}^n}(1) \otimes_{\mathbb{C}} \underline{V} \to T_{\mathbb{P}^n} \to 0$.*

Definition 4.1 (Euler Vector Field)**.** *The Euler vector field is $e = \sum x_i \frac{\partial}{\partial x_i}$ which is a vector field on $V \setminus 0$ which is tangent to the fibers of projectivization.*

The first map in the short exact sequence above is e. This gives the above sequence the name "The Euler Sequence." It determines the tangent space of $\mathbb{P}^n$ by $(\mathscr{O}(1) \otimes_{\mathbb{C}} \underline{V})/\mathscr{O}$ where the top is a sum of $\dim V$ copies of $\mathscr{O}(1)$ modulo the diagonal embedding of $\mathscr{O}$.

Example 4.3. *For $n = 1$, the Euler sequence is $0 \to \mathscr{O} \to \mathscr{O}(1) \oplus \mathscr{O}(1) \to \mathscr{O}(2) \to 0$ with the first map $(x_2, -x_1)$ and the second being $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.*

Proposition 4.2. *Let K be the canonical bundle on $\mathbb{P}^n$. Then $K = \mathscr{O}(-n-1)$*

Proof. Dualizing the Euler sequence we get $0 \to \Omega^1_{\mathbb{P}^n} \to \mathscr{O}(-1) \otimes_{\mathbb{C}} \underline{V^*} \to \mathscr{O} \to 0$. Taking determinants, which is additive for short exact sequences, we get $\det(\Omega^1) \otimes \det(\mathscr{O}) = \det(\mathscr{O}(-1) \otimes_{\mathbb{C}} \underline{V^*})$. The left is K and the right is the determinant $n+1$ copies of $\mathscr{O}(-1)$, and so is $\mathscr{O}(-n-1)$. □

Definition 4.2 (Normal Bundle)**.** *Let X be a variety and $D \subset X$ a smooth divisor. Then we have a short exact sequence $0 \to T_D \to T_X|_D \to N \to 0$ for some vector bundle N. We call N the normal bundle.*

We define N^*, the dual of N, which satisfies $0 \to N^* \to \Omega^1_X|_D \to \Omega^1_D \to 0$ to be the conormal bundle.

Theorem 4.3 (Adjunction Formula)**.** *Let X be a variety, $D \subset X$ a divisor. Then $K_D = K_X(D)$.*

Proof. Let f be a local defining equation for D. Then df is a local generating section for N^*. Change the defining equation $f' = gf$, then $df'|_D = fdg|_D + gdf|_D = gdf|_D$, so conclusion is that df transforms in the same way as f.

So $N^* \cong (\mathscr{O}_X(D))|_D$. Take determinants to get $K_X|_D \cong K_D \otimes \mathscr{O}_X(-D)$ and so $K_D = K_X(D)$. □

Example 4.4. *For $X = \mathbb{P}^2$, and D a smooth curve of degree d, we get $\mathscr{O}_X(D) = \mathscr{O}_{\mathbb{P}^2}(d)$ and so $K_X = \mathscr{O}_{\mathbb{P}^2}(-3)$ and $K_D \cong \mathscr{O}_{\mathbb{P}^2}(d-3)|_D$.*

Definition 4.3 (Projection from a Point)**.** *Let $D \subset \mathbb{P}^n$ be a divisor given by $f = 0$, and let $p \in \mathbb{P}^n \setminus D$. We define the projection from a point map to be $\mathbb{P}^n \setminus \{p\} \to \mathbb{P}^{n-1}$ by taking a line through p to its intersection with $\mathbb{P}^{n-1}$. In coordinates, we can take $p = (1 : 0 : \ldots : 0)$ and so $\pi(x_0, \ldots, x_n) = (x_1, \ldots, x_n)$.*

Proposition 4.4. *This π is the projectivization of the linear projection $\mathbb{A}^{n+1} \to \mathbb{A}^n$ whose kernel is the line in $\mathbb{A}^{n+1}$ corresponding to p.*

We call this map a linear projection. We will study $\pi|_D : D \to \mathbb{P}^{n-1}$.

This map is a branched covering of degree d. We are interested in determining B, the branch locus.

Remark 1. *The branch locus of this map is called the apparent contour due to its connection to how we look at things and the use of projective geometry to draw things in perspective.*

Definition 4.4 (Polar Equation). *As D is given by $(f = 0)$, we have $df = \left(\frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_n}\right)$ and we define $d_p f = \langle p, df\rangle = \sum p_i \frac{\partial f}{\partial x_i}$. As a function of the x_i, this is called the polar equation of f at p. The degree of the polar equation is $d-1$.*

Now, R, the ramification divisor, has codimension 2 in $\mathbb{P}^n$, and it is easier to write down its equations that those of B, the branch divisor.

Lemma 4.5. *R is a complete intersection, and is given by $f = d_p f = 0$.*

Note that this is simply the condition for a multiple root in the case of a polynomial in one variable, $f = f' = 0$.

Example 4.5. *If we have $n = 2$, then D is a curve of degree d in the plane. Then the number of ramification points is $d(d-1)$, which is also the number of branch points. For general n, this argument gives $\deg B = d(d-1)$.*

Proposition 4.6. *Let C be a nonsingular conic. Then $d_p f \cap C$ is the set of points of C whose tangent lines contain p.*

Example 4.6. *For $C \subset \mathbb{P}^2$ a conic, the polar map (as a function of p rather than x_i) takes $\mathbb{P}^2 \setminus C$ to $(\mathbb{P}^2)^*$. For instance, let $f = x_0^2 + x_1^2 + x_2^2$ and let $p = (p_0, p_1, p_2)$, then $d_p f = x_0 p_0 + x_1 p_1 + x_2 p_2$ so we get a line for each p, and for each line there is a p which gives it.*

Proposition 4.7. *Given $f \in \mathrm{Sym}^2(V^*)$ you have a map $d_p f : V \to V^*$ (as a function of p) which is an isomorphism converting a point to a hyperplane. This map extends to all of $\mathbb{P}^2$ and takes points on the conic to their tangent lines.*

4.2 Direct Images

Example 4.7. *Let $f : \mathbb{P}^1_w \to \mathbb{P}^1_z$ where the subscripts are the names of the coordinates. Let f be given by $z = w^2$. Then what is $f_*\mathscr{O}$? It is a rank two vector bundle on $\mathbb{P}^1_z$, and in particular, it is an $\mathscr{O}_{\mathbb{P}^1_z}$-module. We do an even-odd decomposition.*

Then $f(w) = g(w^2) + wh(w^2) = g(z) + wh(z)$, and $g \in \Gamma(\mathscr{O}_{\mathbb{P}^1_z})$ and $h \in \Gamma(\mathscr{O}_{\mathbb{P}^1_z}(-1))$. g is arbitrary and h must vanish at ∞. Thus $\pi_*\mathscr{O}_{\mathbb{P}^1_w} = \mathscr{O}_{\mathbb{P}^1_z} \oplus \mathscr{O}_{\mathbb{P}^1_z}(-1)$.

We want to generalize this example, and get the following:

Theorem 4.8. *Let $\pi : \mathbb{P}^1 \to \mathbb{P}^1$ be the squaring map. Then*

$$\pi_*\mathscr{O}(d) = \begin{cases} \mathscr{O}\left(\frac{d}{2}\right) \oplus \mathscr{O}\left(\frac{d-2}{2}\right) & d \text{ is even} \\ \mathscr{O}\left(\frac{d-1}{2}\right) \oplus \mathscr{O}\left(\frac{d-1}{2}\right) & d \text{ is odd} \end{cases}$$

Proof. The proof is as in the example, by using an even-odd decomposition. □

Even more generally:

Theorem 4.9. *Let π by the n^{th} power map $\mathbb{P}^1 \to \mathbb{P}^1$. Then $\pi_*\mathscr{O}(d)$ is a direct sum of n line bundles, with $\pi_*(\mathscr{O}(-1)) = \oplus_{i=1}^n \mathscr{O}(-1)$ and an increase in d increases the degrees of the terms of the image so that none has degree more than one greater than any of the others.*

Proof. We take $f(w) \mapsto \sum_{i=0}^{n-1} w^i g_i(z)$. So now we get conditions on g coming from $f \in \Gamma(\mathscr{O}(d))$, specifically a pole at infinity with $n\,\mathrm{ord}_\infty(g) + i \leq d$.

Now there are n cases, for $\pi_*\mathscr{O}(-1)$, we get a direct sum of n $\mathscr{O}(-1)$'s. As we increase degree, we increase the degree of the summands alternately, so that none has degree more than one greater than any of the others.

$\deg(\pi_*\mathscr{F}) = \deg \mathscr{F} - (n-1)$, $\pi_*\mathscr{F}$ is a direct sum of line bundles and the distribution of degrees is as tight as possible (only achieves two adjacent values) □

More generally, $\pi : \mathbb{P}^1_w \to \mathbb{P}^1_z$ any algebraic map of degree n, then $z = \pi(w) = P(w)/Q(w)$ where $\deg P \leq n, \deg Q \leq n$, relatively prime and at least one has degree equal to n, and Q is the unique polynomial vanishing at $\pi^{-1}(\infty)$ and P is the unique on vanishing on $\pi^{-1}(0)$. To use this to find them, we use Lagrange interpolation.

So in general, $\pi_*(\mathscr{L}) = \oplus \mathscr{L}_i$, and $\deg \pi_*\mathscr{L} = \deg \mathscr{L} - (n-1)$ and the degrees will be bunched closely.

4.3 Split Sequences and Grothendieck's Theorem

Definition 4.5 (Split Sequence)**.** *Let $0 \to \mathscr{S} \to \mathscr{V} \to \mathscr{Q} \to 0$ be a short exact sequence of vector bundles (as sheaves).*

We say this splits if $\mathscr{V} = \mathscr{S} \oplus \mathscr{Q}$, this is equivalence to there being a map $s : \mathscr{Q} \to \mathscr{V}$ which composes to the identity on Q.

Proposition 4.10. *On the long exact sequence, $H^0(\mathscr{Q}) \to H^1(\mathscr{S})$ is the zero map if the sequence splits.*

If the coboundary map is zero, does the sequence split?

Example 4.8. *Look on $\mathbb{P}^1$ where we have $0 \to \mathscr{O}(-1) \to \mathscr{O} \oplus \mathscr{O} \to \mathscr{O}(1) \to 0$ given by maps $(y, -x)$ and $\binom{x}{y}$. This is certainly not a split sequence. The map $\delta : H^0(\mathscr{O}(1)) \to H^1(\mathscr{O}(-1))$ is a map from a 2 dimensional vector space to a 0 dimensional vector space.*

Conclusion: Split$\Rightarrow \delta = 0$ but not vice versa.

However, there is a variant that works.

Theorem 4.11. *Let $0 \to \mathscr{S} \to \mathscr{V} \to \mathscr{Q} \to 0$ be a short exact sequence of vector bundles. Then this sequence splits if and only if for all $\mathscr{L} \in \mathrm{Pic}(X)$, the map $H^0(\mathscr{Q} \otimes \mathscr{L}) \to H^1(\mathscr{S} \otimes \mathscr{L})$ induced by taking the tensor product and then the long exact sequence on cohomology is zero.*

Proof. Let $\mathscr{L}$ be a line bundle. Then $0 \to \mathscr{S} \otimes \mathscr{L} \to \mathscr{V} \otimes \mathscr{L} \to \mathscr{Q} \otimes \mathscr{L} \to 0$ is also exact. This sequence is split if and only if the original one was.

Remainder of proof omitted. □

More generally, we are talking about the extension problem.

Lemma 4.12. *Let $\mathscr{Q}$ be a vector bundle and $\mathscr{Q}^*$ its dual. Then there exists a natural map $\mathscr{O} \to \mathscr{Q} \otimes \mathscr{Q}^*$.*

Definition 4.6 (Ext)**.** *Let $0 \to \mathscr{S} \to \mathscr{V} \to \mathscr{Q} \to 0$ be a short exact sequence of vector bundles with $\mathscr{Q}$ a line bundle. Then we define $\operatorname{Ext}(\mathscr{Q}, \mathscr{S}) = H^1(\mathscr{Q}^* \otimes \mathscr{S}) = H^1(\mathscr{H}om(\mathscr{Q}, \mathscr{S}))$, as $\delta : H^0(\mathscr{O}) \to H^1(\mathscr{Q}^* \otimes \mathscr{S})$ is equivalent to giving $\delta \in H^1(\mathscr{Q}^* \otimes \mathscr{S})$.*

So then by definition, $\operatorname{Ext}(\mathscr{Q}, \mathscr{S})$ consists of all extensions of $\mathscr{Q}$ by $\mathscr{S}$, and $H^1(\mathscr{H}om(\mathscr{Q}, \mathscr{S}))$ is a cohomology group that classifies extensions.

Theorem 4.13 (Grothendieck)**.** *Every vector bundle on $\mathbb{P}^1$ is a direct sum of line bundles.*

Proof. The idea is that, given $\mathscr{V}$, we can find a line bundle contained in $\mathscr{V}$, $\mathscr{L}$, which is maximal of degree d.

(Note that a subsheaf which is locally free might not be a subbundle, depending on whether the quotient is locally free)

A locally free subsheaf of a locally free sheaf is a subbundle iff it is maximal.

Use Riemann-Roch to prove that the set of possible degrees is bounded above. Now replace $\mathscr{V}$ by $\mathscr{V} \otimes \mathscr{L}^{-1}$. Equivalently, $d = 0$, and than tensoring with $\mathscr{L}^{-1}$ then there are no global sections left.

If $d > 0$, then the sections of $\mathscr{V}$ cannot vanish anywhere.

So we have $0 \to H^0(\mathscr{V}) \otimes_{\mathbb{C}} \mathscr{O} \to \mathscr{V} \to \mathscr{Q} \to 0$ gives a long exact sequence $0 \to H^0(\mathscr{V}) \simeq H^0(\mathscr{V}) \to H^0(\mathscr{Q}) \to H^0(\mathscr{V}) \otimes H^1(\mathscr{O})$, but the isomorphism causes $H^0(\mathscr{Q}) = 0$, and so $\mathscr{Q}$ is a vector bundle of lower rank. Then by induction, $\mathscr{Q} = \oplus_i \mathscr{O}(-d_i)$ for $d_i > 0$.

So we have now that V can be written as an extension $0 \to \oplus_j \mathscr{O} \to \mathscr{V} \to \oplus_i \mathscr{O}(-d_i) \to 0$

We claim that this extension splits.

All such extensions split, because

$$\operatorname{Ext}(\oplus_i \mathscr{O}(-d_i), \oplus_j \mathscr{O}) = \oplus_{i,j} \operatorname{Ext}(\mathscr{O}(-d_i), \mathscr{O}) = \oplus_{ij} H^1(\mathbb{P}^1, \mathscr{O}(d_i)) =$$

$$\oplus_{ij} H^0(\mathbb{P}^1, K(-d_i))^* = \oplus_{ij} H^0(\mathscr{O}(-d_i - 2))^* = 0$$

. □

4.4 Jumping Phenomenon

Definition 4.7 (Family of Vector Bundles)**.** *A family of vector bundles parameterized by $\mathbb{A}^1$ is a vector bundle on $X \times \mathbb{A}^1$. We define $V_t = i_t^* V$ a vector bundle on X.*

Proposition 4.14. *There exists a family of vector bundles such that $s \neq 0, t \neq 0$ implies $V_s \simeq V_t$ but $V_t \not\simeq V_0$ for $t \neq 0$.*

Equivalently, the topology on $\{$isomorphism classes of vector bundles on $X\}$ is non Hausdorff. The point $\{V_0\}$ is in the closure of $\{V_1\}$.

It looks like the quotient $\mathbb{A}^1/\mathbb{C}^*$, or in general $\mathbb{A}^n/\mathbb{C}^*$, which is $\mathbb{P}^{n-1}\coprod\{0\}$.

Example 4.9. *Let E be an elliptic curve. Then $\pi : E \to \mathbb{P}^1$ is two to one. Let $\mathscr{L} \in \mathrm{Pic}^0(E) \simeq E$ be a line bundle. Then we have $0 \to E \to \mathrm{Pic}(E) \to \mathbb{Z} \to 0$ a short exact sequence. So what is $\pi_*\mathscr{L}$? It's a locally free sheaf on $\mathbb{P}^1$ of rank 2.*

So it is $\mathscr{O}(a) \otimes \mathscr{L}$. This has $\deg(\pi_*\mathscr{O}) = \deg \mathscr{L} - \frac{1}{2}(\deg R) = -2$, so $h^0(\mathbb{P}^1, \pi_*\mathscr{L}) = h^0(E, \mathscr{L}) = \begin{cases} 1 & \mathscr{L} = \mathscr{O} \\ 0 & \mathscr{L} \neq \mathscr{O} \end{cases}$.

However, also $h^0(\mathbb{P}^1, \mathscr{O}(a+1)) = \left\{\begin{matrix} a+1 & a \geq -1 \\ 0 & a \leq -1 \end{matrix}\right\} = \max\{0, a+1\}$. So we are left with $\max(0, a+1) + \max(0, b+1) \in \{0, 1\}$, and $a + b = -2$. We will write $b = -2 - a$, and so we have $\max(0, a+1) + \max(0, -(a+1))$ is 1 or zero. So this is $|a+1|$. If $\mathscr{L} = \mathscr{O}$ then $|a+1| = 1$, and so $a = 0$ or $a = -2$. If $\mathscr{L} \neq \mathscr{O}$, then $|a+1| = 0$, so $a = -1$.

Alternatively, tensor with $\pi^*\mathscr{O}_{\mathbb{P}^1}(1)$, $\pi_*(\mathscr{L}) = \mathscr{O}(a) \oplus \mathscr{O}(b)$, and $\pi_*(\mathscr{L} \otimes \pi^*\mathscr{O}(-1)) = \mathscr{O}(a+1) \oplus \mathscr{O}(b+1)$.

This is an application of the following:

Theorem 4.15 (Projection Formula)**.** *Let $\pi : X \to Y$ and let $\mathscr{F}$ a sheaf on X and $\mathscr{G}$ a sheaf on Y. Then $\pi_*(\mathscr{F} \otimes \pi^*\mathscr{G}) = \pi_*\mathscr{F} \otimes \mathscr{G}$.*

Corollary 4.16. *$h^0(E, \mathscr{L} \otimes \pi^*\mathscr{O}(d)) = h^0(\mathbb{P}^1, \mathscr{O}(a+d)) + h^0(\mathbb{P}^1, \mathscr{O}(b+d))$ for $d >> 0$ this gives $0 = a + b + 2$ so $a + b = -2$.*

And, to phrase it as a theorem rather than a remark that has come up,

Theorem 4.17 (Degree Shift)**.** *Let $\pi : X \to Y$ a morphism and $\mathscr{L}$ a line bundle on X. Then $\deg(\pi_*\mathscr{L}) = \deg \mathscr{L} - \frac{1}{2}\deg R$.*

Corollary 4.18. *$\chi(\pi_* L) = \chi(L)$.*

5 Moduli and Deformations

We are interested in looking at families of objects. This leads to notions of moduli spaces, and to infinitesimal moduli, that is, deformations.

Definition 5.1 (Deformation Space)**.** *The deformation space of X is $\mathrm{Def}(X) = T_{[X]}\{moduli\ of\ X's\}$.*

Proposition 5.1. *Let C be a smooth projective curve and let $\mathscr{L}$ be a line bundle. Then $\mathrm{Def}(\mathscr{L}) = H^1(X, \mathscr{O}_X)$*

Proof. Let X be a variety, and $\mathscr{L} \in \mathrm{Pic}(X)$, what is $T_{\mathscr{L}}\,\mathrm{Pic}(X)$? Here we have a solution to the nonlinear problem, that is, $\mathrm{Pic}(X) = H^1(X, \mathscr{O}_X^*)$.

To specify $\mathscr{L} \in \mathrm{Pic}(X)$, give an open cover $\underline{U}$ and gluing function $g_{ij} \in \Gamma(U_{ij}, \mathscr{O}^*)$ satisfying a cocycle condition.

A transition function is a 1-cochain, and a cocycle condition is a 1-cocycle, and so fixing $\underline{U}$, $\mathscr{L}$ corresponds to a set of consistent transition functions. This is more like a line bundle with a trivialization on U_i.

So how about infinitesimal deformations?

Start with $\mathscr{L} \mapsto \{g_{ij}\}$. We want to deform to first order. Rather than looking at $X \to \operatorname{Spec} \mathbb{C}$ as a complex scheme, we look at $X \mapsto \operatorname{Spec} \mathbb{C}[\epsilon]$.

We have natural maps $\mathbb{C}[x] \to \mathbb{C}[\epsilon] \to \mathbb{C}$, and so $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{C}[\epsilon] \to \operatorname{Spec} \mathbb{C}[x]$.

So we want to replace g_{ij} with $g_{ij} + \epsilon h_{ij}$.

So we want to know what the consistent ways of writing h_{ij}'s. So what does the cocycle condition say about the h_{ij}'s?

$(g_{ij} + \epsilon h_{ij})(g_{jk} + \epsilon h_{jk}) = (g_{ik} + \epsilon h_{ik})$

To zeroth order, this gives $g_{ij} g_{jk} = g_{ik}$, which is just saying that $\{g_{ij}\}$ is a line bundle, and so is satisfied.

To first order, it says $g_{ij} h_{ik} + h_{ij} g_{ik} = h_{ik}$.

And we don't care about the second order terms or higher.

The set of solutions are then the solutions to a linear equation, and so we have linearized the problem. Let us define $H_{ij} = g_{ij}^{-1} h_{ij}$.

Multiply by g_{ki}(that is, divide by g_{ik}). Note that $g_{ij}/g_{ik} = g_{kj}$, and so now the equation is $H_{jk} + H_{ij} = H_{ik}$.

This cocycle condition gives $H^1(X, \mathscr{O})$. □

Example 5.1. *So we have objects line bundles on X, moduli space* $\operatorname{Pic}(X) = H^1(X, \mathscr{O}^*)$ *and deformations* $H^1(X, \mathscr{O}_X)$.

Example 5.2. *If we look at rank n vector bundles on a fixed variety X, then the moduli space is* $\operatorname{Bun}_n(X) = H^1(X, GL_n(\mathscr{O}))$*, and the deformations are elements of* $H^1(X, End(V)) = H^1(X, V \otimes V^*)$ *at V.*

Example 5.3. *If our objects are curves of genus g, then $\mathscr{M}_g$, then the tangent space at C a curve is* $T_{[C]}(\mathscr{M}_g) = H^1(C, \mathscr{T}_C) = H^0(C, K_C^{\otimes 2})^*$ *by Serre Duality, for $g \geq 2$.*

Example 5.4. *Let X be any variety, and $\mathscr{M}_X$ the moduli space of varieties with the same underlying complex topology, then* $T_X(\mathscr{M}_X) = H^1(X, \mathscr{T}_X)$.

Example 5.5 (Principal G-bundles)**.** *Then look at the moduli space* $\operatorname{Bun}_G(X)$*, and* $T_V(\operatorname{Bun}_G(X)) = H^1(X, \operatorname{Aut}(V))$ *where* $V \in \operatorname{Bun}_G(X)$.

Example 5.6. *Fix Y a variety and vary X a subvariety. Then we have the moduli space $\mathscr{M}_{X \subset Y}$ and* $T_X(\mathscr{M}_{X \subset Y}) = H^0(X, \mathscr{N}_{X/Y})$.

Example 5.7. *Varying both Y and X. This is no good, but instead we can look at varying both X and a $\mathscr{L}$ a line bundle on X. There should be a map to $\mathscr{M}_X$ forgetting $\mathscr{L}$, but there shouldn't be a map to the moduli of $\mathscr{L}$.*

6 Lecture

Last time: How to calculate $h^{p,q}$ for a hypersurface in $\mathbb{P}^{n+1}$.

Answer: If $p+q \neq n$, then we have $\delta_{p,q}$. The interesting case is $p+q=n$. Then $h^{p,q} \simeq R^{d-n-2+pd}$ where $R = S/\mathscr{J}$, the Jacobian.

Example 6.1. *Let $S = \mathbb{C}[x,y]$ and $f = \prod_{i=1}^{d}(x - a_i y)$.*

Take as an example $f = x^d + y^d$. Then $\mathscr{J} = (x^{d-1}, y^{d-1})$, and so $\dim_{\mathbb{C}} R = (d-1)^2$. (We are taking $\mathbb{C}[x,y]$ as the affine plane here)

Proposition 6.1. $\dim R$ *is constant as f varies as long as X remains smooth. Similarly for R^k.*

So in this case, the hodge numbers are topological invariants. However, this is not true in general, sometimes it depends on the complex structure.

Now let X be a plane curve in $\mathbb{P}^2$, then $h^{1,0} = R^{d-n-2} = R^{d-3}$ and $h^{0,1} = R^{2d-3}$. By Serre duality, $\dim R^{d-3} = \dim R^{2d-3}$.

Again, consider $f = x^d + y^d + z^d$. Then if $d = 5$, we have $g = 6$.

$R^0 = \mathbb{C} \cdot 1$, has dimension 1. $R^1 = \mathbb{C}x + \mathbb{C}y + \mathbb{C}z$ has dimension 3. $R^2 = \mathbb{C}x^2 + \mathbb{C}y^2 + \mathbb{C}z^2 + \mathbb{C}xy + \mathbb{C}xz + \mathbb{C}yz$ is dimension 6. R^3 is generated by cubic monomials, and there are 10. R^4 has dimension 12. R^5 has dimension 12, R^6 is 10, R^7 is 6, R^8 is 3 and R^9 is 1.

Start with a smooth hypersurface of degree d and dimension n. Variation of Hodge structure gives vector spaces R^{d-n-2} thgouth $R^{d(n+1)-n-2}$. We have bilinear maps $R^d \times R^{d-n-2} \to R^{2d-n-2}$, etc, and the top piece has dimension 1.

Can you recover X (ie, its equation f) from this bilinear data?

Sometimes the answer is no.

Example 6.2. *Let $n = 2, d = 3$, that is, we have a plane cubic curve, an Elliptic Curve.*

We're given R^k, which are all one dimensional. There's no information here, any two elliptic curves give the same map.

However, looking at a plane quintic, there are four different groups and nontrivial maps between them, and we can do it.

In most cases, the answer is yes.

Next: Calculate $H^i(\mathbb{P}^r, \mathscr{O}(n))$ from the definition.

Take the standard open cover of $\mathbb{P}^r$ by copies of $\mathbb{A}^r$.

Then take $\mathscr{F} = \oplus_{n\in\mathbb{Z}}\mathscr{O}(n)$. We have $S = \mathbb{C}[x_0, \ldots, x_r] = H^0(\mathbb{P}^r, \mathscr{F}) = \oplus_{n\in\mathbb{Z}} H^0(\mathbb{P}^r, \mathscr{O}(n))$.

$\check{C}^*(\mathcal{U}, \mathscr{F}) : \prod_{i_0=0}^{r} S_{x_{i_0}} \to \prod_{0\leq i_0, i_1 \leq r} S_{x_{i_0}, x_{i_1}} \to \ldots$.

And so for $r = 1$, $H^0(\mathscr{F}) = S = \mathbb{C}[x,y]$, $H^1(\mathscr{F}) = x^{-1}y^{-1}\mathbb{C}[x^{-1}, y^{-1}]$. And now, $H^1(\mathbb{P}^1, \mathscr{O}(n))$ consists of the strictly negative monomials of degree $-n$.

The idea in general is that the Čech complex decomposes into a direct sum of subcomplexes, one for each monomial.

The part of the Cech complex for a given monomial is the Koszul Complex for this Vector Space.

That is, $\mathbb{C} \to V \to \bigwedge^2 V \to \bigwedge^3 V \to \dots$. All that we must do is compute the cohomology of this complex. This complex is exact except at the beginning and end...

Wait...

The answer is SUPPOSED to be...???

7 WEDNESDAY

Calculation: $H^k(\mathbb{P}^r, \mathscr{O}(n))$.

$S = \mathbb{C}[x_0, \dots, x_r] \subset \mathbb{C}[x_i^{\pm 1}] = S_{x_0,\dots,x_r}$.

Let $R = \{0, \dots, r\}$ and $I \in \mathbb{Z}^R$, with x^I monomials. Then $J \subset R$ we get a localization S_J by inverting x_i for $i \in J$.

$S_j = \oplus_{|J|=j} S_J$. The Cech complex for the standard $\mathcal{U}$ and the quasicoherent $\mathscr{F} = \oplus_n \mathscr{O}(n)$ is $S_1 \to S_2 \to S_3 \to \dots \to S_{r+1}$.

We augment the complex and have $S_0 \to S_1 \to \dots \to S_{r+1}$, and this augmented complex $\mathcal{C}_*$ has decomposition $e_* = \oplus_I e^I$ where e_*^I is $S_0^I \to S_1^I \to \dots \to S_{r+1}^I$ where $I = I^- \coprod I^+$ and $I^- = \{r \in R | I_i < 0\}$ and $I^+ = I \setminus I^-$. $a = |I^+|$.

The coefficient of x^I in S_J is 0 or $\mathbb{C}$ (if $J \supset I^-$) and the coefficient in S_j is $\mathbb{C}$, $\{J | J \supset I^-, |J| = j\}$.

So the complex is in fact $0 \to 0 \to \dots \to 0 \to \mathbb{C} \to \mathbb{C}^{r+1-a} = V \to \bigwedge^2 V \to \dots$. So the coefficient in S_j is $\bigwedge^{|J|-a} V$.

We've identified the spaces. What are the maps? V has a basis of degree 1 monomials not in I. $V \ni e$ is a sum of these monomials with coefficient 1. Then $1 \mapsto e \mapsto v \wedge e$ and $v_1 \wedge v_2 \mapsto v_1 \wedge v_2 \wedge e$.

This is the Koszul Complex. In a different basis, we take $e = (1, 0, \dots, 0)$ and $V = \mathbb{C}e \oplus V_0$. Then $\bigwedge^k V = e \bigwedge^{k-1} V_0 \oplus \bigwedge^1 V_0$.

And so we get maps $\bigwedge^k V_0 \to e \bigwedge^k V_1$, etc. And these maps make the complex exact, unless $0 = \dim V = |I^+|$. That is, unless $I < 0$ strictly.

And so $H^*(\check{C}_*^I) = \begin{cases} \mathbb{C} & * = r, I^- = R \\ 0 & else \end{cases}$ and $H^*(\check{C}_*) = \begin{cases} 0 & * \neq r \\ \frac{1}{\prod x_i} \mathbb{C}[x_i^{-1}] & * = r \end{cases}$.

And so $h^r(\mathbb{P}^r, \mathscr{O}(-(r+1) - n)) = h^0(\mathbb{P}^r, \mathscr{O}(n))$. And as $\mathscr{O}(-r-1) = K$, this gives us Serre Duality for $\mathbb{P}^n$.

Now we will sketch proofs of deRham, Dolbeault and Cech theorems.

dR: On an algebraic manifold X, look at $0 \to \mathbb{C} \to A^0 \to A^1 \to \dots$. We want to break dR into a sequence of short exact sequences, $0 \to \mathbb{C} \to A^0 \to Z^1 \to 0$, $0 \to Z^1 \to A^1 \to Z^2 \to 0$, etc. So for each we get a long exact sequence on cohomology. Check things carefully, it works out by induction.

Similarly for Dolbeault. We get an acyclic (fine) resolution of Ω^p, and so we break into SES, and then use long exact sequence of cohomology.

Similarly for Cech, II.3.5 in Hartshorne.

Let M be an A module. Then $\tilde{M}$ is a sheaf of $\mathscr{O}_X$ modules for$A = \Gamma(X, \mathscr{O}_X)$.

We get for $f \in A$, $X_f = \{x \in X | f(x) \neq 0\}$, and $\Gamma(X_f, \tilde{M}) = M_f$. Then sheafify. A sheaf $\mathscr{F}$ of $\mathscr{O}_X$-modules is quasicoherent if locally $\mathscr{F}|_{U_i} = \tilde{M}_i|_{U_i}$.

Coherent is the same, but with M_i finitely generated. (This is in the Zariski topology)

Proposition 7.1. *Take $M = \Gamma(X, \mathscr{F})$ and $\tilde{M} = \mathscr{F}$. On $X = \operatorname{Spec} A$, the functor $M \mapsto \tilde{M}$ from A-modules to quasicoherent sheaves on $\operatorname{Spec} A$ is an equivalence of categories with inverse functor $\Gamma(X, -)$.*

Take an injective resolution of M, $0 \to M \to I^*$. Sheafifying, we get $0 \to \tilde{M} = \mathscr{F} \to \tilde{I}^*$ is an exact sequence of q.c. $\mathscr{O}_X$-modules. And so $\mathscr{F}$ has a resolution by injectives, flasques, acyclic, and so $H^*(X, \mathscr{F}) = H^*(\Gamma(I^*))$.

The proof imitates the proof for dR and Dolbeault.

8 FRIDAY

Today and Wednesday we will do Derived Functors and Categories

Then cancel class for a conference at IAS

And we will finish up on Chern Classes, K-Theory, and Grothendieck-Riemann-Roch

On Monday 11/19, we will finish talking about deformation theory and the Torelli Problem

11/26 no class, IAS

12/3 ???

The Torelli Problem: We are giving a finite number of bilinear forms $A \times B \to C$ and we're told that there exists a vector space V and polynomial $f \in \operatorname{Sym}^d V$ such that $R^i \times R^j \to R^{i+j}$ is isomorphic to $A \times B \to C$. Can we completely recover f?

Easier question: Given $S^i \times S^j \to S^{i+j}$ isomorphic to $A \times B \to C$. Example: $i = j = 1$, then we have $V \times V \to \operatorname{Sym}^2 V$ isomorphic to $A \times B \to C$. Pick $a \in A, b \in B$, and look at $\{a' \in A, b' \in B | a'b' = ab\}$. There are two possibilities. One is that the factorization is unique, up to $\mathbb{C}^*$, and the other is that there exist two factorizations.

Geometrically, $\mathbb{P}(V) \times \mathbb{P}(V) \to \mathbb{P}(\operatorname{Sym}^2 V)$ has image a segre variety projected to the symmetric product. The image is isomorphic to $\operatorname{Sym}^2 \mathbb{P}(V)$, ($\dim V \geq 3$ implies that this is singular)

So bilinear form gives a projected segre variety, which has singular locus $\mathbb{P}(V)$ embedded in $\mathbb{P}(\operatorname{Sym}^2 V)$ by $\mathscr{O}(2)$. And so we get $C \simeq \operatorname{Sym}^2 A$ for dimension ≤ 2, replace singular by "having unique inverse maps"

More talk of Torelli rather than the promised Derived Categories.

8.1 Derived Categories

"Derived Categories for the Working Mathematician" by Thomas. Available on the Arxiv

The idea is to go from objects to complexes to equivalence categories of complexes.

Input category: Abelian.

Definition 8.1 (Additive Category)**.** *For any* A, B *objects of* $\mathcal{A}$*,* $\hom(A, B)$ *is an abelian group and composition is distributive over addition, finite products exist, and there exists a zero object.*

Definition 8.2 (Abelian Category)**.** *An abelian category is an additive category such that any morphism has a kernel and a cokernel,* f *monic implies that* $f = \ker \operatorname{coker} f$ *and* f *epic implies* $f = \operatorname{coker} \ker f$*.*

In an abelian category, we can discuss the following notions:

complexes, exact sequences, and cohomology of a complex, morphisms of complexes, quasi-isomorphism, additive, exact, left and right exact functors.

So if $\mathcal{A}$ is an abelian category, then $C(\mathcal{A})$ is teh category of complexes.

So now $D(\mathcal{A})$, the derived category, has the same objects and the morphisms are "invert all quasiisomorphisms". So a morphism in $D(\mathcal{A})$ is a zigzag...confusing gibberish, will look in book.

Lemma 8.1. *Any morphism is* $A \xrightarrow{f} C \xleftarrow{g} B$*.*

There are more common flavors than the general derived category.

D^+, D^-, D^b are the bounded above, bounded below, and bounded complexes.

9 WEDNESDAY

"Complexes Good, Cohomology Bad"

Note that $D^b(\mathcal{A})$ is an additive category, but is not Abelian, because kernels and cokernels do not necessarily exist.

The actual structure we have is that of a triangulated category.

Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor. A class $R \subset ob(\mathcal{A})$ is adapted if R is stable under direct sums and for any $A \in ob(\mathcal{A})$ injective, there is an element in R, X, such that $0 \to A \to X$ is exact, and if F(exact complex of elements of R) is exact.

Lemma 9.1. $D^+(\mathcal{A})$ *is obtained by inverting quasi-isomorphisms in* $K^*(\mathcal{A})$ *the category of lower bounded complexes in* $\mathcal{A}$ *with objects in* R *with morphisms homotopy classes of chain maps.*

Example 9.1. *Injective objects work for* $F = \Gamma$*.*

There exists an analogue for right exact functors.

If F is left exact, then we can construct $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$, the right derived functor of F.

As $D^+(\mathcal{A})$ is equivalent to $K^+(R)/q.i.$, we can define RF, and do so by applying F.

RF is an exact functor, but what does this mean? It takes "long exact sequences" to "long exact sequences"

In general, $RF : D^+(A) \to D^+(B) \to B$ with the last one by H^i. These compositions are R^iF, the ith derived functor, because they factor through $D^+(A)$ as functors $A \to B$.

Example 9.2. *$F = \Gamma$:Sheaves$\to$ Ab. Then $R^i\Gamma = H^i$ as above, with H^i the sheaf cohomology we defined. $R\Gamma$ contains all this information, and a bit more as well.*

Example 9.3. *$F = \otimes\mathscr{F}$ from sheaves to sheaves. And this is right exact. So $L\otimes$, which we write $\overset{L}{\otimes}$, the ith derived functor is called Tor_i*

Example 9.4. *$\mathscr{H}om(\mathscr{F},\mathscr{G})$ is covariant in $\mathscr{G}$ and contravariant in $\mathscr{G}$, and the first is left exact and the second is also left exact. These derived functors are called Ext^i.*

Not always, but most of the time, $R(F \circ G) \cong RF \circ RG$

In fact, $R^n(F \circ G) \Leftarrow R^iF(R^jG)$ (this is not quite convergence, but spectral sequence convergence)

This is called the Leray Spectral Sequence.

Example 9.5. *Let $p : X \to Y$ and $F = \Gamma(Y, -)$, $G = p_*$, then we have functors Sheaves$(X) \to$ Sheaves$(Y) \to$ Ab, then Leray gives $H^i(Y, R^jp_*\mathscr{F}) \to H^{i+j}(X,\mathscr{F})$.*

The R^ip_* are called the higher direct images, and so we have $Rp_* : D^+(X) \to D^+(Y)$.

In the C^∞ world, we recover the dR, Dolbeault and Čech Theorems.

Cones and Triangles

In topology, take $f : X \to Y$ be a continuous map. Construct $cone(f)$ to be $X \times [0,1] \coprod Y/X \times 0, (x,1) \sim f(x)$.

The use is that it gives a sequence on homology, $H_i(X) \to H_i(Y) \to H_i(C_f) \to H_{i-1}(X) \to \ldots$. So C_f acts like (X,Y). $C_i(C_f) = C_i(Y) \oplus C_{i-1}(X)$.

So, given a chain map $f : A \to B$, we define C_f to be $A[1] \oplus B$, and d_{C_f} is a two by two matrix $\begin{pmatrix} d_A[1] & 0 \\ f & d_B \end{pmatrix}$. (check to make sure)

So if we have $A^* \xrightarrow{c} B^* \xrightarrow{i} C_f^*$, then $C^*(i) = A^*[1]$. So this is really like a triangle, and is the analogue of an exact sequence.

10 Lecture

MISSED

11 Lecture

Previously we looked at varieties and schemes, but last time we started to look at properties of morphisms $f : X \to Y$

Affine, projective, proper, finite

smooth, flat

The top line is proper in Y, the bottom is local in X.

Given $f : X \to Y$, what is the expected behavior of the family of cohomologies $H^i(X_y, \mathscr{F}_y)$, where $\mathscr{F}$ is a coherent sheaf on X and X_y is the fiber $f^{-1}(y)$.

Example 11.1. *Let E be an elliptic curve and $Y = E$, $X = E \times E$. Then $f : X \to Y$ is a map $E \times E \to E$ with the map the second projection. Then let $\mathscr{F} = \mathscr{O}_{E\times E}(\Delta - E_1 - E_2)$ where Δ is the diagonal, and $E_1 = E \times \{0\}$, and similarly for E_2.*

For each $y \in Y$, $X_y \simeq E$.

$\mathscr{F}_y = \mathscr{O}_E(y - 0)$. This has $h^1 = h^0$ 1 if $y = 0$ and 0 else.

Define $\chi(X, \mathscr{F}) = \sum_i (-1)^i h^i(X, \mathscr{F})$ to be the Euler characteristic of the sheaf $\mathscr{F}$.

Assume $X \subset \mathbb{P}^n$ projective. Here we have $\mathscr{O}_X(1)$. We define the Hilbert polynomial $\mathbb{P}(n) = \chi(X, F \otimes \mathscr{O}(n)_X)$.

Note: for $n >> 0$, all h^i vanish for $i > 0$.

$f : X \to Y$, f_* is left exact, so we get $R^i f_*$ the higher derived images.

What is the behavior of $R^i f_x \mathscr{F} \otimes k(y)$ as a function of $y \in Y$ vs $h^i(X_y, \mathscr{F}_y)$?

Theorem 11.1. *Let $f : X \to Y$, $\mathscr{F}$ on X, then $P_{\mathscr{F}_y}(n)$ is independent of y if and only if $\mathscr{F}$ is flat over Y. In particular, $P_{\mathscr{O}_{X_y}}(n)$ is independent of xy iff X is flat over Y.*

If $f : X \to Y$ is any projective morphism of noetherian schemes and $\mathscr{F}$ is coherent over X and flat over Y, then for all $i \geq 0$, the function $h^1(y, \mathscr{F}) = \dim H^(X_y, \mathscr{F}_y)$ is upper semicontinuous.

Assume also that Y is integral. If $h^i(Y, \mathscr{F})$ is constant, then $R^i f_* \mathscr{F}$ is locally free on Y.

$(R^i f_* \mathscr{F} \otimes k(y) \Rightarrow H^i(X_y, F_y)$. (Graouert?)

$f : X \to Y$ projective morphism of noetherian schemes. $\mathscr{F}$ coherent on X and flat over Y. Then if $\rho^i(y) : R^i f_* \mathscr{F} \otimes k(y) \to H^i(X_y, \mathscr{F}_y)$ is surjective at y, then it is an isomorphism on some open set containing y.

Now we have functors $Coh(X) \to Coh(Y) \to VS$ or we can do $Coh(X) \to Coh(X_y) \to VS$, (the last map is global sections) and the obvious diagram commutes. And ρ^i are derived functors of the restriction map

If this holds, then $\rho^{i-1}(y)$ is surjective iff $R^i f_* \mathscr{F}$ is locally free near y.

12 Last Lecture

Another perspective with higher direct images: does taking cohomology commute with base change?

Base chance is, given $f : X \to Y$, and $u : Y' \to Y$, we have $g : X' \to Y'$, with $X' = Y' \times_Y X$. This is called the base change, with $v : X' \to X$. Then define $\mathscr{F} = v^* \mathscr{F}$ a sheaf on X' for a sheaf $\mathscr{F}$ on X. Fiber by fiber this "looks like" $\mathscr{F}$.

In this situation, we get a natural morphism $\exists \phi^i : u^* R^i f_* \mathscr{F} \to R^i g_*(v^* \mathscr{F})$.

Special case: $Y' = \{y\}$, $y \in Y$. Then $X' = X_y$ is the fiber, and $\mathscr{F}_y$ is $\mathscr{F}'$.

In this case, $R^i g_* \mathscr{G} = H^i(X_y, \mathscr{G})$.

The map from above is $\phi^i_y : (R^i f_* \mathscr{F})_y \to H^i(X_y, \mathscr{F}_y)$ from Wednesday.

Q: If ϕ^i an isomorphism?

A: If u is a FLAT base change, the ϕ^i is an isomorphism. The question is local on Y', so we can take $Y = \operatorname{Spec} A$, $Y' = \operatorname{Spec} A'$. So it boils down to $H^i(X, \mathscr{F}) \otimes_A A' \cong H^i(X', \mathscr{F}')$.

This is a straightforward Cech Cohomology computation.

This is also true if u is inclusion of a closed point, then ϕ^i is an isomorphism for the top i.

In more detail, ϕ^i is surjective at y iff isomorphism at y implies that is is an isomorphism at $\tilde{y}$. And then ϕ^{i-1} is surjective iff $R^i f_*$ is locally free.

(Case $i = \dim X - \dim Y + 1$, and i is the top dimension of fibers. Then ϕ^i is an isomorphism. $R^i f_* \mathscr{F} = 0$ is locally free, and so ϕ^{top} is an isomorphism.)

An elliptic fibration is a morphism $\pi : Z \to B$ Choose B, Z smooth and π flat with the generic fiber an elliptic curve. Take a section $u : B \to Z$. We ARE allowing some fibers to be singular.

e.g, $f, g \in \Gamma(\mathbb{P}^2, \mathscr{O}(3))$, then for all $[s,t] \in \mathbb{P}^1$, $sf + tg$ is a plane cubic curve, it is elliptic if smooth. If it is smooth, it is an elliptic curve. We want something like $f/g : \mathbb{P}^2 \to \mathbb{P}^1$ by $[x : y : z] \mapsto [f(x,y,z) : g(x,y,z)]$.

Let $Z = \{((x,y,z),(s,t)) \in \mathbb{P}^2 \times \mathbb{P}^1$ such that $sf(x,y,z) + tg(x,y,z) = 0\}$. This is a variety. The inverse image of the first projection gives $\mathbb{P}^1$ if $f(x,y,z) = g(x,y,z) = 0$ and is a point otherwise. Then $Z\setminus$ nine copies of $\mathbb{P}^1$ is isomorphism to $\mathbb{P}^2\setminus$ 9 points.

So if we take $B = \mathbb{P}^1$, each of the removed $\mathbb{P}^1$'s gives a section, and so we have an elliptic fibration.

This surface is a del Pezzo surface (not quite, actually...del Pezzo only goes up to 8, this is ALMOST del Pezzo)

General construction. Start with B and a line bundle $\mathscr{L}$. Then $\mathbb{P}^2 \to \mathbb{P}_B(L^2 \oplus L^3 \oplus \mathscr{O}) = P$ is a $\mathbb{P}^2$ bundle over B. A plane cubic can be written in Weierstrass form $y^2 = x^2 - g_2 x - g_3$. Choose $g_2 \in \Gamma(B, \mathscr{L}^4)$ and $g_3 \in \Gamma(B, \mathscr{L}^6)$. Then Z is the solution space for the homogeneous weierstrass equation.

Recall: On one elliptic curve, E, we have a Poincare sheaf $\mathscr{P} = \mathscr{O}_{E\times E}(\delta - E_1 - E_2)$. It's a line bundle. It's restriction to $E \times a$ is the line bundle $\mathscr{O}_E(a - 0)$. This gives an isomorphism $E \to \operatorname{Pic}^0(E)$. This works verbatum for elliptic fibrations.

$\mathscr{P}'$ is a sheaf on $Z \times_B Z$ and is $\mathscr{O}_{Z\times_B Z}(\Delta - Z_1 - Z_2)$.

We define $\mathscr{P} = \mathscr{O}_{Z\times_B Z}(\Delta - Z_1 - Z_2) \otimes \tilde{\pi}^*(\mathscr{L}^{-1})$ (where $\tilde{\pi} : Z \times_B Z \to B$ is the composition of projection and structure map)