1 Lecture 1

Possible Topics:

1. Hodge Conjecture. Probably about 5 lectures.
2. Geometric Langlands. Perhaps 5 lectures, and might contain geometry of derived categories
3. Geometry of Derived Categories
4. Grothendieck-Riemann-Roch? Maybe 2 lectures
5. Mirror Symmetry/Calabi-Yau's. As many lectures as needed.

2 Lecture 2

Geometric Langlands Program.

Rather than describe the program right out, we will discuss examples first.

This is essentially a nonabelian phenomenon.

Input: A curve (smooth, compact, genus g) C and G a reductive group. Examples of reductive groups are $GL(n), SL(n), SO(n), Sp(n)$, products, etc.

Baby case: Abelian Geometric Langlands Conjecture. We have $G = (\mathbb{C}^*)^r$, and will take $r = 1$. This will roughly correspond to Class Field Theory in Number Theory, and the full GLC corresponds to nonabelian class field theory.

Roughly we have objects G-bundles versus G local systems. In the case $G = GL(n)$, we have that a G-bundle is a rank n vector bundle and a G-local system is a homomorphism $\pi_1(C) \to G$ which is the same as a rank n vector bundle with a flat connection. (All connections are flat, because curvature is a 2-form, and we are on a 1-dimensional object)

The moduli of rank n vector bundles is $H^1(C, GL_n(\mathscr{O})) = Bun$ and the moduli space of G-local systems is $H^1(C, GL_n(\mathbb{C})) = Loc$.

We know both that $T_V Bun = H^1(C, End_{\mathscr{O}}(V))$ and that $T_{(V,\nabla)}Loc = H^1(C, End_{\mathbb{C}}(V))$.

We want to compute these dimensions, and we will use Riemann-Roch: $h^0 - h^1(V \otimes V^*) = \deg - rank(g-1)$.

If $n = 1$ then $V = \mathscr{L}$. $End(V) = V \otimes V^* = \mathscr{L} \otimes \mathscr{L}^{-1} = \mathscr{O}$, so deg is zero.

If $n = 2$ and split, then $V = L_1 \oplus L_2$. So $EndV = L_1 \otimes L_1^{-1} \oplus L_1 \otimes L_2^{-1} \oplus L_2 \otimes L_1^{-1} \oplus L_2 \otimes L_2^{-1}$ which is $0 + d_1 - d_2 + d_2 - d_1 + 0 = 0$.

And so, $\deg(End(V)) = 0$ for all V. First we note that $\deg V = \deg \det V$, and $\det(\mathrm{End}\, V) = \bigwedge^{n^2}(End(V))$.

And so $\dim_V Bun = h^1(C, End(V)) = n^2(g-1) + h^0(C, End(V))$.

For a "general" vector bundle, $h^0 = 1$, and so generically $\dim_V Bun = n^2(g-1) + 1$

The dimension of $T_{(V,\nabla)}Loc$ is $2 \dim_V Bun$.

Rough Statement of the Geometric Langlands Program: Any G local system E on C determines a local system c_E on a Zariski open subset of $Bun_{C,G}$, the moduli space of bundles with structure group G on C. The key property is that this is an automorphic sheaf. If we extend to the whole space, we get a "perverse sheaf."

We will not be talking about perverse sheaves and D-modules, but will focus on automorphic sheaves and restrict ourselves to open subsets whenever necessary.

Example 2.1. *Case where $n = 1$, so $G = GL(1) = \mathbb{C}^*$. On this, $Bun_{C,GL(1)} = \mathrm{Pic}(C)$. Abelian Langlands tells us that every $\mathbb{C}^*$ local system on C determines a $\mathbb{C}^*$ local system on $\mathrm{Pic}(C)$.*

A local system on C means a map $\rho : \pi_1(C) \to \mathbb{C}^*$. A $\mathbb{C}^*$-local system on $\mathrm{Pic}^0(C) = Jac(C)$ is $c_\rho : \pi_1(Jac(C)) \to \mathbb{C}^*$. We note that $\pi_1(Jac(C)) = H_1(Jac(C), \mathbb{Z}) = H_1(C, \mathbb{Z})$, which is the abelianization of $\pi_1(C)$.

So a map $\rho : \pi_1(C) \to \mathbb{C}^*$ factors through $\pi_1(Jac(C))$ because $\mathbb{C}^*$ is abelian, and so gives us $c_\rho : \pi_1(Jac(C)) \to \mathbb{C}^*$, a local system of $Jac(C)$.

This naturally extends to all of $\mathrm{Pic}(C)$. Choose $L \in \mathrm{Pic}^d(C)$. Then $\otimes L^{-1} : \mathrm{Pic}^d(C) \to \mathrm{Pic}^0(C)$, and we pull back along this map. This doesn't depend on L, because the induced automorphism on $\pi_1(Jac(C))$ given by translation by L_0 is trivial.

Example 2.2. *If we take a rank 2 local system on C, then we get a local system on and open subset of Bun whose rank is 2^{3g-3}. In general, rank n gives $2^{3g-3}3^{5g-5}\dots n^{(2n-1)(g-1)}$.*

We need to discuss Hecke correspondences now.

A correspondence between varieties X,Y is a subvariety $H \subset X \times Y$.

If H is proper over Y, then H induces a map on cohomology $H^*(X) \to H^*(H) \to H^*(Y)$ by first pulling back and then pushing forward.

Example 2.3. *Let $f : X \to Y$ be any morphism and H be the graph of f. Then the operator induced by H is f_*.*

Example 2.4. *Let $g : Y \to X$ be a morphism and H it's graph. Then the operator induced by H is g^*.*

Hecke Correspondence:

$$\mathrm{Bun}_{C,G} \swarrow \mathrm{Hecke} \searrow \mathrm{Bun}_{C,G} \times C$$

The Hecke Correspondence is $\{(V, V', x) | V \subset V' \subset V(x)\}$ where $V(x) = V \otimes \mathscr{O}(x)$, and the inclusions are inclusions of coherent sheaves.

An automorphic sheaf is an eigensheaf of Hecke.

3 Lecture 3

More on the Hecke Correspondence.

Denote $Bun = Bun_{n,\mathbb{C}}$. Then $Hecke \subset Bun \times Bun \times \mathbb{C}$ is the locus $\{(V', V, x) | V \subset V' \subset V(x)\}$.

We call $p_1 : Hecke \to Bun$ the first projection and $p_2 : Hecke \to Bun \times \mathbb{C}$ the second and third projections taken together.

What is $p_2^{-1}(V, x)$?

Example 3.1. *$x \in C$ and $V = \mathcal{O}_C \oplus \mathcal{O}_C$. What is the set of all V' such that $\mathcal{O}_C \oplus \mathcal{O}_C = V \subset C' \subset V(x) = \mathcal{O}_C(x) \oplus \mathcal{O}_C(x)$.*

V and V' work, as do $\mathcal{O}_C \oplus \mathcal{O}_C(x)$ and $\mathcal{O}_C(x) \oplus \mathcal{O}_C$. Is there anything else? Yes! You can take other coordinates. So in fact we get two points and a projective line.

Choose any linear function $f : V_x \to \mathbb{C}$. Let $V' = \ker f$, and this gives the answer.

Back to the question: what is $p_2^{-1}(V, x)$?

It is disconnected. To see this we look at $\ell(V'/V)$, the length of this sheaf. Because V'/V is supported at x, and it's fiber there is V'_x/V_x, then $(V/V)_x \subset V'_x/V_x \subset (V(x)/V)_x$, so the first is zero and the last can be anything up to n.

So the answer is that $p_2^{-1}(V, x) = \coprod_{i=0}^{n} Gr(i, V_x)$, where $n = \dim(V_x)$.

We saw that the vector bundle V' determines a subspace of V_x. Conversely, given a subspace $W \subset V_x$, $\Gamma(U, V') = \{x \in \Gamma(U, V(x)) | pole(s) \subset W\}$.

Summary: Given a vector bundle V and a subspace $W \subset V_x$, we get the "elementary transform" $V' =$

V' is a line bundle, and $V \subset V' \subset V(x)$, $0 \to W \to V_x \to Q \to 0$ and $0 \to Q \to V'_x \to W \to 0$ are short exact

$Hecke = \coprod_{i=0}^{n} Hecke^i$, where $Hecke^i \xrightarrow{p_2} Bun \times \mathbb{C}$ is a fiber bundle with fiber $Gr(i, n)$.

Hecke Operators: $H^i := (p_2^i)_*(p_1^i)^*$. Explicitly, $H^i(\mathscr{F}) := p_{2*}^i p_1^{i*} \mathscr{F}$.

Let $\mathscr{F}$ be a sheaf on X and $\mathscr{G}$ a sheaf on Y, then $\mathscr{F} \boxtimes \mathscr{G}$ is defined to be $(p_X^* \mathscr{F}) \otimes (p_Y^* \mathscr{G})$.

Conjecture 3.1 (Baby Version of GLC)**.** *For each local system E on C, there exists a "local system" c_E on Bun such that $H^i(c_E) = c_E \boxtimes \bigwedge^i E$.*

Case $n = 1$, then $i = 0, 1$. $Bun = Bun_{1,C} = \text{Pic}(C) = \coprod_{d=-\infty}^{\infty} \text{Pic}^d(C)$. Then $Hecke = \{(L', L, x) | L, L' \in \text{Pic}(C), L \subset L' \subset L(x)\}$. So $Hecke^0 = \{(L', L, x) | L' = L\}$ and $Hecke^1 = \{(L', L, x) | L' = L(x)\}$. As abstract varieties, $Hecke^0 = \text{Pic}(C) \times C$ and $Hecke^1 = \text{Pic}(C) \times C$.

What does it mean to be a $Hecke^0$ eigensheaf? Nothing. The condition is $p_1^* c_E = c_E \boxtimes 1 = p_1^* c_E$.

What does it mean to be a $Hecke^1$ eigensheaf? Want c_E on $\text{Pic}(C)$ with the condition $A\partial^* c_E \cong c_E \boxtimes E$ where $A\partial : Hecke^1 \to \text{Pic}(C)$ and takes (L, x) to L'.

It must be E on C, then on $C \times C$ we have $E \boxtimes E$. So in general $E^{\boxtimes N}$ on C^N.

4 Lecture 4

We have been looking at GLCC for the case $G = GL(1)$, the abelian case. Then we have E a local system on C and it induces natural local systems on $C^{\times d} = C \times \ldots \times C$ and $\mathrm{Sym}^d C = C^{[d]} = C^{\times d}/S_d$.

We have $E \boxtimes \ldots \boxtimes E$ on $C^{\times d}$ whose fiber at $(x_1, \ldots, x_d) = E_{x_1} \otimes \ldots \otimes E_{x_d}$.

We claim that there exists a natural local system on $Sym^d(C)$ whose pullback to $C^{\times d}$ is $E^{\boxtimes d}$. The fiber at an unordered d-tuple $\{x_1, \ldots, x_d\} = D$ is $Bij(\{1, \ldots, d\}, D)$.

We do have a map $\pi : C^{\times d} \to Sym^d(C)$, but we can't take $\pi_*(E^{\boxtimes d})$, because it has rank $d!$. We have two actions of S_d, permute the sheets of π and get an action on $\mathscr{F}$ and we can act directly on the fiber $e_1 \otimes \ldots \otimes e_d \mapsto e_{\sigma(1)} \otimes \ldots \otimes e_{\sigma(d)}$.

So we look at $(\pi_* E^{\boxtimes d})^{S_d}$, the sheaf of the S_d-equivariant sections of $E^{\boxtimes d}$. The fiber at D is the assignment $f : Bij \to (\pi_* E^{\boxtimes})|_D$ such that $f(\sigma x) = \sigma f(x)$.

So this gives a sheaf on $\mathrm{Sym}^d C$. It is a local system for $n = 1$ but not otherwise.

Topologically, $\rho : \pi_1(C) \to GL(1)$, and $\pi_1(C^{\times n})$ has maps to $\pi_1(C)$ and $GL(1)$ such that they all commute, and $\pi_1(C^{\times n}) \to GL(1)$ factors through $\pi_1(Sym^n(C))$, because as $GL(1)$ is abelian, the representation of $\pi_1(C^{\times n})$ factors through $\pi_1(Sym^n(C))$.

Algebraically, we need to calculate the local invariants when two of the x_i come together. eg, $E = \mathbb{C} < a > \oplus \mathbb{C} < b >$ and $C = \mathbb{A}^1$. Then $E^{\boxtimes 2} = \mathbb{C}^{\otimes 4}$ and $E^{\boxtimes 2}_{(x,y)} = \mathbb{C} < a_x \otimes a_y, b_x \otimes a_y, a_x \otimes b_y, a_y \otimes b_y >$. S_2 invariants are $a_x \otimes a_y$, $b_x \otimes b_y$, $a_x \otimes b_Y + a_y \otimes b_x$, but $a_x \otimes b_y - a_y \otimes b_x$ goes to minus itself.

Summary: In rank ≥ 2, we get a local system of rank n^d away from the discriminant, its extension across the discriminant has local monodromy: it is not a local system.

For rank $n = 1$, it does extend.

Anyway, on $Sym^d C$ we do get a rank 1 local system. We want a rank 1 linear system p_E on $\mathrm{Pic}(C)$.

We'll use $A\partial_d : Sym^d C \to \mathrm{Pic}^d C$ (Abel-Jacobi d). But for $d >> 0$, ($d > 2g - 2 = \deg K_C$) thus $A\partial_d$ is smooth, that is, it is a morphism, it is surjective, and its differential is everywhere surjective, so it has smooth fibers, in fact, the fibers are $\mathbb{P}^{d-g}$. In particular, the fibers are simply-connected.

Conclusion: we get a natural rank 1 linear system on $\mathrm{Pic}^d(C)$ for $d > 2g-2$. Namely $(A\partial_d)_*(E^{\boxtimes d})^{S_d}$. This is a rank 1 linear system on $\mathrm{Pic}^d(C)$. Finally, this extends to c_E on $\mathrm{Pic}(C)$ as there exists a natural isomorphism $\mathrm{Pic}^d(C) \to \mathrm{Pic}^{d-(2g-2)}(C)$ for all d. For $d > 4g - 4$, the pullback of our $c_{E,d-2g+2}$ is $c_{E,d}$.

5 Lecture 5

Today we'll talk about the arithmetic/number theory version of the Langlands Conjecture.

Dictionary:

"Spec F", or more likely the collection of "places" in the field, corresponds to our curve C.

A number field corresponds to a function field $k(C)$.

$\mathbb{A}$=adeles.

For all $x \in C$, $\mathscr{O}_X \subset F_x$ with quotient field k_x.

Over $k = \mathbb{C}$, $k_x = \mathbb{C}$ and $\mathscr{O}_x = \mathbb{C}[[t]]$ and $F_x = \mathbb{C}((t))$.

Define $\mathscr{O} = \prod_x \mathscr{O}_x \subset \mathbb{A} := \prod'_x F_x \subset \prod_x F_x = F$.

$GL(F) \setminus GL_0(\mathbb{A})/GL_n(\mathscr{O}) \sim Bun_{GL_n,C}$.

Above, $\mathscr{O} \to \mathbb{A}$ an inclusion and $F \to \mathbb{A}$ is also an inclusion. The first is the collection of things with no poles and the second are the global meromorphic functions. And so we can restrict $GL_n(\mathbb{A})$ to each subset, and quotient out. This is Weil's approach to the moduli of vector bundles.

Case $n = 1$: $GL_1(\mathbb{A}) = \mathbb{I}$ is the set of ideles.

This leads naturally to stacks, because for $n > 1$ we don't have a moduli SPACE, but rather need a STACK

5.1 Assigning a Vector Bundle to an Adele Matrix

What we need is an open cover of C and gluing matrices. As we want $GL_n(\mathscr{O})$ to be trivial, we should focus on the poles.

Try listing the poles $x_1, \ldots, x_\ell$. Let $U_0 = C \setminus \{x_1\}$ and let U_i be a small disc containing x_i. Then the vector bundle is trivial on each U_i and U_0, and we need gluing data on $U_0 \cap U_i$ for $1 \leq i \leq \ell$. There is no compatibility condition necessary.

This is given by the adelic matrix at x_i.

Back to the dictionary, we can see that $Bun_{n,C}$ corresponds to $GL_n(F) \setminus \mathrm{GL}_n(\mathbb{A})/\mathrm{GL}_n(\mathscr{O})$.

Local systems correspond to unramified ℓ-adic representations

General ℓ-adic representations correspond to locally systems on curve C

Automorphic functions correspond to the trace of the Frobenius morphism acting on Hecke eigensheafs.

An ℓ-adic representation: take ℓ to be prime and not divide $q = p^n$, and $F = K(C)$. Then $\sigma : Gal(\bar{F}/F) \to GL_n(\bar{\mathbb{Q}}_\ell)$.

(A local system is an ℓ-adic representation where $F = K(C)$, $\bar{\mathbb{Q}}_\ell = \mathbb{C}$, $k = \mathbb{C}$and we consider only unramified representations)

Reasonable conditions:

1. Nowhere ramified (ie, $\sigma(Interia_x) = 1$ for all x)
2. σ is geometrically irreducible, that is, irreducible on $Gal(\bar{F}/Fk)$.

6 Lecture 6

The correspondence is best expressed in three columns, Number Fields, Function Fields and Geometric Langlands.

The base field will be $\mathbb{Q}$ or $\mathbb{F}_q(t)$ or $k(\mathbb{P}^1)$, and we will look at $F \supset \mathbb{Q}$, $F = k(C)$ where $C/\mathbb{F}_q$, and $k(C)$ where C/k.

So we have G is any connected(?) reductive group, and then for function fields, we look at $G(F) \setminus G(\mathbb{A})/G(\mathscr{O})$ and corresponds in the geometric case to $Bun_{G,C}$, and $G(\mathbb{A})$ corresponds to the Bundles with a meromorphic trivialization and a local trivialization at each x. (In terms of schemes, we demand a trivialization at each point, including the generic point)

We note that $G(\mathbb{A}) = \prod'_x G(F_x)$, and this will give us gluing data for the bundles, giving a bijection.

Quotient by $/G(\mathscr{O})$ corresponds to forgetting local trivializations and by $G(F)\setminus$ to forgetting the generic trivialization.

ℓ-adic representations correspond to local systems, that is, $\sigma : Gal(\bar{F}/F) \to G(\bar{\mathbb{Q}}_\ell)$ correspond to $\sigma : \pi_1(V) \to G$.

An automorphic function $f : G(\mathbb{A}) \to \bar{\mathbb{Q}}_\ell$ which is invariant under $G(\mathscr{O}$ and $G(F)$ which satisfies

1. It is cuspidal
2. It is a Hecke eigenfunction.

corresponds to an automorphic sheaf which is irreducible and a Hecke eigensheaf.

Now take $G = GL_n$. Then $0 \le i \le n$, $x \in C$, we have π_x a uniformizer at x, that is, a local coordinate on C at x.

then look at $H^i_x = GL_n(\mathscr{O}_x) diag(\pi_x, \dots, \pi_x, 1, \dots, 1) GL_n(\mathscr{O}_x)$, a double coset in G.

We define a Hecke operator to be $(T^i_x f)(g) = \int_{H^i_X} f(gh_x) dh_x$.

The picture is that $Hecke^i_x$ is a correspondence $Bun \to Bun$, specifically the set $\{(V, V') | V \subset V' \subset V(x)\}$.

So given a function, we are pulling back to $Hecke^i_x$ and then integrating over the fibers to get a function on Bun.

Conjecture 6.1. *For all σ ℓ-adic representations, there exists f_σ an automorphic function, unique up to $\bar{\mathbb{Q}}^*_\ell$ with $T^i_x f_\sigma = |k(x)|^{-i(i+1)/2} \operatorname{tr}(\wedge^i \sigma(Frob_x)) f_\sigma$.*

This suggests a relation between sheaves and functions. $\operatorname{tr}(Frob_x 0)$, but this makes sense only over $\mathbb{F}_q$.

7 Lecture 7

7.1 Extensions and Connections

1. Connection with Shimura-Taniyama-Weil
2. GLC for other reductive grops
3. Ramified versions

4. D-modules and Derived Categories

5. Stacks, Gerbes

6. Hitchen's System and its quantization

7. Connection to Mirror Symmetry

We will do an "impressionistic" look at the first, and it won't be used later.

Can we see geometric objects in the number field version of LC?

Geometrically, we take $x \in C \to \mathbb{P}^1$ and in number theory we have a prime in the ring of integers in a finite extension F of $\mathbb{Q}$.

For a rank n bundle on C, we don't know a good analogue in the number theory, but if n is two, then we have "tate modules" which are associated to elliptic curves.

If E is an elliptic curve over K and ℓ doesn't divide the discriminant, it corresponds to E elliptic over $\mathbb{C}(\mathbb{P}^1)$ with discriminant the set of points where the differential of $E \to \mathbb{P}^1$ vanishes. The discriminant for the number theoretic elliptic curve can be thought of as a family of curves over the primes, and then ℓ not dividing the discriminant is the same as E reduced modulo ℓ is smooth (this is the fiber over ℓ).

Consider the ℓ^k-torsion points of E. Then the Tate module of the curve E is $\varprojlim_k E[\ell^k]$.

And TSW is a special case of Langlands.

Back to geometric Langlands, what about other reductive groups?

A reductive group is the complexification of a connected compact real group. Equivalently (we think) every representation is a direct sum of irreducibles.

Example 7.1. *S^1 is a connected compact real group. $\mathbb{C}^* = GL(1, \mathbb{C})$ is a complexification, because it has an involution whose fixed points are S^1. $\mathbb{C}^*$ is a complexification of $\mathbb{R}^+$ with multiplication.*

Similarly, T^n gets $(\mathbb{C}^*)^n$.

Example 7.2. *$U(n)$, viewed as a real group, has $GL(n, \mathbb{C})$ as a complexification.*

Example 7.3. *$SU(n)$ complexifies to $SL(n, \mathbb{C})$. (possibly quotients by finite subgroups of the center are also complexifications of $SU(n)$)*

Example 7.4. *$SO(n, \mathbb{R})$ complexifies to $SO(n, \mathbb{C})$.*

Example 7.5. *$Sp(n, \mathbb{C}) \cap U(n)$ complexifies to $Sp(n, \mathbb{C})$.*

Any complex simple group is reductive. There exists a complete classification: $SL(n)$, $SO(2n+1)$, $Sp(n)$ and $SO(2n)$ and five exceptional groups F_4, G_2, E_6, E_7 and E_8 and quotients of the above by finite subgroups of the center. When talking about simple Lie groups, we mean no connected normal Lie subgroup.

Two Liegroups G_1, G_2 are isogenous if they are quotients of a third group G by finite subgrousp of its center, that is, $G_1 \cong G/S_1$ and $G_2 = \cong G/S_2$.

We will give names to the groups mentioned before, $SL(n)$ will be A_{n-1}, $SO(2n+1)$ is B_n, $Sp(n)$ is C_n, $SO(2n)$ is D_n, and then there are E_6, E_7, E_8, F_4 and G_2, the Cartan-Killing List, when A_n for $n \geq 1$, B_n for $n \geq 2$, C_n for $n \geq 3$ and D_n for $n \geq 4$.

This classifies teh isogeny classes of simple complex groups, the simply-connected groups, and the centerless groups. (This last class is called groups of adjoint type)

Classification of reductive groups, $R = (A \times S)/F$ where A is abelian, which can be abelian varieties, complex tori, C^n and $(C^*)^n$, S is simple and f is a finite subgroup of the center.

Eg, $GL(n, \mathbb{C}) = \mathbb{C}^* \times SL(n, \mathbb{C})/\mathbb{Z}_n$

The Lie algebra of a Lie group G is $\mathfrak{g} = T_e G$, the left invariant vector fields on G. The operation $[,]$ is the commutator of vector fields. Lie algebra determines the Lie group up to isogeny.

8 Lecture 8

Today we'll talk about GLC for other groups.

Let G be reductive, that is, $(A \times S)/F$ where A is abelian, S is semisimple and F is finite.

A couple of key concepts:

"Maximal Torus" $T \subset G$ is a connected abelian subgroup isomorphic to $(\mathbb{C}^*)^r$ which is maximal among such subgroups consisting of elements that are semisimple in the adjoint representation. (that is, diagonalizable in the adjoint action on the Lie Algebra)

Example 8.1. *If $G = GL(n)$, then T can be taken to be the diagonal matrices. The adjoint representation is a map $ad : GL(n) \to \mathrm{Aut}(\mathfrak{gl}(n)) = GL(n^2)$. It mapes $g \mapsto (A \mapsto gAg^{-1})$. For $g = diag(g_1, \ldots, g_n) \in T$, $ad(g)$ is the diagonal $n^2 \times n^2$ matrix with eigenvalues $g_i g_j^{-1}$.*

Caveat: T is not the same as a maximal abelian subgroup.

Basic Fact: All maximal tori in a group G are conjugate. In particaulr, r is an invariant of G called rank.

$Rank(GL(n)) = n$. $Rank(SL(n)) = n - 1$, and this is why we call this A_{n-1}. $GL(n) = A_{n-1} \times C^*/\mathbb{Z}_n$. $Rank(G \times H) = Rank(G) + Rank(H)$. $Rank((\mathbb{C}^*)^n)) = n$.

Better, replace G by $\mathfrak{g}$ and T by $\mathfrak{t}$, and call this the Cartan Subalgebra.

Aut, Ad, ad. Let $g \in G$ and $A \in \mathfrak{g}$.

Then $Aut_g \in Aut(G)$ takes $h \mapsto ghg^{-1}$, and $Aut : G \to AutG$ is a map.

Then $Ad : G \to \mathrm{Aut}(\mathfrak{g})$, and $ad : \mathfrak{g} \to End(\mathfrak{g})$.

$Aut_g : G \to G$ induces a linear map $T_e G \to T_e G$, that is, $\mathfrak{g} \to \mathfrak{g}$, this is Ad_g. Then $ad = D_e(Ad)$, that is, the induced map $ad : \mathfrak{g} \to End(\mathfrak{g})$, acuting on the tangent space of $Aut(\mathfrak{g})$.

Caveat: a maximal abelian subalgebra of $\mathfrak{g}$ is not necessary a Cartan subalgebra.

$char_G = char_T = \hom(T, \mathbb{C}^*) \cong \mathbb{Z}^r$. $\hom(T, \mathbb{C}^*) \subset \mathfrak{t}^\vee$

On $\mathfrak{g}$, there exists a natural pairding $A, B \mapsto Tr(ad_A \circ ad_B)$. This is the Killing Form, it's a natural bilinear pairing on $\mathfrak{g}$, and hence on $\mathfrak{t}$. It induces one on $\mathfrak{t}^\vee$, and it takes $\mathbb{Q}$ values on $char$.

Example 8.2. *Let $G = SL(3)$, then char is the hexagonal lattice.*

How about isogenous groups?

Let $G_2 = G_1/F$ where F is finite. Then $T_1 \to T_2$ is surjective and the kernel is F. Sp then we have $char_{G_2} \to char_{G_1} \to F^\vee$ and we get lattices of the same rank.

Picture of an isogeny class: Starts with G_{sc} the universal cover and goes to $G_{ad} = Ad(G) \subset Aut(\mathfrak{g}) = G/Z(G)$.

Then $\pi_1(G_{sc}) = 0$ and $Z(G_{sc})$ is the biggest possible. Then G_{ad} has the biggest π_1 possible, and $Z(G_{ad}) = 0$, and there are things in between.

Eg, $SL(n)$. $SL(n)$ is simply connected, and $\mathbb{P}(GL(n))$ is centerless. We call $char(G_{sc})$ the weights and $char(G_{ad})$ the roots.

So then we have $roots \subset chars \subset weights$. Swapping these gives the Langlands Dual Group, which we will discuss next time.

9 Lecture 9

Let G be reductive, that means that $G = (S \times A)/F$ as before.

ASIDE: Complex Torus is a word used inconsistently. In complex geometry, a complex torus is $(S^1)^{2g}$ with a complex structure (this is the same as compact connected abelian (abelian is implied, not a necessary hypothesis) complex Lie group), and in group theory a complex torus is $(\mathbb{C}^*)^r$, this is also called an algebraic torus.

Let $T \subset G$ be a maximal torus and $\mathfrak{t} \subset \mathfrak{g}$ the Cartan subalgebra given by T.

Example 9.1. *If $G = GL(n)$ and T is the collection of invertible diagonal matrices, then $\mathfrak{t}$ is the set of arbitrary diagonal matrices.*

To a reductive group G we associate six lattices as follows. $roots \subset chars \subset weights$, $coroots \subset cochars \subset coweights$.

$chars = \hom(T_G, \mathbb{C}^*) \cong \mathbb{Z}^r$ where r is the dimension of T and is defined to be the rank of G.

The isogeny class of G includes G_{sc} and G_{ad}. The roots are the characters of G_{ad} and the weights are teh characters of G_{sc}

Available structure: Lattice=(free abelian group with a $\mathbb{Q}$ valued pairing)

The dual lattice means the dual group with the dual metric.

So we can define the Langlands dual group ${}^L G$ is the group whose characters are cocharacters of G. We have no justification yet for existence or uniqueness of such.

We will assume that such a group exists and is "reasonably functorial" in gneeral.

But explicitly define N to be the normalizer of T in G and W to be N/T the group of connected components of N.

Example 9.2. *$G = GL(n)$ and T the diagonals. Then N is the collection of invertible matrices with a single nonzero entry in each row and column. Then W is S_n.*

Example 9.3. *Take $G = SO(2n)$. Then if we take this as the group preserving the form $x_1x_{2n} + x_2x_{2n-1} + \ldots + x_nx_{n+1}$, then it is anti-skew-symmetric, and so has diagonal $\alpha_1, \ldots, \alpha_n, -\alpha_n, \ldots, -\alpha_1$ as the maximal torus. Thus, $SO(2n)$ has rank n.*

To determine W, we see that it fits into the short exact sequence $1 \to (\mathbb{Z}/2)^{n-1} \to W \to S_n \to 1$ where the first is the group of even sign changes.

Facts

1. W depends on $\mathfrak{g}$, not G.
2. $W(G) = W({}^LG)$.
3. W acts (by conjugation) on T. Therefore on characters $\hom(T, \mathbb{C}^*)$. W preserves the pairing and so is a group of isometries of the lattice.

Example 9.4. *$G = GL(n)$, then the characters are $\mathbb{Z}^n$ the square lattice, so we really have $(\mathbb{Z}, standard)^{\oplus n}$. So $W = S_n$. The obvious permutation action on an orthonormal basis preserves the pairing only up to $\pm$.*

The maximal torus of ${}^LG$ is the dual of hte maximal torus of G. $T_{{}^LG} = (T_G)^\vee$, that is, $\mathfrak{t}_{{}^LG} = (\mathfrak{t}_G)^\vee$.

General yoga: The group G can be recponstructed from $\mathfrak{t}, char, W$ action, *metric*. A_n, D_n, E_n are "simply laced" and B_n corresponds to C_n, G_2 to G_2 and F_4 to F_4.

10 Lecture 10

Example 10.1. *If $G = GL(n)$ then ${}^LG = GL(n)$*
If $G = SL(n)$ then ${}^LG = PGL(n)$

In general, with Lie Algebras, each type maps to itself except that B and C swap.

At the group level, ${}^LG_{sc}$ is the appropriate adjoint group.

Example 10.2. *$Z(SO(2n)) = \mathbb{Z}/2$ and $\pi_1(SO(2n)) = \mathbb{Z}/2$. So there is a double cover of $SO(2n)$ which we call $Spin(2n)$ and is simply connected. So we have $Spin(2n) \to SO(2n) \to PSO(2n)$ and this has fundamental group either $\mathbb{Z}/4$ (if n odd) and $\mathbb{Z}/2 \times \mathbb{Z}/2$ (if n even).*

So now, let C be a smooth compact complex curve of genus g. Define $Bun_{G,C}$ to be the moduli space of semistable principal G-bundles on C and $Loc_{G,C}$ to be the moduli space of semistable principal G local systems on C.

Theorem 10.1. *TFAE*

1. $\rho : \pi_1(C) \to G$
2. (P, ∇) *where* P *is a principal holomorphic* G*-bundle over* C *and* ∇ *is a flat holomorphic connection.*
3. *Locally constant sheaf of groups on* C *with fiber isomorphic to* G.

Roughly, what Geometric Langlands says is that points of Loc_G correspond to Hecke Eigensheaves on $Bun_{^LG}$.

A better version is: introduce the notion of $\mathscr{D}$-modules. Define $\mathscr{D}$ to be the sheaf of rings on $Bun_{^LG}$ given by the holomorphic differential operators. On any X, we have $\mathscr{O}_X$ the functions, $\mathscr{T}_X$ the vector fields, and $\mathscr{D}_X \subset Hom_{\mathbb{C}}(\mathscr{O}_X, \mathscr{O}_X)$. We define a filtration $\mathscr{D}_X^0 \subset \mathscr{D}_X^1 \subset \dots$ and the union is $\mathscr{D}_X$. Then $\mathscr{D}_X^0 = \mathscr{O}_X$. We say that $L \in D^i$ if $[L, \mathscr{O}_X] \subset D_X^{i-1}$. That is, $L(fg) - gL(f) \in D^{i-1}$.

In local coordinates, a section of D_X is a differential operator $f_0(z) + f_1(z)\frac{\partial}{\partial z} + f_2(z)\frac{\partial^2}{\partial z^2} + \dots$.

Conjecture 10.1 (Geometric Langlands Conecture)**.** *There exists a natural equivalence of catagories* $c : D^b(Loc, \mathscr{O}) \to D^b({}^LBun, \mathscr{D})$. *c of a structure sheaf of a pont is a Hecke eigensheaf on* ${}^LBun$ *with eigenvalues corresponding to the original local system.*

Next time: Hecke Operators.

11 Lecture 11

As a point x, we have $Hecke_x = \{(V, V', \beta) | V, V' \in Bun$ and $\beta : V_{C\setminus x} \simeq V_{C\setminus x}\}$

We define $Hecke$ to be $\{(V, V', \beta, x) | x \in C, \dots\}$ a relation in $Bun \times (Bun \times C)$.

Note: The fibers of p, q, p_x, q_x are infinite dimensional. They are affine grassmanians, that is, V is a principal G-bundle on C with trivialization on $C \setminus x$.

These are Ind-schemes.

How do we get finite dimensional fibers?

Let $\rho : G \to GL_N$ be any representation. Let $T \subset G$ be a maximal torus. Then $\rho|_T$ can be simultaneously diagonalized: there exists $w_1, \dots, w_N$ simultaneous eigenvectors.

Then there exist functions $\lambda_i : T \to \mathbb{C}^*$ such that for all $A \in T$, $\rho(A)w_i = \lambda_i(A)w_i$. Clearly, λ_i is a group homomorphism, and so it is a character.

Upshot to an N-dimensional representation ρ of G, we associate an N-tuple of characters. This set of characters is invariant under the Weyl group W and uniquely characterizes an irreducible ρ.

So now we have finite dimensional Heckes, and we have

$${}^LHecke_x^\mu = \{(V, V', \beta) | \forall \text{ finite dimensional representations } \rho : {}^LG \to GL_n(\mathbb{C})$$

$$\text{with } \rho(\beta) : \rho(V) \to \rho V' \otimes \mathscr{O}_C(\langle \mu, \lambda^\rho \rangle, x)\}$$

where λ^ρ is the highest weight of ρ.

This has finite dimensional fibers, and $\mu_1 \leq \mu_2$ implies that ${}^L Hecke_x^{\mu_1} \subset {}^L Hecke_x^{\mu_2}$. So we can form $\varinjlim_\mu {}^L Hecke_x^\mu = {}^L Hecke_x$.

Example 11.1. *For $G = SL(n)$, this runs over $\mathbb{Z}^{n-1}/S_n$. This is gneerated by $n-1$ elements $\mu_1, \ldots, \mu_{n-1}$. The fiber of $Hecke^{\mu_i}_{SL(n),x} = Gr(i, n)$.*

Unfortunately, for all other μ, the fibers of $Hecke^\mu_{SL(n),x}$ are singular.

For general G and μ, the fibers are smooth iff the representation with highest weigth μ has only $W\mu$ and 0 as weights, this is phrased as μ being miniscule.

Hecke Operators

$L^\mu_{\mathscr{H}} : D^b({}^L Bun, \mathscr{D}) \to D^b({}^L Bun, \mathscr{D})$ by $m \mapsto q^\mu_!(p^{\mu^*} m \otimes IC^\mu)$.

These operators generate an algebra. This algebra is commutative.

12 Lecture 12

Conjecture 12.1 (GLC, tentatively)**.** *There exists a natural equivalence $c : D^b(Loc, \mathscr{O}) \to D^b({}^L Bun, \mathscr{D})$ sending structure sheaves of a point $\mathscr{V}$ to automorphic $\mathscr{D}$-modules $\mathscr{M}$, ie, to the Hecke eigensheaf ${}^L Hecke^\mu(\mathscr{M}) = \mathscr{M} \boxtimes \mathscr{V}$ on ${}^L Bun \times C$.*

Define $p^\mu : Hecke^\mu \to {}^L Bun$ to be the first projection, and $q^\mu : Hecke^\mu \to {}^L Bn \times C$ the second. Define $\mathscr{H}ecke^\mu(\mathscr{M}) = q^\mu_!((p^\mu)^* \mathscr{M} \otimes IC^\mu)$.

So ${}^L \mathscr{H}eche^\mu(\mathscr{M}) = \mathscr{M} \boxtimes p^\mu(\mathscr{V})$ on ${}^L Bun \times C$.

Discrepancy: ${}^L Bun$ is disconnected. $\pi_0({}^L Bun) = H^2(C, \pi_1({}^L G)) = \pi_1({}^L G) = (Z(G))^\vee$.

Example 12.1. *$G = GL(n) = {}^L G$. $\det GL(n) \to \mathbb{C}^*$ gives $\pi_1(GL(n)) \to \pi_1(\mathbb{Z}^*) = \mathbb{Z}$.*

So topological types of rank n bundles on C are parametrized by their degree $d = c_1 \in \mathbb{Z}$.

There exist infinitely many components of n "types"

Example 12.2. *${}^L G = SL(n)$, then $\pi_1({}^L G) = 0$, and so $Bun_{SL(n)}$ is connected, so we have $Bun_{SL(n)} \to Bun^0_{GL(n)} \to \mathscr{J}$, with the first map inclusion and the second the surjective determinant map.*

Example 12.3. *${}^L G = \mathbb{P}GL(n)$. A $\mathbb{P}GL(n)$ bundle on C is an equivalence class of $GL(n)$ bundles modulo $\otimes LB$. Then we have $C^* \to GL(n) \to \mathbb{P}GL(n)$, and so the number of components of $Bun_{\mathbb{P}GL(n)} = n$.*

So now, we note that $D^b({}^L Bun, \mathscr{D}) = \prod_{d \in \pi_1({}^L G)} D^b({}^L Bun^d, \mathscr{D})$. However, we get no such statement for Loc, due to the flat connection! So either ${}^L Bun$ is too big or Loc is too small.

A solution is to replace the moduli space Log by the moduli stack $\mathscr{L}oc$. (Actually, it's a $\mathbb{C}^*$ gerbe)

The new objects are "twisted sheaves":

A "sheaf" on X can be twisted by a cohomology class in $H^2(X, \mathscr{O}^*)$. Fix a Čech 2-cocycle $\{U_i\}$ an open cover $\alpha_{ijk} \in \Gamma(U_{ijk}, \mathscr{O}^*)$ with $\alpha_{ijk}\alpha_{jk\ell}\alpha_{k\ell i}\alpha_{\ell ij} = 1$ on $U_{ijk\ell}$.

An α-twisted sheaf $\mathscr{F}$ on X is an assignment of a sheaf of abelian groups $\mathscr{F}(U_i)$ to each U_i. and transition functions $g_{ij} : F(U_i)|_{U_{ij}} \to \mathscr{F}(U_j)|_{U_{ij}}$ satisfying $g_{ij}g_{jk}g_{ki} = \alpha_{ijk}$.

This is precisely the same thing as a sheaf, except the condition involving α.

The case with $\alpha_{ijk} = 1$ is an ordinary sheaf.

If α_{ijk} and α'_{ijk} are cohomologous, there is an obvious equivalence between α-twisted sheaves and α'-twisted sheaves.

So we can talk about sheaves twisted by classes in $H^2(X, \mathscr{O}^*)$.

Sheaves on $\mathscr{L}oc$ are twisted sheave on Loc, and under GLC, we have

Sheaves on Loc (sheaves on $\mathscr{L}oc$ with $\alpha \equiv 1$) correspond to the neutral component of ${}^L Bun$ and genuinely twisted sheaves correspond to the other components.

It's complicated...

easier to consider only "regularly stable" local systems. They form Zariski open Loc^{rs}, and $\mathscr{L}oc^{rs}$ is a gerbe over Loc^{rs} with structure group $Z(G)$ acting trivially.

So twists correspond to characters of $Z(G) = \pi_1({}^L G)$

Next time will be preempted by a physicist, and Monday we start Hodge Conjecture.

13 Lecture 13 - Hodge Theory Begins Here

Let X be a smooth projective variety. Let $Z \subset X$ a subvariety of complex codimension p. Then we have $[Z] \in H^{2p}(X, \mathbb{Z}) \to H^{2p}(X, \mathbb{C})$.

Properties of $[Z]$

1. It is integral.
2. It is of Hodge type (p, p).
3. It is positive.

The set of classes in $H^{2p}(X, \mathbb{Z})$ that arise from Z's is a cone. Characterizing it is very complicated. Instead, ask for its linear span:

Conjecture 13.1 (Hodge Conjecture)**.** *Any Hodge class (ie, (p, p) class in $H^*(X, \mathbb{Q})$) is a $\mathbb{Q}$-linear combination of algebraic cycle classes.*

The original version, stated by Hodge in 1952 was this over $\mathbb{Z}$. It is false.

There is a generalized Hodge Conjecture which was shown false by Grothendieck, but is fixable. There is also a variant by replacing $[Z]$ by $c_i(V)$, the chern classes of a vector bundle. Variants for compact Kähler. Variants for other cohomology theories.

Why would we conjecture this? The conditions above are necessary, but no one has found any examples of sufficient conditions beyond them.

The main "evidence" is the case that $p = 1$, due to Solomon Lefshetz.

This case holds over $\mathbb{Z}$ and for compact Kähler X.

The basic ideas is to look at the exponential sequence $1 \to \mathbb{Z} \to \mathscr{O} \to \mathscr{O}^* \to 1$. Taking cohomology, we get $0 \to H^0(X,\mathbb{Z}) \to H^0(X,\mathscr{O}) \to H^0(X,\mathscr{O}^*) \to H^1(X,\mathbb{Z}) \to H^1(X,\mathscr{O}) \to H^1(X,\mathscr{O}^*) \to H^2(X,\mathbb{Z}) \to H^2(X,\mathscr{O})$. The map $H^1(X,\mathscr{O}^*) \to H^2(X,\mathbb{Z})$ is the first chern class.

In the case where X is a curve, $c_1(L) = \deg(L)$. In general $c_1(L) : H_2(X,\mathbb{Z}) \to \mathbb{Z}$ takes c to $\deg L/C$.

Plan: Characterize the image of c_1, and then identify with the image of codimension 1 cycles.

Part 1: The image of c_1 is $\ker(H^2(X,\mathbb{Z}) \to H^2(X,\mathscr{O}))$ and this map is $H^2(X,\mathbb{C}) \to H^{20} \oplus H^{11} \oplus H^{02}$ to H^{02}. So then $H^2(X,\mathbb{Z}) \cap H^{11}$ is the Hodge $(1,1)$ classes.

Given an effective divisor $Z \subset X$, we get a line bundle $L = \mathscr{O}_X(Z)$. Conversely, any algebraic line bundle L has rational sections so it is $\mathscr{O}(Z)$ for some Z. For holomorphic bundles, use GAGA for $Pic^{alg} X \cong \mathrm{Pic}^{an} X$.

Summary: on a smooth projective X

1. Any Hodge (1,1) class is c_1 of a line bundle
2. Algebraic line bundles are analytic line bundles
3. Any algebraic line bundle is $\mathscr{O}(Z)$ for some divisor Z.
4. $c_1(\mathscr{O}(Z)) = [Z]$

So the Hodge conjecture is true for $p = 1$ over $\mathbb{Z}$.

This turns out to be most of what is known about the Hodge conjecture.

(Consider the subring of $H^*(X,\mathbb{C})$ generated by $H^{1,1} \cap H^2(X,\mathbb{Z})$. It follows that any class here is algebraic.)

What would a counterexample be?

Not a curve, not a surface, not a 3-fold...so problem starts on a 4-fold for (2,2) classes.

Start with a 4-fold, for $(2,2)$ classes. In a typical family of 4-folds, eg, all smooth hypersurfaces of fied degree in $\mathbb{P}^5$ (or any other 5-fold) $H^{2,2}$ is a vector space of fixed dimension $H^4(X,\mathbb{Z})$ is a lattice of fixed rank, but $H^4(X,\mathbb{Z}) \cap H^{2,2}(X)$ jumps (we call this $Hodge^2$.)

For generic X, $Hodge^2(X)$ is as small as possible, but for special X, it jumps up.

The Noether-Lefshetz or Jump locus is a countable union of algebraic subvarieties (usually dense) of the parameter space.

Fix a class in $H^4(X,\mathbb{Z})$. What is the locus in the base where it becomes Hodge? That is, in $H^{2,2}$?

The Hodge Conjecture implies that this is algebraic.

Cattani, Kaplan, and Deligne proved this corollary.

The most likely counterexample would be this type.

A Weil descrbed such classes on Abelian 4-folds: for special abelian 4-folds, $Hodge^2$ is not generated by $Hodge^1$.

14 Lecture 14

NO CLASS FRIDAY

Let X be a complex manifold (compact, often). Let $A^k(X)$ be the C^∞ k-forms. Then these decompose into $\oplus_{p+q=k} A^{p,q}(X)$, and elements of these are $w = \sum_{|I|=p,|J|=q} w_{IJ} dz_I d\bar{z}_J$.

$H^k_{dR}(X, \mathbb{C})$ is homology with respect to $d = \partial + \bar{\partial}$, with each being the differential in the $A^{p,q}$'s.

On a general compact complex manifold, there is no Hodge decomposition.

Example 14.1 (Hopf Surface)**.** *Look at $T^2 \to S \to \mathbb{P}^1$ surfaces S. That is, $\mathbb{C}^* \to \mathbb{C}^2 \setminus \{0\} \to \mathbb{P}^1$. Choose $q \in \mathbb{C}^*$, $|q| \neq 1$. Then we have $(\mathbb{C}^2 \setminus 0)/(q)$. This still maps to $\mathbb{P}^1$, but it's fiber isn't $\mathbb{C}^*$, it's $\mathbb{C}^*/(q)$, which gives T^2.*

Let h be a hermitian matrix, $\sum_{i,j} h_{ij} dz_i \otimes d\bar{z}_j$. Define $\omega = \frac{i}{2} \sum_{i,j} h_{ij} dz_i \wedge d\bar{z}_j$. Then ω is the Kahler form of h, it's a $(1,1)$ form on X. So then X (or (X,h)) is Kahler if $d\omega = 0$. Any projective manifold is compact and Kahler.

Basic results:

1. (X,h) is Kahler iff the complex structure is parallel with respect to Levi-Civita connection.

2. $\Delta_d = 2\Delta_{\bar{\partial}}$ on a Kahler manifold.

3. A form on a Kahler manifold is harmonic iff all its (p,q) components are harmonic. Thus, on a compact Kahler X, we have the Hodge Decomposition.

So the Hodge conjecture is that any (p,p) rational cohomology class is "algebraic". Counterexamples on compact Kahler manifolds:

Example 14.2. *Let $X = T^4 = \mathbb{C}^2/\Lambda$. Where Λ is any discrete lattice isomorphic to $\mathbb{Z}^4$. Pick any constant hermitian form. It gives a $(1,1)$ form on $\mathbb{C}^2$. This may or may not descent to a rational form on X. It will be the class of a divisor (curve) in X iff h is positive.*

Don't impose this, take an indefinite h. Deformation tehory works: for some complex structures, this is a rational $(1,1)$-form. So there exists a holomorphic Line bundle L with $\omega = c_1(L)$, but $h^0(L^n) = 0$ for all $n \in \mathbb{Z}$ nonzero. On a general such X, this is the only line bundle or the only hodge class.

so for Kahler manifolds, need to replaces algebraic by "chern class of a vector bundle"

notions of algebraic: Hodge classes $\langle\{c_i(\mathscr{F})|\mathscr{F}$ is a coherent sheaf on $X\}\rangle$. This contains both the classes coming from subvarieties and those from vector bundles.

15 Lecture 15

16 Chern Classes

Let X be a topological manifold and E a complex vector bundle over X.

Then $c_i \in H^{2i}(X, \mathbb{Z})$.

If X is a complex manifold and E is a holomorphic bundle, then c_i are Hodge classes, in $H^{i,i}(C, \mathbb{Z}) \cap H^{2i}(X, \mathbb{Z})$.

Chern-Weil: ∇ any complex connection on E, $\nabla : E \to E \otimes A^1$, then R_∇ the curvature is in $A^2(End(E))$.

Then let σ_i be the ith symmetric polynomial of the eigenvalues of a matrix. Then $\sigma_1(A) = \operatorname{tr}(A)$. $\sigma_2(A) = \frac{1}{2}(-\operatorname{tr}(A^2) + (\operatorname{tr} A)^2)$, etcetera.

Applied to R_∇, we get $c_k = \left(\frac{i}{2\pi}\right)^k \sigma_k(R_\nabla) \in A^{2k}(X)$.

R_∇ is an endomorphism, so defined only up to conjugation. But $\sigma_k(R_\nabla)$ is basis independent. This depends on a choice, ∇.

Effect of change: $\nabla \mapsto \nabla + \omega$ for $\omega \in A^1(End(E))$ is $R_\nabla \mapsto R_\nabla + Exact$, and so $c_k \mapsto c_k + Exact$.

The upshot, $[c_k(E, \nabla)] \in H^{2k}(X, \mathbb{C})$. This is independent of ∇, and so we call it $c_k(E)$. It's actually in $H^{2k}(X, \mathbb{Z}) \to H^{2k}(X, \mathbb{C})$'s image.

Another approach: $Gr^k(n)$ the set of k-dimensional quotients of $\mathbb{C}^n$. Over $Gr^n(N)$, there is a universal rank n vector bundle Q (in fact, we have a short exact sequence of vector bundles $0 \to S \to V \to Q \to 0$ over $Gr^n(N)$).

Let $f : X \to Gr^n(N)$ by any C^∞ map. Then we get a pullback vector bundle f^*Q on X. (In a sense, grassmannians are universal...make this precise.)

Facts:

1. Any rank n vector bundle E on X is f^*Q for some f and some $N >> 0$.
2. Any two such (f, N), (f', N') are stably equivalent. That is, if we have the same bundle arising from two such maps, there is another N'' larger than N, N' and a map $f'' : X \to Gr^n(N'')$ such that the diagram with these four objects and the natural maps commutes.

So then, for any class $c \in H^*(Gr^n(\infty))$ the pullback f^*c is the characeristic class of X, E.

$Gr^n(\infty)$ is a classifying space for vector bundles. A topological vector bundle $E \to X$ is equivalent to $[f : X \to Gr^n(\infty)]$.

Example 16.1. *For $n = 1$, $Gr^1(\infty) = \mathbb{P}^\infty$. As $H^*(\mathbb{P}^{N-1}, \mathbb{Z}) = \mathbb{Z}[H]/H^n$, we obtain $H^*(\mathbb{P}^\infty, \mathbb{Z}) = \mathbb{Z}[H]$.*

So if $E = f^*\mathscr{O}(H)$, then $c_1(E) = f^*H \in H^2(X, \mathbb{Z})$ is the degree of E. If X is a complex curve, then this is in $H^2(X, \mathbb{Z}) = \mathbb{Z}$.

Splitting Principle: Formally, any $E \to X$ a vector budnle, there exists $\pi : Y \to X$ smooth map such that π^*E is a direct sum of line bundles.

Informally, any fact that holds for direct sums of line bundles holds for all vector bundles.

$c_k(E) \in H^{2k}(X, \mathbb{Z})$, because this is true for line bundles and the Whitney formula $c(E) = c(E_1)c(E_2)$ if $E = E_1 \oplus E_2$ or even $0 \to E_1 \to E \to E_2 \to 0$.

We want the Hodge classes to contain a type $*$, which contains as special cases the chern classes of vector bundles and the cycle classes of algebraic subvarieties. The possibilities for $*$ are the chern classes of coherent sheaves, the chern classes of complexes, and teh chern classes of $D^b(X, Coh)$.

There is a natural map $c : D^b(X, Coh) \to H^*(X, \mathbb{Q})$.

We define the chern class of a complex to be $c(complex) = \prod c(F_i)^{(-1)^i}$.

Check that this is well defined up to quasi-isomorphism.

17 Lecture 16

c_*(coherent complexes) containes c_*(VB Complexes), which contains the classes of algebraic sycles and $c_*(VB)$, whic is contained in c_*(coherent sheaves) which is also contained in c_*(coherent complexes).

Why are classes of algebraic cycles related to chern classes of VBs or coherent sheaves?

Due to resolutions: to an algebraic subvariety $Z \subset X$ we associate $I_Z \subset \mathscr{O}_X$. Then $[Z] = c(i_*\mathscr{O}_Z)$.

Use $0 \to I_Z \to \mathscr{O}_Z \to \mathscr{O}_Z \to 0$. This gives a quasi-isomorphism $0 \to I_Z \to \mathscr{O}_Z \to 0$ with $0 \to 0 \to i_*\mathscr{O}_Z \to 0$, and so this chern class is $c(I_Z \to \mathscr{O}_X)$. So we have algebraic cycles contained in coherent sheaves, contained in coherent complexes. But NOT contained in VB. However, it can be written as a VB complex:

If $\operatorname{codim} Z = 1$, then I_Z is a LB, and so $I_Z \to \mathscr{O}_X$ is a complex of VB.

On smooth n-dimensional X, any coherent sheaf has an n-step locally free resolution.

In fact, let $\mathscr{O}(1)$ be ample, then any coherent sheaf has resolution by vector bundles of the form $\oplus\mathscr{O}(d_i)$.

Example 17.1. *$X = \mathbb{P}^2$, and Z is a point. Then we have $0 \to \mathscr{O}(-2) \to (\mathscr{O}(-1)^{\oplus 2}) \to \mathscr{O}_{\mathbb{P}^2} \to \mathscr{O}_{pt} \to 0$.*

$H^0(\mathscr{O}_{\mathbb{P}^2}(1)) =< x, y, z >$, and say the point is $x = y = 0$. Then $H^0(I_Z) =< x, y >$.

So what is $c(\mathscr{O}_{pt})$? It is $c(\mathscr{O}(-2) \to \mathscr{O}(-1)^2 \to \mathscr{O}) = c(\mathscr{O})c(\mathscr{O}(-2))/c(\mathscr{O}(-1))^2 = \frac{1(1-2H)}{(1-H)^2} = \frac{1-2H}{1-2H+H^2}$. This is $\frac{1-2H+H^2}{1-2H+H^2} - \frac{H^2}{1-2H+H^2}$, and so we have $1 - H^2$ (by expanding the denomenator and remembering that $H^3 = 0$.)

Similarly for $\mathbb{P}^3$, except we have to go another step (and get $\binom{3}{k}$ instead of $\binom{2}{k}$ as the expondents). We get that the chern class of the complex is $1+2H^3$. The class of I_{pt} is $1-2H^3$. In general, $c(I_Z)=(-1)^k/(k-1)![Z]$.

Cohomology of Tori

Let $\Gamma\cong\mathbb{Z}^{2g}$, and $\Gamma_{\mathbb{Q}}=\Gamma\otimes\mathbb{Q}$, and $\Gamma_{\mathbb{C}}=\Gamma\otimes\mathbb{C}^{2g}$.

So then set $X=\Gamma\setminus\Gamma_{\mathbb{C}}/W$, with $W=\mathbb{C}^g$, and $W\cap\bar{W}=\{0\}$.

Interpretation: $\Gamma=H_1(X,\mathbb{Z})$, $\Gamma_{\mathbb{Q}}$ is cohomology with $\mathbb{Q}$ coefficients, etc.

$H^k(X,\mathbb{Z})=\bigwedge^k\Gamma$.

The Hodge Filtration $H^{p,q}=\bigwedge^p W\otimes\bigwedge^q\bar{W}\subset\bigwedge^{p+q}\Gamma_{\mathbb{C}}$.

Are there Hodge classes? In general (this doesn't make sense on a moduli space, it makes sense before dividing by isomorphisms) there are none. $(\bigwedge^p W\otimes\bigwedge^p\bar{W})\cap\bigwedge^{p+q}\Gamma=0$. So the Hodge Conjecture is true, because it there are no Hodge classes, then every hodge class is algebraic.

However, by moving W into special position, we can arrange to have Hodge classes.

The number of complex parameters is $g(2g-g)=g^2$. For $p=1$, the complex tori that have a $Hodge^1$ class depend on $\frac{g(g+1)}{2}$ complex parameters

There exist several components distinguished by teh signiture of the $(1,1)$ form.

Positive⇒abelian varieties

Indefinite⇒non-algebraic tori having Pic $\neq 0$.

The interesting case is $g=4$, $p=2$. Generically, $Hodge^2=0$. Always, $(Hodge^1)^{\otimes 2}\to Hodge^2$. So the Weil tori are the ones where $Hodge^2$ is strictly bigger than the image of $(Hodge^1)^{\otimes 2}$. We will look at these more tomorrow.

They exist for both abelian varieties and for nonalgebraic tori. This will show that the Hodge conjecture is false for Kähler manifolds

18 Lecture 17

Weil Tori:

Look at the ring $\mathbb{Z}[i]$. This acts on Γ, the lattice. Then the torus is $\Gamma_{\mathbb{Z}}\setminus\Gamma_{\mathbb{C}}/H^{1,0}$, where $\gamma\cong\mathbb{Z}^{2g}$ with a bilinear form.

Let $g=2n$.

i acts on Γ, hence on $\Gamma_{\mathbb{C}}$, with eigenvalues $\pm i$.

Let C be a curve of genus g. Then $Jac(C)=H^1(C,\mathscr{O}^*)=H^1(C,\mathscr{O})/H^1(C,\mathbb{Z})=\Gamma$.

And so $0\to\mathbb{C}\to\mathscr{O}\to W\to 0$. Thus he have $H^{1,0}=H^0(C,W)\to H^1(C,\mathbb{C})\to H^1(C,\mathscr{O})=H^{0,1}$, and take $H^1(C,\mathbb{Z})\setminus H^1(C,\mathbb{C})/H^{1,0}$.

$X=\Gamma\setminus\Gamma_{\mathbb{C}}/H^{1,0}$, $W=W_i\oplus W_{-i}$.

Stability of Vector Bundles

We saw examples of jumping phenomenon for X a variety, $0\in P$ a parameter space and V a vector bundle over $X\times P$, with $p,q\in P$, $p\neq 0\neq q$, then $V|_{X\times p}\cong V|_{X\times q}$, but $V|_{X\times p}\not\cong V|_{X\times 0}$.

So in the natural topology, the set of isomorphism classes of vector bundles is not separated.

How to construct a reasonable moduli space?

Restrict to stable bundles, then we get a nice moduli space. Or we can consider all vector bundles and then we have to work with a moduli stack. We will focus on the first.

In the case that X is a curve, C, then we define the slope of a vector bundle $V \to X$ to be $\mu(V) = \deg(V)/\operatorname{rank}(V)$. If $V = \oplus L_i$, then $\mu(V)$ is the average of the $\deg L_i$. A vector bundle V is stable if for all proper subbundles $0 \subsetneq V' \subsetneq V$ then $\mu(V') < \mu(V)$.

A better, equivalent definition: allow V' to be any coherent subsheaf, then $rankV'$ is defined to be additive in exact sequences, and so take a locally free resolution. Equivalently, it is the rank at the generic point of X.

Example 18.1. *On $\mathbb{P}^1$, and bundle is a sum of line bundles. So the only stable bundles are line bundles.*

Because of this, we introduce the following:

Definition 18.1 (Semistable)**.** We call V semistable if $\forall V' \neq 0 \subset V$, we have $\mu(V') \leq \mu(V)$.

So now on $\mathbb{P}^1$, $\mathscr{O} \oplus \mathscr{O}$ is semistable, but $\mathscr{O}(1) \oplus \mathscr{O}(-1)$ is instable.

For a curve, a projective moduli space $\mathrm{Bun}_{C,n}$ exists paramaterizing S-equvialence classes of semistable bundles.

It contains a Zariski open subset of isomorphism classes of stable bundles.

Definition 18.2 (S-equivalence)**.** S-equivalence is the relation generated by $0 \to V_1 \to V \to V_2 \to 0$ where $\mu(V_1) = \mu(V_2)$, then $V \cong V_1 \oplus V_2$.

This does extend to higher dimensions of X.

The only thing we must do is define the slope of a vector bundle on X. We need to fix a Kahler class or an Ample Line Bundle on X. That is, fix $\omega \in H^{1,1}(X)$ greater than 0. Then we define $\deg V = c_1(V) \wedge \omega^{n-1}/\omega^n$, where $n = \dim X$.

If X is projective, then this is literally the projective degree of a divisor representing $c_1(V)$.

Typical Behavior:

The Kahler cone is partitioned into subcones, and inside of each subcone, we get the same moduli space, but as you cross between subcones, you get a different moduli space, and points on the boundary give something singular which is dominated by each.

We are going to a theorem of Donaldson-Uhlenbeck-Yau which gives a differential geometric interpretation of stability.

19 Lecture 18

Hodge Classes on 2n-Tori

Let $\Gamma \cong \mathbb{Z}^{4n}$, $X = \Gamma \setminus \Gamma_{\mathbb{C}}/W$. Note that $\Gamma = H_1(X,\mathbb{Z}) = H^1(X,\mathbb{Z})^*$ and $\Gamma_{\mathbb{C}} = H_1(X,\mathbb{C}) = H^1(X,\mathbb{C})^*$.

To we have $0 \to H^{1,0} \to H^1(X,\mathbb{C}) \to H^{0,1} \to 0$ and $0 \to W = H^{0,1^*} \to H_1(X,\mathbb{C}) \to H^{1,0^*} \to 0$.

How common are Hodge classes?

There is no "moduli space' of complex tori of dim > 1, and so we have to work in "period space". that is, we parameterize (X,i), $i : \Gamma \to \mathbb{Z}^{4n}$ an iso.

This period space is open (in the complex topology) in $Gr(N,2N)$ consisting of W such that $W \cap \Gamma_{\mathbb{R}} = 0$.

the dimension is N^2. So now we fix $\gamma \in H^{2n}(X,\mathbb{Z}) = \bigwedge^{2n} H^1(X,\mathbb{Z}) = \bigwedge^{2n} \Gamma^* = \bigwedge^{2n} H_1(X,\mathbb{Z}) = \bigwedge^{2n} \Gamma$ (by Poincare Duality)

When is it of type (n,n)?

(Variants among N not necessarily even, $\gamma \in H^{2k}(X,\mathbb{Z})$, $0 \leq k \leq N$)

So $\gamma \in Gr(N,2N)$. $T_\gamma Gr \cap Gr \subset \mathbb{P}(\bigwedge^N \Gamma_{\mathbb{C}})$, the Plücker Space.

This is a Schubert cycle $\sigma_1 = \{S \in Gr | S \cap \gamma \neq 0\}$.

Claim: σ_1 is a singular divisor in Gr. It's singular locus σ_{22} is the locus of W where γ is Hodge.

General element of Plucker Space is linear combination of basic verieties $e_{i_1} \wedge \ldots \wedge e_{i_n}$

Say that $\gamma = e_1 \wedge \ldots \wedge e_N$. Then $T_\gamma Gr = e_{i_1} \wedge \ldots \wedge e_{i_N}$ where at least one i_j is in $\{1,\ldots,N\}$.

So $Sing(T_\gamma Gr \cap Gr)$ is teh set where at least two of the i_j's are in $\{1,\ldots,N\}$. Thus, it is σ_{22} which has codimension 4.

Explicitly, $W \cap \sigma$ should have vector space dimension 2, and so $W|_{W \cap \sigma}$ has $N-2$.

$Gr(2,N) + Gr(N-2, 2N-2) = 2(N-2) + (N-2)N = N^2 - 2n + 2N - 4 = N^2 - 4$.

Do the variant, then you will see that a generic torus on which a given $\gamma \in H^4(X,\mathbb{Z})$ is of type $(2,2)$ has no $\mathbb{Z}$ classes of type $(1,1)$.

Weil Tori

$X = \Gamma \setminus \Gamma_{\mathbb{C}}/W$. Add the assumtion that Γ is a $\mathbb{Z}[I]$-module, where $I^2 = -1$. Then $\Gamma_{\mathbb{C}} = W \oplus \bar{W}$ is the Hodge decomposition, and so is $\mathbb{C}_i^{2n} \oplus \mathbb{C}_{-i}^{2n}$.

$\mathbb{Z}[i]$ acts on X iff W is $\mathbb{Z}[i]$-stable iff $W = W_i \oplus W_{-i}$ and $W_{\pm i} = W \cap \mathbb{C}_i^{2n}$.

Take $\dim W_{\pm i} = n$. Then we gave $\Gamma_{\mathbb{C}} = W \oplus \bar{W} = W_i \oplus W_{-i} \oplus \bar{W}_i \oplus \bar{W}_{-i} = \mathbb{C}_i^{2n} \oplus \mathbb{C}62n_{-i}$.

Our parameter space is $Gr(n, \mathbb{C}_i^{2n}) \times Gr(n, \mathbb{C}_{-i}^{2n})$. The only condition $W = W_i \oplus W_{-i}$ satisfies $W \cap \bar{W} = 0$ or $W \cap \Gamma_{\mathbb{C}} = 0$, which is an open condition.

Claim; $Hodge^2$ has rank ≥ 2 on a Weil Torus, and it is generically equal to 2.

$H^{2n}(X,\mathbb{Q}) \cong H_{2n}(X,\mathbb{Q}) \cong \bigwedge_{\mathbb{Q}}^{2n} \Gamma_{\mathbb{Q}} \supset \bigwedge_{\mathbb{Q}[i]}^{2n} \Gamma_{\mathbb{Q}} \cong \mathbb{Q}[i]$.

Claim: This is in $H^{n,n}$.

$\bigwedge^{2n} \mathbb{C}_i^{2n} = \bigwedge^{2n}(W_i \oplus \bar{W}_{-i}) = \bigwedge^n W_i \oplus \bigwedge^n \bar{W}_{-i} \subset \bigwedge^n W \oplus \bigwedge^n \bar{W} = H^{n,n}(X)$. Now we note that the first part is the image of $\bigwedge_{\mathbb{Q}[i]}^{2n} \Gamma_{\mathbb{Q}[i],i} \to \bigwedge^n \Gamma_{\mathbb{C}}$ over $\mathbb{C}$.

20 Lecture 19

The analytic part can be read on our own in "Voisin"

Obstructions to the Hodge Conjecture:
There are three types

1. Torsion: $H^{2k}(X,\mathbb{Z})_{tor} = \ker H^{2k}(X,\mathbb{Z}) \to H^{2k}(X,\mathbb{Q})$, which is also $\ker(H^{2k}(X,\mathbb{Z}) \to H^{2k}(X,\mathbb{C}))$. This is then contained in $\ker(H^{2k}(X,\mathbb{Z}) \to F^{k-1})$ of the Hodge filtration. And so, anything torsion is automatically Hodge. So then any torsion class should be algebraic, and there exists counterexamples: torsion classes on smooth projective varieties that cannot be algebraic.

2. Integrality: Let $X \subset \mathbb{P}^{n+1}$ a general hypersurface of deg d, eg $n = 3, k = 2$, then there exists a curve $C \subset X$ such that $[C] \in H^4(X,\mathbb{Z})$ is $a\gamma$ for $a \in \mathbb{Z}$, $a > 1$ and $\gamma \in H^4(X,\mathbb{Z})$, but γ is not algebraic.

3. Generalized Hodge: Let X be projective, smooth, $\gamma \in Hodge^k(X) = H^{k,k}(X) \cap H^{2k}(X,\mathbb{Q})$. Then there exissts $Z \subset X$ a subvariety of codimension k with $\gamma \in Im(H^*(Z) \to H^*(X))$. So Generalized Hodge is, given $\gamma \in H^{2k}(X,\mathbb{Q}) \cap F^{k-i}$, then it is in the image of $H^*(Z) \to H^*(X)$ for some alg subvariety Z of codim $k - i$.

Note that if we have Z of codimension k, we have $H^{2i}(Z,\mathbb{Q}) \to H^{2k}(X,\mathbb{Z})$. Anything in $H^{2i}(X,\mathbb{Q})$ is within i steps of center, that is, $H^{2k}(X,\mathbb{Q}) \cap F^{k-i}$. The GHC is the converse.

Grothendieck shows that this is false, but a quick fix is to replace "class γ" by "Hodge substructure"

So the corrected version is Any Hodge Structure $A \subset H^{2k}(X,\mathbb{Q}) \cap F^{k-i}$ is in the image of $H^*(Z,\mathbb{Q})$ for some Z of codimension $k - i$.

A Hodge substructure is $A = \oplus_{p+q=2k}(A \cap H^{p,q})$.

Grothendieck's example is $\gamma \in H^{2k}(X,\mathbb{Q}) \cap F^k$ which is not contained in a Hodge substructure.

Kähler's Example: $X \subseteq \mathbb{P}^4$ a general hypersurface of degree d. (IE, it holds for $X \in \mathbb{P}(\mathbb{C}[x_0,\ldots,x_4]^d) \setminus \cup_{countable} V_i$ with V_i a proper subvariety).

$H^k(\mathbb{P}^4,\mathbb{Z})$ has $\mathbb{Z}$'s down the middle of the Hodge diamond and zeroes elsewhere. For $H^*(X,\mathbb{Z})$, we have (rotate 45 degrees clockwise)

$\mathbb{Z}$	0	0	?
0	$\mathbb{Z}H$	?	0
0	?	$\mathbb{Z}H^2/d$	0
?	0	0	$\mathbb{Z}H^2/d$

Claim: Assume $p > 3$ prime and $p^3|d$, then the degree of any curve $C \subset X$ general is divisible by p. (And so, HC over $\mathbb{Z}$ fails)

We'll prove this claim next time.

21 Lecture 20

Today, we will prove a theorem which has the following corollary:

Corollary 21.1. *The Hodge conjecture over $\mathbb{Z}$ is false for X as described in the theorem.*

Theorem 21.2. *Let $p \geq 5$ be prime, $d = p^3 s \in \mathbb{N}$ and X a general hypersurface in $\mathbb{P}^4$ of degree d. If $C \subset X$ is a curve, then p divides $\deg C$.*

Proof. We will recall the construction, due to Grothendieck, of the Hilbert scheme parameterizing all subschemes of a given projective variety with a given Hilbert Polynomial.

We can also define the Hilbert scheme of pairs $C \subset X \subset \mathbb{P}^4$ to be the subset of $Hilb(X_0) \times Hilb(C_0)$ with $\{(X, C) | C \subset X\} = Hilb(C_0, X_0)$.

By looking at fiber products of the universal families over $Hilb(C_0)$ and $Hilb(X_0)$, we can see that $Hilb(C_0, X_0)$ is algebraic and parameterizes all pairs $C \subset X$. It also has a universal curve and hypersurface, with $\mathscr{C} \subset \mathscr{X}$.

Now consider the projection $Hilb(C_0, X_0) \to Hilb(X_0)$. We can break this into $Hilb^{gen} \cup Hilb^{spec}$, the former of which are the curves which exist over any X, and the latter the ones that only exist over some X. By assumption, we have X general, and so it is in $Hilb^{gen}$. Thus, we can deform $C \subset X$ to a cruve in any other X.

As X moves towards any particular X', C will move with it. That is, if $\mathscr{X} \to B$ is any appropriate family, we have a family $\mathscr{C} \to B$ such that $\mathscr{C}_b \subset \mathscr{X}_b$ for all $b \in B$.

Now let $Y \subset \mathbb{P}^4$ have degree s, and look at $H^0(\mathbb{P}^4, \mathscr{O}(p))$. Choose generic sections $\phi_0, \ldots, \phi_4$. These give a map $\phi : \mathbb{P}^4 \to \mathbb{P}^4$ with $\phi^* \mathscr{O}(1) = \mathscr{O}(p)$. Let $X_0 = \phi(Y)$.

Then $\deg X_0 = \# X_0 \cdot H^3 = \# Y \cdot (\phi^* H)^3 = \# Y \cdot p^3 H^3 = p^3 \deg Y = p^3 s = d$.

By choosing ϕ sufficiently general, we can assume the following:

1. $\phi : Y \to X_0$ is a birational map.
2. There is a surface in Y which maps 2-to-1 to the image
3. There is a curve in Y which maps 3-to-1 to the image
4. There are finitely many points which map 4-to-1
5. Nothing maps 5-to-1

So $C \subset X$ specialies to $C_0 \subset X_0 = \phi(Y)$ with $\deg C = \deg C_0$. Now, $\phi^{-1}(C_0) \subseteq Y$ is a curve, and $\phi_*[\phi^{-1}(C_0)]$ divides $6[C_0]$, and so the degree divides $6 \deg C_0$.

Define $\tilde{C} = \phi^{-1}(C_0)$. Then $\deg C_0 = \# C_0 \cdot H = \# \tilde{C} \cdot \phi^* H = p \deg \tilde{C}$, and so as long as p is not 2 or 3, it must divide $\deg C_0 = \deg C$. □

Note: The generator of $H^4(X, \mathbb{Z}) = \mathbb{Z}$ is algebraic on a dense subset of $Hilb(X_0)$, though not open (it is fails to be so on a countable collection of proper subvarieties).

Idea of how to see this: Look at surfaces, $W \subset X$ of degree d on W. Any integral Hodge class is algebraic. CAn get a countable collection of classes in $H^2(W,\mathbb{Z})$, each is Hodge somewhere and so is algebraic (as HC is true in codimension 1) which map to $1 \in H^4(X,\mathbb{Z})$.

22 Lecture 21

Grothendieck's counterexample: "Generalized Hodge is false"

Consider $X = (S^1)^8$ a 4 complex dimensional torus

$H^1(X,\mathbb{Z}) \xrightarrow{\sim} \Lambda = \mathbb{Z} < \alpha_1, \dots, \alpha_4, \beta_1, \dots, \beta_4 >$ with polarization, which is the same as $c_1(L) \in H^2(X,\mathbb{Z}) = \bigwedge^2 \Lambda$

So then fix $\alpha \in H^4(X,\mathbb{Z}) = \bigwedge^4 \Lambda$.

We want to describe:

1. all complex structures on X
2. which are principally polarized abelian varieties
3. on which α is in $H^{2,2}$
4. on which α is in $H^{3,1} \oplus H^{2,2} \oplus H^{1,3}$

1. Look at $Gr(4,8) = Gr(4, \Lambda \otimes \mathbb{C})$. A complex structure with a trivialization of Λ gives a point of Gr. To recover X, we have $\Lambda \setminus \Lambda \otimes \mathbb{C}/\text{span}W$. This works iff $\text{span}W \cap \Lambda_{\mathbb{R}} = 0$. The dimension is then 16.
2. Riemann's bilinear relations: we describe W via the period matrix, which is four by eight: choose a complex basis $w_1, \dots, w_4$ for W such that $< \alpha_i, w_j >= \delta_{ij}$. But them $< \beta_i, w_j >= \Omega_{ij}$, the period matrix. Ω is symmetric and has imaginary part positive definite. Thus, $\dim \mathscr{A}_4 = 4 \cdot 5/2 = 10$. This has a Hodge interpretation: we get an abelian variety iff the polarization is in $H^{1,1}$. In $Gr(4,8)$, W is a Lagrangian subspace with a positivity condition.
3. Now, for each $\alpha \in \bigwedge^4 \Lambda$, (eg $\alpha = \alpha_1 \wedge \alpha_2 \wedge \beta_1 \wedge \beta_2$). Sometimes α is algebraic, for instance, $X = S_1 \times S_2$ where S_i are 2-dimensional complex tori. Then if $H^1(S_i) = \Lambda_i$, we have $\Lambda = \Lambda_1 \oplus \Lambda_2$, and we can fix $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \Lambda_1$. Then $\alpha \in H^4(S_1) \times H^0(S_2)$. So then $\alpha = [S_2]$.

Now a geometric description, is that $\alpha \in H^4(X,\mathbb{Z}) = \bigwedge^4 \Lambda \subset H^4(X,\mathbb{C})^*$. And so we have the Plucker embedding of the grassmanian. So $\alpha = 0$ is a hyperplan section of $Gr(4,8)$, call it Y_α. Then $\alpha \in (H^{31} \oplus H^{22} \oplus H^{13})$ iff $w \in Y_\alpha$. And so $\alpha \in H^{22}$ iff $w \in Sing(Y_\alpha)$

In general, H^{40} is the wedge of four things, all of them in W, H^{31} is three from W one from $\bar{W}$, etc. So $\alpha \in H^{31} \oplus H^{22} \oplus H^{13}$ says that α has no components in H^{40} of H^{04}, which is the same as α having no component in H^{40}, so then $\alpha \wedge W = 0$, which is $W \in Y_\alpha$. Similarly, $\alpha \in H^{22}$ means that it has no component in H^{40} or H^{31}, which means that $\alpha = 0$ is tangent to $Gr(4,8)$ at W, an so $W \in SingY_\alpha$.

So now we claim that there exists a principally polarized abelian variety (PPAV) on which there exists $\alpha \in H^4(X,\mathbb{Z}) \cap (H^{31} \oplus H^{22} \oplus H^{13})$ which is unique up to scalar and not in H^{22}. So then the span of α is not a Hodge substructure.

Dimension Count: To be in $H^{31} \oplus H^{22} \oplus H^{13}$ is one condition, so we have 10-1=9 parameters left. H^{22} is 6, and so having imposed the first one, we have a lot of space to choose from. Now, the second Riemann bilinear condition holds along H^{22}, and is an open condition, so it holds.

23 Lecture 22

Two more Hodge topics:

First we will talk about Hodge loci and absolute Hodge classes, and then we will prove the integral Hodge conjecture for cubic fourfolds/intermediate Jacobian and rationality.

Next: Calabi-Yaus and Mirror Symmetry. We will mostly follow notes with Mark Gross from a book "Calabi-Yau Manifolds and their friends"

Maybe also Cox's "Toric Varieties", and Cox and Katz's "Mirror Symmetry"

So there is a notion of an absolute Hodge class, and algebraic implies absolute Hodge, which implies Hodge, so we will ask if we can show that Algebraic implies absolute Hodge.

Corollaries:

Corollary 23.1. *X a complex projective variety over $\mathbb{C}$. Then there exists a finitely generated $\mathbb{Q} \subset K \subset \mathbb{C}$ such that X is defined over K.*

For all embeddings $\sigma : K \to \mathbb{C}$, we get a new complex variety X_σ

GAGA says that algebraically, on X we have $\mathbb{C}$ qis Ω^*_{dR}, by $\mathbb{H}^*(X,\Omega^*)$ and analytically we have $H^*(X_{an},\mathbb{C})$ and $\mathbb{H}^*(X,\Omega^*_{dR}) \xrightarrow{\sim} H^*(X_{an},\mathbb{C})$.

Moreover, Hodge filtration can be defined algebraically.

We define a filtration F^* of Ω^* by subcomplexes $F^k\Omega^*$ with $F^k\mathbb{H}^*(X,\Omega^*) = Image(H^*(X,F^k\Omega^*) \to \mathbb{H}^*(X,\Omega^*))$. Intuitively, F^k should mean the at least k holomorphic differentials. So we define $(F^k\Omega^*)^\ell = \begin{cases} \Omega^\ell & \ell \geq k \\ 0 & \ell < k \end{cases}$.

If Z is a codimension k subvariety of X, then it has a class $[Z] \in F^k\mathbb{H}^{2k}(X,\Omega^*)$, which tells us that $[Z]$ is Hodge in $H^{2k}(X^{an}_\sigma,\mathbb{C})$ for all $\mathbb{C} \to \mathbb{C}$.

Thus, Algebraic implies Absolute Hodge.

Note: There exist examples such that X^{an}_σ is not diffeomorphic to X^{an}.

The reason is that except for the identity and complex conjugation, none of these automorphisms are continuous.

So now, let $\pi : \mathcal{X} \to B$ a deformation of X. Look at $R^n\pi_*\mathbb{Z}$ is a sheaf on B, and $R^n\pi_{0*}\mathbb{Z}$ is locally constant, and so is $R^n\pi_{0*}\mathbb{C}$.

So it is a vector bundle over B_0 with a flat connection $\nabla : R^n\pi_{0*}\mathbb{C} \to R^n\pi_{0*}\mathbb{C} \otimes \Omega^1_B$ the "Gauss-Manin Connection"

It also has the Hodge filtration $F^kR^n\pi_{0*}\mathbb{C} \subset R^n\pi_{0*}\mathbb{C}$. This is a vector subbundle, but it is not a FLAT subbundle. $\nabla F^k \subset F^{k-1}\otimes\Omega^1_B$, "Griffiths Transversality"

Now, on graded pieces, we have $\nabla^k : (F^k/F^{k+1}) \to (F^{k-1}/F^k)\otimes\Omega^1_B$. Now ∇ is $\mathbb{C}$-linear, but ∇^k is $\mathscr{O}$-linear, and so commutes with multiplication by a holomorphic function. This is because the extra term in the Leibniz formula goes to F^k, which we mod out by.

Equivalently, we get $T_B \otimes H^{k,n-k} \to H^{k-1,n-k+1}$. This is called the Kodaira-Spencer map. This was a key ingredient in studying the Torelli problem and proving injectivity of the period map last semester.

Let $\tilde{B}$ be the universal cover ob $B_0 \ni b$. Then any integral class $c \in H^n(X,\mathbb{Z})$ extends to a flat section of $R^n\tilde{\pi}_*\mathbb{Z}$. Over B_0, we get a multivalued section. Is this locus algebraic?

Assume that c is Hodge. Consider $Hodge_c = \{\tilde{b} \in \tilde{B} | c_{\tilde{b}} \in H^{k,k}(X_{\tilde{b}},\mathbb{Z})\}$. Is this algebraic? The Hodge Conjecture implies yes. In fact, if c is absolutely Hodge, then yes.

Major known results:

Theorem 23.2 (Cattani-Deligne-Kaplan). *The Hodge locus is always algebraic.*

Theorem 23.3 (Deligne). *Hodge implies Absolute Hodge for abelian varieties.*

24 Lecture 23

<u>Cubics</u>

For cubic curves in $\mathbb{P}^2$, they are elliptic curves.

For a cubic surface in $\mathbb{P}^3$, the Hodge diamond is zero except $1, 7, 1$ along the middle. This is a rational surface, which is a blowup of $\mathbb{P}^2$ at 6 points in general position (no three on a line and not all six on a conic). It contains 27 lines. Look in Lecture Notes 777.

We embed into $\mathbb{P}^3$ via $\mathscr{I}_{p_1+\ldots+p_6}(3)$ where the p_i are the points blown up. So then a line in X is a curve in X such that $L\cdot C = \deg(L/C) = 1$.

So we have a map $X \to \mathbb{P}^2$ from the blowup. $L \in \mathrm{Pic}(X)$, $\pi_* L = J_{p_1+\ldots+p_6}(3)$. So the lines will be the 6 exceptional divisors, the 15 lines between pairs of points, and there are 6 conics through five of the six points which become lines.

If $C \subset X$ has image $\bar{C}$ in $\mathbb{P}^2$ which has degree d and multiplicity m_i at p_i, then $\deg(X \subset \mathbb{P}^3) = 3d - \sum m_i$. To see how this gives us 27, if we take $d = 0$, we get the six exceptional curves which have multiplicities -1 at one and zero else. We get 15 lines containing 2 of them in degree 1, and in degree 2 we get 6 which pass through five points. No singular conics work because they are two lines intersecting, and similarly for higher degrees.

So now we've shown that there are at least 27 lines. This is precise, but we will prove the following easier result:'

Theorem 24.1. *There are finitely many lines in a smooth cubic surface.*

Proof. Let $\mathscr{M}$ be the space of all smooth cubic surfaces. This is an open subset of $\mathbb{P}^{19}$. Take $\mathscr{X} \subset \mathbb{P}^{19} \times \mathbb{P}^3$ the universal family of cubic surfaces in $\mathbb{P}^3$.

Let $\mathscr{L}$ be the space of cubics and line, and let $\mathscr{X}_{\mathscr{L}} = \{(x, \ell, X) | x \in \ell \subset X\} \subset \mathbb{P}^3 \times Gr(2,4) \times \mathscr{M}$. So we have the forgetful maps in a commutative square.

We also have a map $\mathscr{X}_{\mathscr{L}} \to Gr(2,4)$. Note that for a cubic to contain a line is a linear condition in the coefficients. In fact, four linear conditions. And so we have $6 + 1 + (19 - 4) = 22$, a so there are finitely many lines on a cubic surface. $\square$

Now we move on to the cubic threefold in $\mathbb{P}^4$. X is unirational, that is, there is a dominant rational map $\mathbb{P}^3 \to X$.

In the case of curves, rational is equivalent to unirational. In the case of surfaces Lüroth proved it. However, for cubic threefolds, many proofs were claimed, but Serre showed that they were all wrong. The irrationality of the cubic threefold was proved by Clement and Griffiths, and a second proof was given by Mumford.

How about lines on X? There will be infinitely many, because a generic hyperplane section is a smooth cubic surface. (Bertini?)

Define $F(X) = \{\ell \in Gr(2,5) | \ell \subset X\}$. Now, we can see that the codimension $\operatorname{codim}(F(X); Gr(2,5)) = 4$, and so we have $\dim F(X) = 2$.

For the moment, we will look at $\{(\ell, \pi, X) | \ell \subset \pi \cap X$ where ℓ is a line, π is a space and X is a cubic threefold in $\mathbb{P}^4\}$

We can map this to the locus of $(\ell, \pi \cap X)$, lines in a cubic surface.

To see unirationality, choose a line $\ell \subset X$, then the projection $X \to \mathbb{P}^2$ is a conic bundle. The general fiber is a smooth conic, and also there exists a curve $\Delta \subset \mathbb{P}^2$, $p \in \Delta$ such that $\pi^{-1}(p)$ is a pair of lines, where Δ is the discriminant curve.

Exercise 24.1. *Find a cover of X which is rational.*

25 Lecture 24

Cubic surfaces: there exist 27 lines.

Cubic threefolds in $\mathbb{P}^4$: there exists a "Fano" surface of lines. (for cubic 4folds, there is a "fano" 4fold of lines)

Let $X \subset \mathbb{P}^4$ be cubic, smooth, and $F \subset Gr(2,5)$ the Fano surface of lines in X.

$\tilde{X}$ is a conic bundle (where $\tilde{X}$ is the blowup of X along a line). There exists a $\mathbb{P}^2$ bundle $\mathbb{P}^2 \to P \to B$ and an embedding $X \to P$ over B such that the fibers of $\pi : \tilde{X} \to B$ are conics in $\mathbb{P}^2$. Generically this is a smooth conic, but over some $\Delta \subset B$, we have line pairs. But no double lines.

For each line, we have a different conic bundle structure.

$$\begin{array}{ccccc} & E & \longrightarrow & \tilde{X} & \\ \swarrow & & \swarrow & & \searrow \\ \ell \longrightarrow & X & \dashrightarrow & & \mathbb{P}^2 = B \\ & \uparrow & & & \uparrow \\ & \mathbb{P}^4 & \longleftarrow & & \tilde{\mathbb{P}}^4 = P \end{array}$$

The fibers of $\tilde{X} \to \mathbb{P}^2$: a point of $\mathbb{P}^2$ correspondes to a plane $A \subset \mathbb{P}^4$ containing ℓ, and so $A \cap X$ is $\ell \cup$ a conic.

Intermediate Jacobian: Given any Hodge structure H, of odd weight $2k-1$ (that means that H is a free abelian group over $\mathbb{Z}$ with $H_{\mathbb{C}} = H^{2k-1,0} \oplus H^{2k-2,1} \oplus \ldots \oplus H^{0,2k-1}$.) then $F^k H_{\mathbb{C}}$ is a "half", that is $F^k \oplus \bar{F^k} = H_{\mathbb{C}}$.

Example 25.1. *$k = 1$: $H_{\mathbb{C}} = H^{10} \oplus H^{01}$ acts on H^1 (curve) or in fact H^1 of anything smooth and projective, then $F^1 = H^{10}$.*

$k = 2$: $H_{\mathbb{C}} = H^{30} \oplus H^{21} \oplus H^{12} \oplus H^{03}$, so $F^2 = H^{30} \oplus H^{21}$.

So given any Hodge structure of odd weight, we form the intermediate Jacobian $\mathscr{J}^k(H) = H \setminus H_{\mathbb{C}}/F^k H_{\mathbb{C}}$. This is a complex torus, isomorphic to $(S^1)^{\dim_{\mathbb{C}} H_{\mathbb{C}}}$.

Etymology: For any X smooth projective of dimension d, $\mathscr{J}^1 = \mathscr{J}^1(H^1(X)) = \mathrm{Pic}(X)$. The Italians also studied $\mathscr{J}^d = \mathscr{J}(H^{2d-1}(X)) = Alb(X)$ which is "dual" to the Picard. So when X is a 3fold, it has $\mathscr{J}^1 = \mathrm{Pic}$, $\mathscr{J}^3 = Alb$ and $\mathscr{J}^2$, the "intermediate" Jacobian.

Theorem 25.1 (Clemens and Griffiths)**.** *If X and X' are birational 3folds, then there exist curves C, C' such that $\mathscr{J}^2(X) \times \mathscr{J}(C) \cong \mathscr{J}^2(X') \times \mathscr{J}(C')$.*

For general 3fold X, $\mathscr{J}^2(X)$ is a polarized complex torus, but no more. For cubic threefolds, however, $\mathscr{J}^2(X)$ is a ppav of dimension 5. It is NOT a Jacobian. (Let $\mathscr{A}_5$ be the moduli space of ppav's of dimension 5 and $\mathscr{M}_5$ the moduli space of genus 5 curves, we have a map $\mathscr{M}_5 \to \mathscr{A}_5$. Is this map surjective? No. $\dim \mathscr{M}_5 = 12$ and $\dim \mathscr{A}_5 = 14$, the Torelli question is whether the map is injective, and last semester we discussed the case of curves, where it is known to be injective.)

So the Clemens-Griffith result implies that $\mathscr{J}^2(X) \times \mathscr{J}(C)$ is $\mathscr{J}(C')$, and it follows that $\mathscr{J}^2(X) \cong \mathscr{J}(C' \setminus C)$, but $\mathscr{J}^2(X)$ is not the Jacobian of any curve, so this fails.

To get $\mathscr{J}^2$ of a cubic 3-fold, we compute the Hodge diamon, which is 1 down the center and $0, 5, 5, 0$ in the middle row (which we compute with last semester's techniques)

Signs of the polarization are given by the Hodge Index theorem, which says that in the middle dimension, it alternates signs and so is $+-+-$.

We look at the first half of the cohomology, and the pairing is positive on h^{21} and negative on h^{30}. In our case, $h^{30} = 0$, and so we have a definite polarization.

General description of $\mathscr{J}^2(X)$ when $\pi : X \to B$ is a conic bundle and Δ the discriminant curve in B. Let $\tilde{\Delta}$ be the Stein factorzation of $\pi^{-1}(\Delta) - Sing \to \Delta$.

$\mathscr{J}(\tilde{\Delta}) \to \mathscr{J}^2(X)$ is a amp $H^1_{\mathbb{Z}} \setminus H^1_{\mathbb{C}}(\tilde{\Delta})/F^1 \to H^2_{\mathbb{Z}} \setminus H^3_{\mathbb{C}}(X)/F^3$.

Now, a map $H^1(\tilde{\Delta}) \to H^3(X)$ is given by $\mathscr{L}$ a line bundle over $\tilde{\Delta}$ and over X, we pullback and then push forward to get a map $H^i(\tilde{\Delta}) \to H^i(\mathscr{L}) \to H^{i+2}(X)$ (must go through homology via Poincare duality). This is the Gysin map.

Under mild conditions, this induces an isomorphism $\mathscr{J}(\tilde{\Delta})/\mathscr{J}(\Delta) \to \mathscr{J}^2(X)$. This is cally $Prym(\tilde{\Delta}/\Delta)$, the Prym variety.

26 Lecture 25

Last time we defined the intermediate Jacobians, and we will be working with $\mathscr{J}^2(X)$, which is constructed from the third cohomology by $H^3(X,\mathbb{C})/H^3(X,\mathbb{Z})+F^2H^3(X,\mathbb{C})$, which is $F^2H^3(X,\mathbb{C})^*/H^3(X,\mathbb{Z})$. It can also be described as the kernel of the map $Delignian \to H^4(X,\mathbb{Z})$, which is more algebraic. That is, there is an object called the Delignian of X such that $0 \to J^2(X) \to Delignian(X) \to H^{2;2}_{\mathbb{Z}}(X) \to 0$.

Properties

1. Let $\tilde{X} = Bl_C X$ where C is a curve in X. Then $H^3(\tilde{X}) = H^3(X) \oplus H^1(C)$, and $H^{30}(\tilde{X}) = H^{30}(X)$, $H^{03}(\tilde{X}) = H^{03}(X)$, but $H^{21}(\tilde{X}) = H^{21}(X) \oplus H^{10}(C)$ and $H^{12}(\tilde{X}) = H^{12}(X) \oplus H^{01}(C)$.

   Thus, $\mathscr{J}^2(\tilde{X}) = \mathscr{J}^2(X) \times \mathscr{J}(C)$, which is an isomorphism of ppav's.

   Corollary 26.1. *The "non-Jacobian part" of $\mathscr{J}^2(X)$ is a birational invariant.*

   The idea: $Bl_p(X)$ doesn't change H^3 or $\mathscr{J}^2$, but $Bl_C(X)$ add a Jacobian.

   Need: The category of ppav is semisimple.

2. Let $\mathscr{C} \subset B \times X$ be a family of curves in X parameterized by B. The choice of base point $b_0 \in B$ gives a map $A\mathscr{J} : B \to \mathscr{J}^2(X)$ which is Abel-Jacobi-like. A variant is to allow $\mathscr{C}$ to be a family of cycles in X (not necessarily effective) and assume that for all $b \in B$, C_b is homologous to zero, then $A\mathscr{J} : B \to \mathscr{J}^2(X)$ is independent of base point.

   Analytically, the former is a special case of the latter by taking C_b to $C_b - C_{b_0}$.

   So now each C_b is $\partial\Gamma_b$ where Γ_b is a real 3-chain in X with homology C_b. $\int_{\Gamma_b}$ is a well defined linear function from harmonic 3-forms to $\mathbb{C}$, and $F^2H^3(X)$ is the set of harmonic $(3,0)$ and $(2,1)$ forms, so we can restrict the integration to this subset. Thus, for all b, we get $\int_{\Gamma_b} \in F^2H^3(X,\mathbb{C})^*$.

   The choice of Γ_b changes only by $H^3(X,\mathbb{Z})$. So we get a well-defined image independent of the choice of Γ_b in $\mathscr{J}^2(X)$.

   The Deligne group is defined to be $\mathbb{H}^4(\mathbb{Z} \to \mathscr{O} \xrightarrow{d} \Omega^1)$. We call this complex the Deligne complex $\mathscr{D}$, and we have $K \to \mathscr{D} \to \mathbb{Z}$ morphisms of complexes with K the complex $\mathscr{O} \to \Omega^1$ in degrees 1 and 2, and $\mathbb{Z}$ the complex $\mathbb{Z} \to 0 \to 0$ with $\mathbb{Z}$ in degree zero.

So we get a long exact sequence on hypercohomology, and so we have $H^3(\mathbb{Z}) \to H^4(K) \to \mathbb{H}^4(\mathscr{D}) \to H^4(\mathbb{Z})$. We can also get that $\mathscr{E} \to dR \to K$ for some complex $\mathscr{E}$. We don't want to spend too much time on this, but the end result is $\mathbb{H}^4(K) = F^2H^{3*}$.

The upshot is that $\mathbb{H}^4(K) = H^3(\mathbb{C})/F^2 = F^{2*}$, and $\mathbb{H}^4(K)/H^3(\mathbb{Z}) = \mathscr{J}^2$. So we have $0 \to \mathscr{J}^2 \to \mathbb{H}^4(\mathscr{D}) \to H^{2,2}_{\mathbb{Z}} \to 0$.

A general result is that given any family $\mathscr{C} \to B$ of curves there exists a natural $A\mathscr{J} : B \to \mathscr{D}el$ which, when composed with c_2 into $H^{2,2}_{\mathbb{Z}}$, sends each point to the second chern class of the ideal sheaf of the curve.

The image of this $A\mathscr{J}$ lands in $shJ^2(X)$, which is the kernel of c_2, if and only if C_b are homologous to zero.

For $\mathscr{J} = \text{Pic}$, the analogue is $0 \to \mathscr{J}^0 \to \mathscr{J} \to H^{1,1}_{\mathbb{Z}} \to 0$. So given a family of codimension k subvarieties of X, $\mathscr{S} \to B$, we get a natural Abel-Jacobi map $B \to \mathscr{D}^k(X)$, which then has $0 \to \mathscr{J}^k(X) \to \mathscr{D}^k(X) \to H^{k,k}_{\mathbb{Z}}(X) \to 0$. In the case $k = 1$, there exist families such that this Abel Jacobi map is surjective. For $k > 1$, the image is very thin. So Jacobi Inversion fails.

Third important property: General fact: if $\alpha : \mathbb{P}^n \to A$, an Abelian variety, then α is constant.

Proof. $H^0(A, \Omega^1_A)$ generates Ω^1_A. But $\alpha^*(H^0(\Omega^1_A))$ consists of projective forms on $\mathbb{P}^n$, and so $d\alpha = 0$. This in fact holds for any rational projective variety. □

Example 26.1. *Let X be a conic bundle, so we have a map $\pi : X \to B$ where the general fiber is a smooth conic and there is a divisor $\Delta \subset B$ over which the fibers are pairs of lines. We get a double cover $\tilde{\Delta} \to \Delta$ from the stein factorization of the lines, and now we let $\mathscr{C} \to \tilde{\Delta}$ be the $\mathbb{P}^1$-bundle. Apply the Abel-Jacobi map, and we get $\tilde{\Delta} \to \mathscr{J}^2(X)$ which factors through $\mathscr{J}(\tilde{\Delta})$ (this is GLC for $GL(1)$) (any map to an abelian variety factors through the Jacobian)*

So we have a map $\mathscr{J}(\tilde{\Delta})/\mathscr{J}(\Delta) = Prym(\tilde{\Delta}/\Delta) \to \mathscr{J}^2(X)$, which is an isomorphism in general.

Exercise 26.1. *Describe Δ and $\tilde{\Delta}$ explicitly when X is a cubic 3-fold.*

27 Lecture 26

Last time we defined the intermediate Jacobians and the Delignian $Del^k(X) = \mathbb{H}^{2k}(\mathbb{Z} \to \mathscr{O} \to \ldots \to \Omega^{k-1})$.

For $k = 1$, we have the Picard group of X.

If k is $\dim_{\mathbb{C}} X$, then we get $Alb(X)$, which has the property that any map from X to an abelian variety factors through an inclusion $X \to Alb(X)$.

More generally, if $\mathscr{C} \subset B \times X \to B$ is a family of codimension k cycles for any k, then there exists a natural map $B \to Del^k(X)$. (It goes into $\mathscr{J}^k(X)$ if $\mathscr{C}_b$ is always homologous to zero)

In the case $k = \dim X$, then $\mathscr{J}^k = Alb$, and $\mathscr{C} = X$ as a family of dimension 0 cycles over X. This gives a map on $Alb(X)$, but depends on choice of basepoint.

Example 27.1. *Let X be a conic bundle over B. then there is a subvariety of B, Δ, which is the locus where the fiber is a pair of lines intersecting. Then $\mathscr{L} \to \tilde{\Delta}$ is a family of codimension two cycles in X. So we get a map $B \to \mathscr{J}^2(X)$ which is constant. Thus, $\Delta \to \mathscr{J}^2(X)$ is constant, and so $\mathscr{J} \to \mathscr{J}^2$ is zero. So we get a natural map $Prym(\tilde{\Delta}/\Delta) \to \mathscr{J}^2(X)$, which is often an isomorphism.*

Example 27.2. *Let X be a Cubic three fold and F a Fano surface. Then $\mathscr{C} \to F \times F$ a family of cycles with fibers $\ell - \ell'$. Then we get a natural map $F \times F \to \mathscr{J}^2(X)$.*

It can be shown that Δ is going to be a plane quintic and thus has arithmetic genus 6, and $\tilde{\Delta} \to \Delta$ is an unramified double cover. Thus, $g(\tilde{\Delta})$ has genus 11. So the dimension of $Prym$ is $11 - 6 = 5$.

So we have a smooth cubic surface $S \subset \mathbb{P}^3$ containing a line ℓ_0 an the question is: if the pencil of planes in $\mathbb{P}^3$ through ℓ_0, how many planes meet S in a pair of lines?

One way to find out is by looking at the euler characteristic. We note that $\tilde{S} = Bl_{\ell_0} S = S$, and so $\chi(S) = 9$. On the other hand, it is also $\chi(\mathbb{P}^1)\chi(Conic) + \deg(\Delta)(\chi(V(xy)) - \chi(conic))$, and this is equal to $2 * 2 + \deg$, so $\deg = 5$.

Another way: say $S = Bl_{p_1,\ldots,p_6}\mathbb{P}^2$, and ℓ_0 is exceptional over p_1. It meets $C_{\hat{i}}$ iff $i \neq 1$, and it meets C_{ij} iff $1 \in \{i, j\}$, and additionally, it meets p_i never.

More generally, $S = dP_n = Bl_n\mathbb{P}^2$ for $n \leq 8$ is a Del Pezzo surface. These contain finite numbers of "lines", that is, curves ℓ such that $\ell \cdot (K_S^{-1})$ is 1.

The intersection configursation of these have symmetry groups $W(E_n)$, with the root system E_n.

Cases: $n = 6$ is the cubic surface, for $n = 8$, we have a rational elliptic surface. For $n = 0$ (that is, $\mathbb{P}^2$), there are no "lines", and for $n = 1$, there is a unique line, the exceptional curve.

For $n = 2$, we have 3 lines, and the Weyl Group will interchange two of them which intersect the third (the two exceptional ones)

For $n = 3$, there are six lines in a hexagonl, etc.

Look at $H^2_0(dP_n, \mathbb{Z})$, that is, the part perpendicular to the anticanonical bundle. This is the lattice of characters of E_n.

So anyway, the imag of $F \times F \to \mathscr{J}^2(X)$ is the theta divisor of $\mathscr{J}^2(X)$.

Next: discuss the singularities of Θ and deduce the Torelli Theorem and the irrationality of cubic threefolds.

Finally, we'll consider cubic 4folds and improve this slightly better to get the Hodge conjecture for them.

28 Lecture 27

Let X be a 3fold, $\mathscr{C} \to B$ a family of curves in X, then we have a natural map to $Del^2(X)$, which factors the natural map to $H^{2,2}_{\mathbb{Z}}(X)$.

The map lands in $\mathscr{J}^2(X)$ iff the curves are homologous to zero.

Example 28.1. *Conic bundles* $X \to \mathbb{P}^2$ *gives* $Prym(\tilde{\Delta}/\Delta) \to \mathscr{J}^2(X)$.

Example 28.2. *If* F *is a Fano surface in* X*, a cubic three fold, then we have* $F \times F \to \mathscr{J}^2(X)$ *which is dimension 5, and the image is the Theta Divisor.*

That is, the dimension of the image is $4 = 5 - 1$, the class of the image is in $H^{1,1}_{\mathbb{Z}}(\mathscr{J}^2(X)) = \bigwedge^2(\text{rank 10 lattice})$, and the elementary divisors equal 1.

This implies that $\mathscr{L} = \mathscr{O}_{\mathscr{J}^2(X)}(image)$ is a line bundle andb $h^0(\mathscr{L}) = 1$. The divisor of a nonzero section is unique Θ.

We may as well describe all the fibers of $F \times F \to \mathscr{J}^2(X)$. Recall that $F \times X \subset \mathscr{L} \to F$ where $\mathscr{L}$ is the universal line. Then we have $\mathscr{C} \to F \times F$ with $\mathscr{C}_{\ell_1,\ell_2} = \ell_1 - \ell_2$. So this map is the Abel-Jacobi map for the family of cycles.

The dagonal $F \subset F \times F$ is sent to 0. All other fibers are finite. $i(\ell_1 - \ell_2) = i(\ell_1' - \ell_2')$. The basi crelation is: take a cubic surface $S \subset X$ and take all four of these lines to be in S. Then we have $E_i + C_{\hat{i}} = E_j + C_{\hat{j}}$. Precisely, they will have teh same index under the map $F \times F \to \mathscr{J}^2(X)$.

Exercise 28.1. *Show that this relation generates the fibers of* i *(except on the diagonal)*

Checking that this is the Θ-divisor is completed by a topological calculation.

Applications

Theorem 28.1 (Torelli)**.** *The principlally polarized abelian variety* $(\mathscr{J}^2(X), \Theta)$ *uniquely determines* X.

Analize $i : F \times F \to \Theta$. Conclude that Θ has a singuilar point at 0 and that Θ is nonsingular elsewhere, so $Bl_F(F \times F) \to Bl_0(\Theta)$ is a finite map, and a nonsingular quotient of $Bl_F(F \times F)$.

The projectivized tangent space at 0 to $\mathscr{J}2(X)$ is isomorphic to the ambien $\mathbb{P}^4$, and so the projectivized cone at 0 to Θ is X.

Idea: $\ell \in F$. What is the exceptional curve in $Bl_F(F \times F)$ over ℓ?

Then we take $\mathbb{P} \subset$ Exceptional Divisor$\subset Bl_F(F \times F)$ to $\ell \subset F \subset F \times F$. The exceptional divisor is the universal line, and the $\mathbb{P}^1$ is ℓ.

Proof: $T_\ell(F \times F) = T_\ell F \times T_\ell F$, and this contains $T_\ell(F)$, So then $N_\ell(F/F \times F) = T_\ell F$, and projectivizing, we get ℓ.

The rest is an IOU

The map from the exceptional divisor to $\mathbb{P}^4$ is dominant onto $X \subset \mathbb{P}^4$ and has image contained in it.

A second application is teh irrationality of X: $\mathscr{J}^2(X)$ is not a Jacobian.

Proof: C a curve of genus five, then $\mathscr{J}(C) \supset \Theta \supset \Theta_{sing}$, which is a curve of double points. Recall Riemann's Singularity Theorem, which states that $\text{Pic}^{g-1}(C) \supset \Theta = \{L \in Pic^{g-1}(C) | h^0(L) > 0\}$, that is, the image of $AJ : Sym^{g-1}(C) \to Pic^{g-1}(C)$. So Θ_{sing} consists of $\{L | h^0(L) > 1\}$, and so $mult_L(\Theta) = h^0(L)$.

On a curve of genus 5, there is a one parameter family of such L's. In fact, on C of any genus, $\dim(\Theta_{sing}) = \begin{cases} g-3 & C \text{ hyperelliptic} \\ g-4 & \text{else} \end{cases}$.

In our case, there exists a 1-parameter family of line bundles of degree 4 on $C \in \mathscr{M}_5$.

In fact, for a curve $C \in \mathscr{M}_5$, $\Delta = \Theta_{sing}$ is a plane quintic, and we have $\tilde{\Delta} \to \Delta$, so $\mathscr{J}(C) = Prym(\tilde{\Delta}/\Delta)$.

We have a map $C \to \mathbb{P}^4$ of degree $2g-2 = 8$, and C is a complete intersection of 4 quadrics. So $\mathbb{P}^3$ =linear system of quadrics in $\mathbb{P}^4$ contains Δ, the singualr quadrics which is a quintic in $\mathbb{P}^2$. Looking at $\tilde{\Delta}$ is the ruling in these quadrics. Each ruling cuts out a g^1_4 on C, and induces an involution on $\Theta_{sing} : L \mapsto K_C \otimes L^{-1}$

29 Lecture 28

Question: Calculate $T_\ell F$ where $X \subset \mathbb{P}^4$ is a cubic three fold and $\ell \subset X$ is a line with F the Fano surface of lines in X.

This is the same as deformations of $\ell \in X = H^0(\ell, N_{\ell \subset X})$.

$$\begin{array}{ccccc} T\ell & \longrightarrow & TX|_\ell & \longrightarrow & N_{\ell/X} \\ \| & & \downarrow & & \downarrow \\ T\ell & \longrightarrow & T\mathbb{P}|_\ell & \longrightarrow & N_{\ell/\mathbb{P}} \\ & & \downarrow & & \downarrow \\ & & (N_{X/\mathbb{P}})|_\ell & = & (N_{X/\mathbb{P}})|_\ell \end{array}$$

Now we use the Euler sequence $0 \to \mathscr{O} \to \mathscr{O}(1)^5 \to T_{\mathbb{P}^4} \to 0$. Upon restriction to ℓ, we have $0 \to \mathscr{O} \to \mathscr{O}(1)^5 \to ?? \to 0$. What is the missing term? It has to be rank 4, and degree five, so it is $\mathscr{O}(a) \oplus \mathscr{O}(b) \oplus \mathscr{O}(c) \oplus \mathscr{O}(d)$ with $a+b+c+d = 5$. Now the long exact sequence gives us $0 \to 1 \to 10 \to ? \to 0$ so there should be 9 independent global sections. Now, how can we find four numbers which have h^0 summing to 9. We can see by computation that we cannot have any -2 or lower, so some of the possibilities are $(1,1,1,2), (0,1,1,3), (0,0,1,4), (0,0,0,5), (-1,1,1,4)$, etcetera.

We claim that is it $(1,1,1,2)$. We tensor with $\mathscr{O}(-1)$ in the ses, and get $0 \to \mathscr{O}(-1) \to \mathscr{O}^5 \to \mathscr{O}(a-1) \oplus \mathscr{O}(b-1) \oplus \mathscr{O}(c-1) \oplus \mathscr{O}(d-1) \to 0$. Taking long exact sequence and looking at dimensions of global sections, we get $0 \to 0 \to 5 \to h^0 \to 0$, so we need to get a 5 after decreasing all the numbers. We can now exclue the sequences with a -1. Tensoring with $\mathscr{O}(-1)$ again rules out anything involving a 0.

With hindsight, we can tensor with the sheaf $\mathscr{O}(-2)$. This gives us $0 \to \mathscr{O}(-2) \to \mathscr{O}(-1)^5 \to T \otimes \mathscr{O}(-2) \to 0$, which gives us $0 \to 0 \to 0 \to h^0 \to 1 \to 0$ in the long exact sequence. Thus, $h^0 = 0$ after tensoring with $\mathscr{O}(-2)$. So $1 = h^0(a-2) + \ldots + h^0(d-2)$, which implies that the largest number is two, and that it can only appear once. Thus, $a = b = c = 1$ and $d = 2$.

So the map $T\mathbb{P}|_\ell = \mathscr{O}(1)^3 \oplus \mathscr{O}(2) \to (N_{X/\mathbb{P}})|_\ell = \mathscr{O}(3)$ comes from $\mathbb{TP}_X \to$

$N_{X/\mathbb{P}}$. The map $\mathscr{O}(1)^5 \to N_{X/\mathbb{P}}$ is given by the five partial derivatives of the defining equation of X. So now we need to restrict this to the line.

On ℓ, then, we have $\mathscr{O}(1)^5 \to \mathscr{O}(3)$ down to $\mathscr{O}(1)^3 \oplus \mathscr{O}(2) \to \mathscr{O}(3)$.

Conclusion so far: $N_{\ell/X} = \ker(\mathscr{O}(1)^3 \to \mathscr{O}(3))$ given by 3 of the five partials X_i, namely the three transversal derivatives.

It is of rank two, so we have $0 \to \mathscr{O}(a) \oplus \mathscr{O}(b) \to \mathscr{O}(1)^3 \to \mathscr{O}(3) \to 0$, with $a + b = 0$. Taking the long exact sequence and dimensions, we have $0 \to h^0 \to 6 \to 4 \to h^1 \to$, now $h^0 = \begin{cases} a+1 & a > 0 \\ 2 & a = 0 \end{cases}$ and $h^1 = \begin{cases} a-1 & a > 0 \\ 0 & a = 0 \end{cases}$. So we have $a = 0, 1, 2, 3, 4, 5$ as possiblities.

We will use a similar trick to rule things out. We will tensor with $\mathscr{O}(-2)$. Then we have $0 \to \mathscr{O}(a-2) \oplus \mathscr{O}(-a-2) \to \mathscr{O}(-1)^3 \to \mathscr{O}(1) \to 0$, which gives us $\mathscr{O}(a-2) \oplus \mathscr{O}(-a-2)$ has no global sections. Thus $a - 2 < 0$, so $a < 2$, this gives $a = 0, 1$. So we have $0 \to 0 \to 0 \to 2 \to ? \to 0$, so $? = 2$. Thus $h^1(\mathscr{O}(a-2) \oplus \mathscr{O}(-a-2)) = 2$. If $a = 0$, then we have $h^1(\mathscr{O}(-2) \oplus \mathscr{O}(-2)) = 1 + 1 = 2$. If $a = 1$ we have $h^1(\mathscr{O}(-1) \oplus \mathscr{O}(-3)) = 0 + 2$, so both are still possibilities.

If $a = 1$, we get that $T_\ell F = H^0(\ell, N_{\ell/X}) = H^0(\ell, \mathscr{O}(1))$ which gives the coordinate ring of ℓ, as we wanted.

We have that $\ell \subset S \subset X$, and we know $N_{\ell/S} = \mathscr{O}(-1)$, and so we have $0 \to N_{\ell/S} \to N_{\ell/X} \to \mathscr{O}(1) \to 0$, that is, $0 \to \mathscr{O}(-1) \to N_{\ell/X} \to \mathscr{O}(1) \to 0$, but still, either $a = 0$ or $a = 1$ works. So we just need to determine if this ses splits. Still not done...

Example 29.1. *Let X be the zero set of $\sum x_i^3 = 0$. Look at $[t : -t : s : -s : 0]$, a line in here. We're stuck. Continued next time.*

30 Lecture 29

We have $\ell \subset S \subset X \subset \mathbb{P}^4$, and we can think of ℓ as a point of F, the Fano surface of X.

We are looking at the short exact sequence $0 \to N_{\ell/S} \to N_{\ell/X} \to N_{S/X} \to 0$, which is $0 \to \mathscr{O}(-1) \to N_{\ell/X} \to \mathscr{O}(1) \to 0$. So it is an extension. Now $\operatorname{Ext}^1(\mathscr{O}(1), \mathscr{O}(-1)) = H^1(\mathbb{P}^1, \mathscr{O}(-2)) = H^1(\mathbb{P}^1, \mathscr{O}) = \mathbb{C}$, so if the extension is 0, we have $\mathscr{O}(1) \oplus \mathscr{O}(-1)$, and otherwise it is $\mathscr{O}^2$.

We now look a $T_{\mathbb{P}^4/\ell} = \mathscr{O}(2) \oplus \mathscr{O}(1)^3$, which gives Euler sequence $0 \to \mathscr{O}(2) \to \mathscr{O}(2) \oplus \mathscr{O}(1)^3 \to \mathscr{O}(1)^3 \to 0$. Now we also have $0 \to N_{\ell/X} \to N_{\ell/\mathbb{P}^4} \to N_{X/\mathbb{P}^4} \to 0$, and the middle term we know is $\mathscr{O}(1)^3$, and the last is $\mathscr{O}(3)$, so we have $0 \to N_{\ell/X} \to \mathscr{O}(1)^3 \to \mathscr{O}(3) \to 0$.

Now, which of the two candidates is this? We will try to calculate $\hom(\mathscr{O}(1), N_{\ell/X})$. Whether or not is it zero tells us the answer. Note that the map $\mathscr{O}(1)^3 \to \mathscr{O}(3)$ is given by the partials X_i which are nonzero along ℓ.

So all we need to do is determine if there is a normal direction in $\mathbb{P}^4$ such that $X_i = 0$.

Example 30.1. *Let X be $ux^2+vy^2+u^3+v^3+w^3$ and ℓ given by $u=v=w=0$. Then X is a nonsingular cubic, with partials $ux, x^2+3u^2, vy, y^2+3v^2, w^2$. The partials x^2, y^2 are nonzero along the line, and so $\hom(\mathcal{O}(1), N) \neq 0$, and $N \cong \mathcal{O}(1) \oplus \mathcal{O}(-1)$, because the extension involved is zero.*

Example 30.2. *Define $X_\epsilon = X + \epsilon xyw$. Still contians ℓ, and is still nonsingular.*

So we originally wanted an identification $\mathbb{P}(T_\ell F)$ with ℓ.

Let $p \in \ell$, we send it to $T_pX \cap X$, a singular cubic surface with its singularity at p. Behavior of the "27" lines on a singular surface is that the conic through five points and the exceptional divisor at the sixth point coincide, so there are 15+6=21 distinct lines, 6 of which have multiplicity 2.

(Check it for 6 points lying on a conic)

Our set up: $F \cap Gr(2, T_p(X))$ is $2\ell + \ldots$. So it gives a well-defined tangent vector to F at ℓ.

Conversely, consider a curve $\ell \in C \subset F$. Each point $\ell' \in C$ determines a hyperplane $\langle \ell, \ell' \rangle$ (there might be a finite number where it isn't a hyperplane), so this gives us a map $C \to (\mathbb{P}^4)^*$ defined away from ℓ. This extends to ℓ, however. So we get $\mathbb{P}^3$ through ℓ such that ℓ counts twice in $F \cap Gr(2, \mathbb{P}^3)$.

So $\mathbb{P}^3 \cap X$ must be singular, so it is the tangent space at X for some $p \in X$, which is in fact in ℓ.

Back in context: We were looking at $\mathcal{AJ} : F \times F \to \mathcal{J}^2(X)$, which has image the Θ divisor.

We get two main results:

1. Irrationality of X
2. Torelli for X

Both follow from geometry of Θ. The irrationality follows by Θ having a unique singular point (the image of the diagonal), and if this were the Jacobian of a curve, we'd have a curve of singularities.

To get Torelli, we look at $\mathbb{P}(T_0\Theta) \subset \mathbb{P}(T_0\mathcal{J}^2(X))$, where 0 is the singular point, and so the tangent cone is a cubic, and in fact the X we started with.

Less-Canonical Version of Abel-Jacobi: $F \to \mathcal{J}^2(X)$, which depends on a choice of basepoint, but its derivative is intrinsic (the Gauss map $G : F \to Gr(2, T_0\mathcal{J}^2(X)) = Gr(2,5)$)

Claim: G is the tautological map.

This would imply that for $\mathcal{AJ} : F \times F \to \mathcal{J}^2(X)$ that its Gauss map $G : F \times F \to Gr(4,5)$ taking $(\ell, \ell') \mapsto span\langle \ell, \ell' \rangle$. Inside there is F, the diagonal, and we need to blow this up in order to have the map defined everywhere.

$\tilde{F} = \mathbb{P}(TF)$, the projectivization of the tangent bundle, and points of this are sent to tangent space to X at a point of ℓ.

The image of $\tilde{F}$ under the Gauss map is the set of tangent hyperplanes to X, and so the tangent cone to the image will in fact be X.

The only thing that still needs to be proved is that $G : F \to Gr(2,5)$ is the tautological map.

To see the identification of $\mathcal{AJ}$ with the Gauss map, we need to restrict attention to $\tilde{\Delta}$. $\ell \to \tilde{\Delta} = \{\ell' | \ell' \cap \ell \neq \emptyset\}$, and $\tilde{\Delta}$ maps down to $\Delta \subset \mathbb{P}^2$.

$\tilde{\Delta}$ sits naturally in $\tilde{F}$.

Next time: Prym Varieties and their Abel-Jacobi map.

31 Lecture 30

Prym Varieties

General situation: if $\tilde{C} \to C$ is a double cover, and there are r ramification points, then $g(C) = g$ and $g(\tilde{C}) = 2g - 1 + r/2$.

We define $Nm(\pi) : \mathscr{J}(\tilde{C}) \to \mathscr{J}(C)$ for $\pi^* : \mathscr{J}(C) \to \mathscr{J}(\tilde{C})$, then $Nm(\pi)(p_1 \pm \ldots \pm p_k) = \pi(p_1) \pm \ldots \pm \pi(p_k)$

In general, we define $\mathscr{O}(Nm(\pi)(D)) = \det(\pi_* \mathscr{O}(D)) \left(\frac{1}{2} Branch\right)$. How do we take half of the branch locus?

Well, we need a choice of $L = \sqrt{\mathscr{O}(B)}$. That is, a line bundle L along with an isomorphism $L^{\otimes 2} \to \mathscr{O}(B)$. This gives a double cover $\tilde{C} \to C$.

Take the inverse image under $\otimes^2$ of teh natural section of $\mathscr{O}(B)$, and we get this double cover, branched along B.

Exercise 31.1 (in Harshorne). *The converse is true. That is, the data of a double cover is equivalent to a choice of square root of the branch divisor.*

So $\pi_*(\mathscr{O}_{\tilde{C}}) = \mathscr{O}_C \oplus L^{-1}$.

(Explicitly, we need to see that if a divisor D on $\tilde{C}$ is linearly equivalent to zero, then so is $Nm(\pi)$ on C)

There exists f rational function on $\tilde{C}$ with $(f) = D$, we need a rational function $g = Nm(\pi)(f)$ on C such that $(Nm(f)) = \pi(D)$, Then $g(p) = f(p_1)f(p_2)$ where p_1, p_2 are the points in $\pi^{-1}(p)$.

Relations: $Nm(\pi) \circ \pi^* = 2\,\mathrm{id}$. $\pi^* \circ Nm(\pi) = 1 + i*$ where $i : \tilde{C} \to \tilde{C}$ is the involution over C.

$\ker^0 Nm(\pi) \subset \mathscr{J}(\tilde{C})$ maps down to $\operatorname{coker} \pi^*$. We have abelian varieis of dimension $g - 1 + r/2$. If $r = 0$, then ther kenreal has 2 components, for $r > 0$ it is connected.

Back to cubic threefolds:

$\tilde{\Delta}_\ell = \tilde{\Delta} = \{\ell' \in Fano(X) | \ell' \cap \ell \neq \emptyset\} \to \Delta$, a plane quintic. So we have $\mathscr{J}(\tilde{\Delta}) \to \mathscr{J}^2(X)$. But also $\pi^* : \mathscr{J}(\Delta) \to \mathscr{J}(\tilde{\Delta})$ which sends this locus to zero in the composition. So $\mathscr{J}^2(X)$ is the Prym variety.

Example 31.1. *If $g = 0$ and $r = 0$, then $\tilde{C} \to C$ is an unramified map of elliptic curves, so the Jacobians are the curves. thus, $Nm(\pi) = \pi : \tilde{C} \to C$ and $\pi^* : C \to \tilde{C}$ is the dual map, so $\tilde{C} \to C \to \tilde{C}$ is multiplication by 2, and $C \to \tilde{C} \to C$ is also multiplication by 2. Finally, the Prym variety is zero.*

Example 31.2. *$g = 0$, $r = 2b$, then $C = \mathbb{P}^1$ adn $\tilde{C}$ is a hyperelliptic curve of genus $b - 1$. Then $Prym = \mathscr{J}(\tilde{C})$.*

If $r = 0$, we can describe $Prym$ as a map $R_g \to A_{g-1}$, where A_{g-1} consists of abelian varieties of dimension $g-1$ and R_g is the moduli of pairs (C, L) where $C \in M_g$ and $L \in \mathscr{J}(C)[2]$.

Consequence:

$$\begin{array}{ccc} M_g & \xhookrightarrow{AJ} & A_g \\ {\scriptstyle Prym}\uparrow & & \\ R_g & \xrightarrow{Prym} & A_{g-1} \end{array}$$

Prym has positive dimensional fibers for $g \leq 5$.

g	$\dim R_g$	$\dim A_{g-1}$
1	1	0
2	3	1
3	6	3
4	9	6
5	12	10
6	15	15
7	18	21

It is, in fact, never injective. (Though it is generically injective for g large enough)

For $g = 6$, we have $\deg(Prym) = 27$, and $Gal(R_6/A_5) = W(E_6)$, the symmetry group of the lines on a cubic surface.

$g = 5$: fiber of $Prym$ is a surface. In fact, it is a souble cover of a Fano surface of lines in a cubic threefold.

A_4 is the moduli space of abelian varieties of dimension 4, which is isomorphic to the moduli of cubic 3folds X and a point of order 2 in $\mathscr{J}^2(X)$.

Last time: we looked at $F \times F \to \mathscr{J}^2(X)$, $F \to \mathscr{J}^2(X)$ projectivized.

We have $TF \to F$ the tangent bundle, which is a rank two vector bundle, and we projectivize it, which gives the Gauss map $G : \mathbb{P}(TF) \to \mathbb{P}^4$.

We used the fact that the image of $\mathbb{P}(T_\ell F)$ is $\ell \subset X \subset \mathbb{P}^4$.

Choose any $\ell_0 \in \tilde{\Delta}_\ell$, and consider $\tilde{\Delta}_{\ell_0} \subset F(X)$, and a point b on it.

Then $T_\ell(\tilde{\Delta}_{\ell_0}) \subset T_\ell(F(X))$, and so we may as well describe $\mathbb{P}(T(\tilde{\Delta}_{\ell_0}))$, which is a curve in the threefold $\mathbb{P}(TF)$. It is isomorphic to $\tilde{\Delta}_{\ell_0}$.

So we have a map $\tilde{\Delta}_{\ell_0} \to \mathbb{P}^4$.

This is the Abel-Prym Map!

Abel-Jacobi gives $C \to \mathbb{P}(H^0(\omega_C))^*$, and we get a map $\mathscr{J}(C) \to \mathbb{P}(H^0(\omega_C))^*$.

The Abel-Prym map, then goes $\tilde{C} \to \mathscr{J}(\tilde{C}) \to Prym(\tilde{C}/C) \to \mathbb{P}(H^0(C, \omega_C \otimes L)^*)$.

So note: $\deg(L) = 0$ (unramified case) and so $\deg(\omega_C \otimes L) = \deg \omega_C = 2g-2$, and so $h^0(\omega_C) = 2g-2-g+1+h^0(0)$ and $h^0(\Omega_c \otimes L) = 2g-2-g+1+h^0(L)$. Now $h^0(0) = 1$ and $h^0(L) = 0$, so $h^0(\omega_C) = g$ and $h^0(\omega_C \otimes L) = g-1$, and so the Abel-Jacobi map goes to $\mathbb{P}^{g-1}$ but Abel-Prym goes to $\mathbb{P}^{g-2}$.

32 Lecture 31

We now look at $X \subset \mathbb{P}^5$ a cubic fourfold. It has hodge diamond

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 1 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
? & & ? & & ? & & ? & & ? \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 1 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}$$

So now we need to determine the middle row. From last semester, we have $H^{40} \sim R^{-3} = 0$, $H^{31} \sim R^0$ has dimension 1, $H^{22}_0 \sim R^3$, and so $h^{22} = 21$, because $\mathscr{J}^3 \to S^3 \to R^3$ consists of a 36 dimensional space into a 56 dimensional space, and so R^3 is 20 dimensional, but this is the primitive Hodge number, so we add one. Thus the Hodge diamond is

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 1 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
0 & & 1 & & 21 & & 1 & & 0 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 1 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}$$

The classes in $H^{p,p}$ for $p \neq 2$ are clearly algebraic, so we only need to look at $H^{2,2}$. (Why is it clear? H^{00} is the whole thing, H^{11} generated by a hyperplane, H^{22} we need to do work, H^{44} is generated by a point, and H^{33} is generated by the class of a line, which we know exist because it is a cubic)

Let Σ be a cuibc surface, and let $Y = Bl_\Sigma X$, and $Y_t = \pi^{-1}(t)$.

$$\begin{array}{ccc}
Y_t & \longrightarrow & t \\
\downarrow & & \downarrow \\
Y & \xrightarrow{\pi} & \mathbb{P}^1 \\
\downarrow{\scriptstyle \tau} & & \\
X & &
\end{array}$$

Then choose $\alpha \in H^{2,2}(X, \mathbb{Z})$. We want to show that α is algebraic.

We define $\alpha_t = (\tau^*\alpha)|_{Y_t} = \beta[\ell]$ for $\beta \in \mathbb{Z}$.

Now, set $\alpha' = \tau^*\alpha - \beta[\ell \times \mathbb{P}^1]$ This is a class on Y, and $\alpha'|_{Y_t} = 0$. (Note that the exceptional divisor E is a product $\Sigma \times \mathbb{P}^1$, so the inverse image of a line $\ell \subset \Sigma$ is $\ell \times \mathbb{P}^1$.

So we can restrict to worrying about the classes α' being algebraic.

Consider the intermediate Jacobian fibration $\mathscr{J} = \mathscr{J}(Y/\mathbb{P}^1) \to \mathbb{P}^1$ which has fibers intermediate Jacobians. We do need to be careful, however. As we have $\mathscr{J}(Y_t) = H^3(Y_t)/F^2H^3 + H^3(Y_t,\mathbb{Z})$, we define $\mathscr{J}$ to be $R^3\pi_*\mathbb{C}$ modulo $R^3\pi_*(0 \to 0 \to \Omega^2 \to \Omega^3) + R^3\pi_*\mathbb{Z}$.

We can also describe it as $\ker(H^4_{Delignian}(Y_t,\mathbb{Z}[2]) \to H^4(Y_t,\mathbb{Z}))$.

If $K = \mathscr{O}_{Y_t} \to \Omega^1_{Y_i}$, then we have $0 \to K \to \mathbb{Z}_D[2] \to \mathbb{Z} \to 0$, this gives $H^3(Y,\mathbb{Z}) \to \mathbb{H}^3(Y,K) \to H^4_D(Y,\mathbb{Z}[2]) \to H^4(4,\mathbb{Z})$.

So the plan is to take α and get α', which then gives us a section ν_α of $\mathscr{J} \to \mathbb{P}^1$, a "normal function", which then gives us a cycle by Jacobi inversion.

We pick a topological cycle A representing α'. Then $A|_{Y_t}$ is $\partial\Gamma_t$ for some 3-chain $\Gamma_t \subset Y_t$. Then $\int_{\Gamma_t} \in \mathscr{J}(Y_t)$, and this gives the normal function.

So we now have an element of $H^0(\mathbb{P}^1,\mathscr{J}) = H^4_D(Y,\mathbb{Z}[2])$. This lifts α.

So we now must convert a section ν of $\mathscr{J}$ to a cycle.

Analogue for a surface: $Z \to \mathbb{P}^1$ a surface and Z_t the fibers. Then $\mathscr{J} = \mathrm{Pic}^0(Z/\mathbb{P}^1) \to \mathbb{P}^1$ and ν a section. Jacobi inversion tells us that $Sym^g(Z/\mathbb{P}^1) \to \mathscr{J}$ is a surjective map over $\mathbb{P}^1$.

So a section ν lifts to a section of $Sym^g(Z/\mathbb{P}^1)$ which IS a curve in Z (of degree g over $\mathbb{P}^1$)

In our case, we need a family of cycles in $Y/\mathbb{P}^1$ which dominate $\mathscr{J}/\mathbb{P}^1$.

Rationally, this is easy. We have $F \times F \to \Theta \subset \mathscr{J}$, and $F \times F \times F \to \mathscr{J}$ surjective, but we want it over $\mathbb{Z}$.

Theorem 32.1 (Markushevich, Tikhmirov,...). *Let M_t be the moduli space of semistable rank 2 torsion free sheaves on Y_t with $c_1 = 0$ and $c_2 = 2\ell$. Then M_t is birational to $\mathscr{J}(Y_t)$ (its Abel-Jacobi map, in fact, is a birational morphism).*

The Abel-Jacobi map for a moduli space of bundles is given by looking ath $\mathscr{U} \to M_t \times Y_t$ the universal bundle, and $L >> 0$ a surfficiently ample line bundle on Y_t. Then look at $\pi_*(\mathscr{U} \otimes L)$ where $\pi : M_t \times Y_t \to M_t$ is the projection. It is a sheaf on M_t, and for $L >> 0$, it is a vector bundle, V. We look at $\mathbb{P}(V)$, a projective bundle over M_t. Now, $\mathbb{P}(V)$ parameterizes a family of codimension 2 cycles in Y_t. So we have $\mathcal{AJ} : \mathbb{P}(V) \to \mathscr{J}(Y_t)$, which factors through M_t, since the fibers are rational.

Thus, Jacobi inversion works over $\mathbb{Z}$.

33 Lecture 32

We will be looking at X with $c_1(X) = 0$. There exists a classification:

There exists a finite cover $Y \to X$ with $Y = Z \times \prod S_i \times \prod C_j$ where $Z = \mathbb{C}^n/\mathbb{Z}^{2n}$ is a complex torus, S_i is holomorphic and symplectic, irreducible and $h^0(S_i,\Omega^2) = 1$.

(Any closed two-form α on a manifold X gives $i_\alpha : TX \to T^*X$, and α is symplectic if it is an isomorphism. Equivalently, α is everywhere of maximal rank.)

Also, C_j is Calabi-Yau, that is, $c_1(C_j) = 0$, $h^1(C_i, \mathscr{O}) = h^2(C_j, \mathscr{O}) = 0$. (A CY is a manifold X such that $K_X = \mathscr{O}$)

In dimension 1, the only possibilities are elliptic curves

In dimension 2, we have complex tori and K3 Surfaces (like quartics in $\mathbb{P}^3$)

In dimension 3, we can have a complex torus, an elliptic curves cross a K3 modulo a finite group action, or a Calabi-Yau threefold.

The classification continues, and we will choose to stop at 3 (in physics, sometimes 4 dimensional ones come up)

Complex tori and K3-surfaces are classical objects, so we're just going to focus on studying Calabi-Yau 3folds.

The Hodge diamond of a CY3 is
$$\begin{matrix} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{11} & & 0 & \\ 1 & & h^{21} & & h^{21} & & 1 \\ & 0 & & h^{11} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{matrix}$$
. So there are two parameters, h^{11}, h^{21}.

Note that as $h^{20} = 0$ we have $H^{11} = H^2$ is generated by $\mathbb{Z}$-classes, and so all classes in H^2 are algebraic.

Now $H^{21} = H^1(C, \Omega^2)$. As we have a nonvanishing 3-form, this gives us an isomorphism $T \to \Omega^2$, so we have $H^{21} = H^1(C, T_C)$, the set of formal deformations of C.

For Calabi-Yau's, we have

Theorem 33.1 (Bohomolov-Tian-Toderov). *Deformations are unobstructed.*

That is, for all CY, C, there exists a moduli space $\mathscr{M}$ and a family $\mathscr{C} \to \mathscr{M}$ (proper, flat, smooth) and a point $0 \in \mathscr{M}$ with C the fiber over 0, such that the Kodaira-Spencer map $T_0\mathscr{M} \to H^1(C, T_C)$ is an isomorphism.

So, topological, there are two invariants with very different behaviors.

Conjecture 33.1 (Mirror Symmetry version 1). *For any CY3, there exists C' another CY3, such that $h^{11}(C) = h^{21}(C')$ and $h^{21}(C) = h^{11}(C')$.*

Example 33.1 (Projective Hypersurfaces). *Let $H \subset \mathbb{P}^n$ be a hypersurface. Then adjuction says that $K_X = 0$ if and only if $\deg X = n + 1$. So for $n = 4$, we need quintic threefolds.*

In this case, $h^{11} = 1$ and $h^{21} = \dim R^5 = \binom{9}{4} - 5 \times 5 = 126 - 25 = 101$.

Example 33.2 (Complete Intersection of k Hypersurfaces). *Look in $\mathbb{P}^{3+k}$, and take the hypersurfaces of degree d_i with $\sum d_i = 4 + k$.*

For $k = 1$, we get the previous. For $k = 2$, we have $H \cap H' \subset \mathbb{P}^5$, we get degree 6=5+1=4+2=3+3, the new cases are 4+2 and 3+3. For $k = 3$, we get

7, which partitions into 3+2+2. For $k = 4$, we get 2+2+2+2=8, and that's it, it can't be done anymore without 1s.

So this gives us 5 new classes of examples.

Example 33.3 (Weighted Projective Spaces)**.** *Consider complex intersections in a weighted projective space. We get more Calabi-Yau's here, in fact, hundreds of them.*

Better: we replace $\mathbb{P}^3$ by a toric variety, that is, $\mathbb{C}^{n+k}/(\mathbb{C}^*)^k$. Now we have thousands or tens of thousands of topological types of Calabi-Yaus

This set of examples is known to be closed under mirror symmetry.

Example 33.4 (Elliptic Fibrations)**.** *Choose a base B and a line bundle $\mathscr{L}$ on B, look at $\mathbb{P}(L^2 \oplus L^3 \oplus \mathscr{O})$, which is a $\mathbb{P}^2$ bundle, write the Weierstrass equation here, and we get an Elliptic Fibrartion over B. To make sure this is CY, then $\mathscr{L} = -K_B$. Mostly in the case where B is a Del-Pezzo surface (that is, Fano). But also, where B is the blow up at 9 points, a rational elliptic surface. So this X is $B \times_{\mathbb{P}^1} B'$ for two rational elliptic surfaces.*

34 Lecture 33

A Calabi-Yau is a variety with $K_X \cong \mathscr{O}_X$, and $h^1(\mathscr{O}_X) = h^2(\mathscr{O}_X) = 0$ (really, only the first, but we can factor the variety by the theorem to get down to one satisfying the latter)

We looked at the Hodge diamond of a CY3, and gave the first approximation of mirror symmetry

Conjecture 34.1 (Mirror Symmetry, version two)**.** $\mathscr{M}_{Complex}(X) \cong \mathscr{M}_{Kahler}(X)$ *naturally*

That is, the moduli space of complex structures on X is naturally isomorphic ot the moduli space of Kähler structures on X.

By Bogomolov-Tian-Todorov unobstructedness, $T_X\mathscr{M}_{Complex}(X) = H^1(T_X) = H^1(\Omega^2_X) = H^{21}$.

$\mathscr{M}_{Kahler}$ is the set of complexified Kähler classes (B-fields added), that is $B + i\omega$ with ω a Kähler form and $B \in H^2(X,\mathbb{R})/H^2(X,\mathbb{Z})$.

We will discuss two techniques: degnerations and Quotients

35 Quotients

Proposition 35.1. *Let G be a finite group acting freely on CY3 X. Then X/G is a CY3.*

Proof. $H^{pq}(X/G) = H^{pq}(X)^G$.

Thus, $0 \leq h^{pq}(X/G) \leq h^{pq}(X)$, and so $h^{00}(X/G) = 1$, $h^{10}(X/G) = 0 = h^{20}(X/G)$, and $h^{30}(X/G) = 1$, though this takes a bit of work: Look at $\chi(\mathscr{O}_{X/G})$, as we have an etale cover, this is $\chi(\mathscr{O}_X)/|G| = 0$, and so we must have $h^{30} = 1$. □

Example 35.1 (Fermat Quintic). *Let X be the Fermat quintic, that is $\sum_{i=0}^{4} x_i^5 = 0$. What are the automorphisms? Well, there are the S_5 permutations, and there's also $(\mathbb{Z}/5)^5/(\mathbb{Z}/5)$. Take a cyclic permutation $(x_0, \ldots, x_4) \mapsto (x_1, \ldots, x_4, x_0)$. If ρ is a fifth root of unity, then $(1, \rho, \rho^2, \rho^3, \rho^4)$ are the fixed points in projective space, but none of these points lie on X, as if you plug them in, you get 5, not zero.*

There's also $(x_0, \ldots, x_4) \mapsto (\rho x_0, \ldots, x_4)$, which has as fixed points $(1, 0, 0, 0, 0)$ and $(0, x, y, z, w)$. So this is not fixed point free. But take $G = \ker((\mathbb{Z}/5)^5/\mathbb{Z}/5 \to \mathbb{Z}/5)$ then the fixed points are $(1, 1, 1, 1, 1)$. So we want the fixed points to be isolated.

So there are various possibilities, like $\mathbb{Z}/5 \times \mathbb{Z}/5$ with the first factor a cyclic permutation and the second multiplication by $(1, \rho, \rho^2, \rho^3, \rho^4)$. This acts freely on X (and also $\sum x_i^5 + \psi \pi x_i$). So we look at the quotient.

$h^{11} = 1$ and $h^{21} = 5$, because the original 101 came from $1 + 4 \times 25$ in terms of representations, and there is one invariant in each.

So X/G has Hodge numbers $(1, 5)$. This is one of the smallest known Calabi-Yau's.

Question from physics: There are three generations of particles, and a CY gives absolute value of $h^{11} - h^{21}$ generations, which is $\frac{1}{2}\chi(X)$, so we need one that has three. This ones says 4.

This was used in a paper by Candelas, (other authors), Witten around 1985 as the basic example of how to compactify string theory.

Example 35.2. *Take $G = (\mathbb{Z}/5)^3$, that is, the kernel from above. This doesn't act freely. Thus, the quotient is singular, and $h^{11}(X) = 1$ and $h^{21}(X) = 101$, but now $h^{11}(X/G) = 1$ and $h^{21}(X/G) = 1$. These need to be carefully defined using the fact that X/G is an orbifold rather than a manifold.*

The number of singular points is the number of orbits of fixed points, which is 100. (we'll do this next time)

The resolution has $h^{11} = 101$ and $h^{21} = 1$, and this is the mirror of the quintic.

36 Lecture 34

1. Our first example is the Quintic Threefold (1,101)

2. We looked at a quintic containing a place, which is singular at 16 points, we find a small resolution and get a new CY3 which is (2,86)

Local Models (Locally in the ANALYTIC topology)

Let X_t be a family of 3folds and X_0 has an ordinary double point: $\sum_{i=1}^{4} x_i^2 = t$. Then we have a diagram

$$\begin{array}{ccccc} & & \hat{X}_0 & & \\ & \nearrow & & \searrow & \\ \hat{\hat{X}}_0 & & & & X_0 \\ & \searrow & & \nearrow & \\ & & \hat{X}_0' & & \end{array}$$

these are isomorphisms away from the inverse image of 0.

Note, in $\mathbb{C}^4$ with coordinates x_i, we have two places $x_1 = ix_2$, $x_3 = ix_4$, π, and $x_1 = ix_2$, $x_3 = -ix_4$, π'. Blow up π and π' in $\mathbb{C}^4$, and take the proper transform of X_0. Get $\hat{X}$, $\hat{X}'$. The effect on cohomology is that $\hat{X}_0, \hat{X}_0'$ is $\chi + 1$ and $\hat{\hat{X}}_0$ is $\chi + 3$.

In a global model, if one exists, (Paper by Bob Friedman ?) then $\hat{X}_0 \to X_0$ a small resolution of n ordinary double points. Then $\chi(\hat{X}_0) = n + \chi(X_0)$.

Options: h^{11} goes up by n, because we are creating algebraic cycles, but similarly, h^{21} goes down by n, and a third possibility is some combination.

1. Quintic with isolated triple point $(1, 0, 0, 0, 0) \in \mathbb{P}^4$, by $x_0^2 \subset_3 (x_1, x_2, x_3, x_4) + x_0 F_4 + \ldots + G_5$. We resolve by blowing up the point. Replace 0 by a non-singular cubic surface. Exercise: This is a crepand reoltuion, and it is a CY3 with (2,90)

2. Let X be singular quintic along $\mathbb{P}^1$ given by $x_0 = x_1 = x_2 = 0$, Then $X \in (x_0, x_1, x_2)^2$. Blow up the line to get $\hat{X} \to X$. The exceptional divisor which maps to $\mathbb{P}^1$ will be a conic bundle over $\mathbb{P}^1$. Each $c_i = c_i(x_0, x_1, x_2)$ is a conic, the fiber. We have a map $\mathbb{P}^1_{(x_3,x_4)} \to \mathbb{P}^5 = \mathbb{P}(\mathrm{Sym}^2\,\mathbb{C}^3) =$ conics in $\mathbb{P}^2$, and it's a twisted cubic.

Wilson's results on the Kahler cone: The set of Kahler classes is an open convex cone in $H^{1,1}_{\mathbb{R}}$. It is polyhedral away from the topological intersection divisor: $\omega^3 = 0$. The faces have three types of faces, and each face correponds to a contraction of a curve to a point, surface to a point or surface to a curve.

37 Lecture 35

Bogomolov decomposition theroem: $c_1(X) = 0$ implies that $Y \to X$ can be written $Z \times \prod S_i \times \prod C_j$

Examples of CYs

Constructing techniques - Quotients with free actions and resolutions of quotients by nonfree actions.

Examples: Degenerations of types I, II, and III

Shape of the Kähler Cone (results mostly due to P. Wilson)

Let X be a projective CY3

(NOTE: Here K_X is the Kähler cone, not the canonical divisor)

$K_X \subset \bar{K}_X \subset H^{1,1}_{\mathbb{R}}(X) = H^2_{\mathbb{R}}() \supset W = \{D \in H^2 | D^3 = 0\}$.

Now, $W \subset U$ open implies that in the complement of U, $\bar{K}_X$ is a rational polyhedral cone, $\sigma \subset \bar{K}_X$ a codimension 1 dace (or facet) and $\sigma \not\subset W$, implies that there exists a primitive contraction $f : X \to Y$ with Y normal and f a birational map such that $span(\sigma) = f^*(\operatorname{Pic} Y) \otimes \mathbb{R}$.

Theorem 37.1 (Classification of Primitive Birational Contraction)**.**

1. *Finite number of curves collapse to points implies that the irreducible components are $\mathbb{P}^1$. (Example: 16 lines to 16 points)*
2. *Irreducible surface$\to$ point, implies that the surface is a del Pezzo, possibly singular (Example: Cubic surface)*
3. *Surface to a nonsingular curve with conic fibers*

Behavior of a facet (iff contraction) under deformations of complex structure:
Type I - Deforms to type 1
Type II deforms to type II
Type III deforms to type III if $g = 0$ and for $g > 1$,can deform to type I or III. The contraction can disappear if $E \to C$ as a fibration has either all smooth fibers or all nodal fibers.

Corollary 37.2. *K_X is deformation invariant unless X contains a surface $E \to C$ as above.*

Let S be a monoid (associative with identity) with the finite partition property (that is, given any element z there are finitly many (x, y) such that $x + y = z$.) This gives a formal monoid ring (with coefficients in R) of $R[[q; S]] = \{\sum_{\eta \in S} a_\eta q^\eta | a_\eta \in R\}$.

Example 37.1. $S = (\mathbb{Z}_{\geq 0})^n \Rightarrow R[[q_1, \ldots, q_n]]$.

Example 37.2. *S is integral Mori monad, which is $N\bar{E}F(X, \mathbb{Z})$.*

NEF is NOT Numerically Effective, it is $Nef(X, \mathbb{Z}) = \{\eta \in H_2(X, \mathbb{Z}) | \omega \cdot \eta \geq 0, \forall \omega \in \bar{K}_X\}$.

Yukawa Coupling: input is three classes D_1, D_2, D_3 in H^2, and have $\langle D_1, D_2, D_3 \rangle = D_1 D_2 D_3 + \sum_{\eta \in H_2(X,\mathbb{Z}) \setminus 0} \sum_{m=1}^{\infty} q^{m\eta} \phi_\eta(D_1, D_2, D_3)$ where ϕ_η is the Gromov-Witten Invariant (which count rational curves $\mathbb{P}^1 \to X$ of class η meeting D_1, D_2, D_3)

Imaigne that each D_i is the class of an actual divisor (which we will identify with D_i). We want $\phi_\eta(D_1, D_2, D_3)$ =the number of maps $u : \mathbb{P}^1 \to X$ such that $u(0) \in D_1$, $u(1) \in D_2$ and $u(\infty) \in D_3$, that is, roughly, the numebr of points in the quotient $\mathscr{M}(\eta, \mathbb{P}^1)/\mathbb{P}GL(2)$ times $(D_1 \cdot \eta)(D_2 \cdot \eta)(D_3 \cdot \eta)$.

Options:

1. The best option is to be lucky, and the above expression makes sense and is finite
2. If you aren't lucky, then you can work hard - $\mathscr{J}$-holomorphic curves which is essentially analytic and nonalgebraic.

3. An algebraic method due to Barbara Fentechi, Seibert and others is to interpret $\mathscr{M}(\eta, \mathbb{P}^1)$ as an algebraic stack and it will be zero dimensional as a stack, and then count the "points", that is, we integrate a virtual fundamental class, and that gives the number.

38 Lecture 36

Last time we defined the $(1, 1)$ Yukawa coupleing on $H^{1,1}$, $W + \sum_{\eta \neq 0}$, with the instanton corrections counting rational curves of class $\eta \in H_2(X, \mathbb{Z})$ meeting D_1, D_2, D_3, with $D_1 \cdot D_2 \cdot D_3$ the classical intersection. This is taking place on the "A-side"

Next: "B-model", or (1,2)-Yukawa coupling on $H^{1,2}$.

VHS (Variation of Hodge Structure) is $(\mathscr{H}_{\mathbb{Z}}, \nabla, S, F)$ where S is a complex manifold (the base), $\mathscr{H}_{\mathbb{Z}}$ is a local system of abelian groups (generally taken to be free, $\mathbb{Z}^{\ell}$), ∇ is a flat holomorphic connection on $\mathscr{H} = \mathscr{H}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathscr{O}_S$, $\mathscr{H}_{\mathbb{Z}}$ flat with respect to ∇ and F induces a Hodge structure on each fiber, that is, $\nabla : \mathscr{H} \to \mathscr{H} \otimes \Omega^1_S$ restricts to $F^p \to F^{p-1} \otimes \Omega^1_S$. This is "Griffiths Transversality".

A smooth proper map $\mathscr{H} \to S$ with X_s the fiber over $s \in S$ with projective fibers (or at least Kähler) then we have VHS for all $n \in \mathbb{Z}$ given by $\mathscr{H}_{\mathbb{Z}} = R^n \pi_* \mathbb{Z}$.

F^* is the family of Hodge filtrations on the fibers.

$G^T : \nabla_S : H^n \to H^n$ is given by $\alpha \mapsto \frac{\partial \alpha}{\partial s}$ where $\alpha(s)$ is a local section of $\mathscr{H}$, and by differentiating, we lose at most one holomorphic differential.

Combine this with the Kodaira-Spencer map, the natural connection factors through KS, and so we have $H^1(T_X) \times F^{p+1} \to F^p$. We have a map down to $H^1(T_X) \times F^p \to F^{p-1}$, which maps to $H^1(T_X) \times H^{p,n-p} \to H^{p-1,n-p+1}$ by the cup product.

Now we assume that X is a CYn and $\mathscr{X} \to S$ is a family. Then Ω_s is a holomorphic n form on X_s, and $\Omega_s \neq 0$, so Ω is unique up to $f \in \Gamma(S, \mathscr{O}_S^*)$. It induces $T_X \xrightarrow{\sim} \Omega_X^{n-1}$.

This induces an isomorphism $H^1(T_X) \xrightarrow{\sim} H^1(\Omega_X^{n-1})$ from deformations of X to $H^{n-1,1}$.

So now we get the $(1, n-1)$-Yukawa coupling $H^{n-1,1} \times H^{p,n-p} \to H^{p-1,n+1-p}$.

Example 38.1. *For $n = 3$, we have $H^{2,1} \times H^{3,0} \to H^{2,1}$, $H^{2,1} \times H^{2,1} \to H^{1,2}$, $H^{2,1} \times H^{1,2} \to H^{0,3}$. A vector $v \in H^{2,1}$ gives three maps, and each map depends linearly on v.*

Their composition $H^{30} \to H^{03}$ depends cubically on v, we se have a cubic map $H^{2,1} \to \hom(H^{3,0}, H^{0,3}) \cong \mathbb{C}$. The upshot is that we get a cubic polynomial on H^{21} (or it is an element of $\otimes^3 H^{12}$, and it is an exercise to check that it is in $\operatorname{Sym}^3 H^{12}$)

Situation so far:

	classical	*quantum*
$A - model$	*intersection*	$+\sum_\eta gromov - witten$
$B - model$	$VHS, Yukawa$	0

Conjecture 38.1 (Mirror Symmetry)**.** *Let $f : \mathscr{X} \to (\Delta^*)^s$ (the product of s punctured discs) a family of Calabi-Yau threefolds with* 0 *a maximally unipotent degeneration (think as singular as possible, called a "large complex structure" by phyicists) then there exists a "mirror" Calabi-Yau threefold $\check{X}$ and a choice of framing $\Sigma \subset \bar{K}_{\check{X}}$ (ie, choose s linearly independent Kähler classes and consider the simplex they space, this corresponds to canonical coordiantes $q_1, \ldots, q_s$ on $(\Delta^*)^s$)*

The e_i's give coordinates $\check{q}_i$ on $\mathscr{M}_{Kah,\Sigma}(\check{X})$ iff $m : (\Delta^*)^s \to \mathscr{M}_{Kah,\Sigma}(\check{X})$ is a mirrir map such that

$$\langle \frac{\partial}{\partial q_i}, \frac{\partial}{\partial q_j}, \frac{\partial}{\partial q_k} \rangle_p = \langle \frac{\partial}{\partial \check{q}_i}, \frac{\partial}{\partial \check{q}_j}, \frac{\partial}{\partial \check{q}_k} \rangle_{m(p)}$$

39 Lecture 37

Today we're going to talk about Borcea-Voisin Calabi-Yaus and Mirror Symmetry

Let $Y = Y_d$ be a d-dimensional compact Kähler manifold and $X_d^\sigma \subset Y_d$ a smooth divisor in class $2c_1(Y)$. ($c_1(Y)$ is $c_1(T_Y)$.)

Then there exists $X_d \to Y$ is a double cover branched over X_d^σ where X_d is nonsingular and $c_1(X_d) = 0$.

Example 39.1. *If $d = 1$, we have $Y = \mathbb{P}^1$. Then X_d^σ is some set of $2k$ points. Then $X \to Y$ branched at these $2k$ points is a curve of genus $k-1$. To have $K_X = 0$, we need $k = 2$, because $\deg K_X = 2g - 2 = 4(k-1)$. So X is CY iff the branche locus is in $2c_1(\mathbb{P}^1)$.*

We call any Calabi-Yau defined this way to be a Calabi-Yau with Involution.

Nikulun's Classification of K3 surfaces with involution:

Let X be a K3, and $L = H^2(X, \mathbb{Z})$. This is $U^3 \oplus (-E8)^2$ where U is a 2 dimensional lattice with bilinear form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ where $-E8$ is the weight lattice for the Lie group of type E8 (the - is a matter of taste, choosing the negative definite part). Define $L^\sigma = \{\ell \in L | \sigma(\ell) = \ell\}$.

Deformation classes of K3s with σ correspond to pairs (L, L^σ). Classication: (r, a, δ) where $r = rank(L^\sigma)$, $L^\sigma \subset (L^\sigma)^* \subset (L^\sigma) \otimes \mathbb{Q}$. Now $(L^\sigma)^*/L^\sigma \cong (\mathbb{Z}/2\mathbb{Z})^a$. (the reason is that $2(L^\sigma)^* \subset L^\sigma$), because $\pi : X \to Y$ gives $\pi_*\pi^* = 2$)

$$\delta = \left\{ \begin{cases} 0 & \text{if } \forall x^* \in (L^\sigma)^*, (x^*)^2 \in \mathbb{Z} \\ 1 & \text{else} \end{cases} \right. = \left\{ \begin{cases} 0 & \text{if } [X_d^\sigma] \text{ is even} \\ 1 & \text{else} \end{cases} \right.$$

This does give us a complete classification of K3's with involution.

GET PICTURE FROM RON

Reflection through the middle line is mirror symmetry for K3's.

Example 39.2. $(10, 10, 0)$ *has the property that $X^\sigma = \emptyset$. This is called the Enriques involution, it is the only example of a fixed-point free involution on a K3. The quotient by σ is called an Enriques Surface, and all 11 dimensions of them arise in this way.*

Example 39.3. $(10, 8, 0)$, *$X^\sigma = E_1 \cup E_2$ where E_i are elliptic curves.*

These are the exceptional cases. In all other cases, $X^\sigma = C_g \cup \bigcup_{i=1}^k E_i$ where this is a disjoint union, C_g is genus g, and E_i are rational.

We have $(g, k) \leftrightarrow (r, a)$ with $g = (22 - r - a)/2$ and $k = (r - a)/2$.

Then Mirror Symmetry is $(r, a, \delta) \leftrightarrow (20 - r, a, \delta)$ and $g \leftrightarrow k + 1$.

Now, let $(X_m, \sigma_m), (X_n, \sigma_n)$ be two CY's with involution. Thne we can construct (X_{m+n}, σ_{m+n}), a resolution of $X_m \times X_n/(\sigma_m, \sigma_n)$. Namely, $Y_{n+m} \to Y_n \times Y_m$ containing $X^\sigma_{n+m} \to (X^\sigma_n \times X^\sigma_m)$, where the first is the proper transform of $X^\sigma_n \times Y_m \cup Y_n \times X^\sigma_m$.

Application: $n = 2, m = 1$ takes a K3 with inv and an elltiptic curve with inv, and you get the BV 3folds.

Then we have $h^{11} = 5 + 3r - 2a = 1 + r + 4(k + 1)$ (1 class from the elliptic curve, r from the K3, and $4(k + 1)$ from the resoltuion)

$h^{21} = 65 - 3r - 2a = 1 + (20 - \gamma) + 4g$ (1 modulus for the elliptic curve, $20 - r$ complex def in the K3, and $4g$ moduli for the curve in the K3)

40 Lecture 38

We've discussed BV CY3's.

Classification (Nikulin) of K3's with involution as (r, a, δ).

A BJ is K3×Elliptic/$\{1, -1\}$.

Now, $L = H^2(X, \mathbb{Z}) = U^3 \oplus (-E8)^2$ with $r = L^\sigma$, a is $(L^\sigma)^*/L^\sigma \cong (\mathbb{Z}/2)^a$ and $\delta \in \{0, 1\}$.

Then $h^{11} = 5 + 3r - 2a$ and $h^{21} = 65 - 3r - 2a$.

Example 40.1. $(10, 8, 0)$ *has* $(19, 19)$ *Hodge numbers. In fact, these are the fiber products of rational elliptic surfaces.*

$RES \to \mathbb{P}^1$ in general 12 I_1. Specialize to $2 \times I_0^*$. We need to resolve the singularities in order to get a nonsingular space (see earlier)

So for BV's, the mirror should be $(20 - r, a, \delta)$.

For K3's, we have lattice polarizing K3's:

Let $M \subset L$ be a fixed sublattice

Example 40.2. *$rankM = 1$, then $= \mathbb{Z}$ and has generator squaring to $2d$ corresponding to the moduli of K3's of degree d*

Consider the moduli space of lattice polarized K3's. $\{(S, \alpha) | S$ a K3, $\alpha : M \to Pic(S) = H^{11}(S, \mathbb{Z})\}$.

Define T to be the transcendental lattice (the lattice of nonalgebraic classes), then set $D_M = \{[\Omega] \in \mathbb{P}(T \otimes_{\mathbb{Z}} \mathbb{C}) | \Omega^2 = 0, \Omega\bar{\Omega} > 0\}$.

Then D_M is the period space of the M-polarized K3's.

Choose an embedding $U \oplus M \to L$ and let $\check{M}$ be $(U \oplus M)^\perp_L$. Then $rk(\check{M}) = 20 - rk(M)$ and we get $U \oplus \check{M} \to L$.

The Complexified Kähler Cone is $T_\mu = \{B + i\omega \in M \otimes_{\mathbb{Z}} \mathbb{C} | \omega^2 > 0\}$.

Lemma 40.1. *$\phi : T_{\check{M}} \to D_M$ is an isomorphism, the mirror map. It depends on a generator E of U, such that $U = E \oplus E'$.*

Proof. $\phi(\check{B} + i\check{\omega}) = \check{B} + E' + \frac{\check{\omega}^2 - \check{B}^2}{2} E + i(\check{\omega} - (\check{\omega}\check{B})E)$. □

Context: SYZ (Strominger-Yau-Zader) Conjecture: Explicit construction of the mirror.

$T^n \to X_n \to S^n$ where π is a fibration with fibers "Special lagrangian" tori in X.

Structure on X: ω is Kähler in H^{11} and Ω is holo volume in H^{n0}. So $Z \subset X$ is Lagrangian if $\dim_{\mathbb{R}} Z = n$ and $\omega|_Z \equiv 0$. It is special lagrangian if $Im\Omega|_Z \equiv 0$ iff $Re\Omega|_Z = vol|_Z$.

Maclean: deformation theory of smooth special lagrangians is unobstructed, so deformations are $H^1(Z, \mathbb{R})$.

$Z \simeq T^n$ impliues that $h^1(Z, \mathbb{R}) = n$.

SYZ: Whenever a mirror exists, there is a SLAG fibration $\pi : X \to S^n$ and then the mirror $\check{X}$ "is" the dual SLAG fibration.

The case $n = 2$ has $X = K3$. So a SLAG fibration IS an elliptic fibration. After a 90 degree roation in twistor space, we have $\omega \in H^{11}$, $\Omega \in H^{20}$ and $\bar{\Omega} \in H^{02}$ so $(X, \omega) \to S^2$ becomes $(X, \Omega) \to \mathbb{CP}^1$ with a section. The first is SLAG, the second is holo.

So an elliptic K3 with section corresponds to a K3 with U-structure.

Back to BV's:

There is a short list of exceptions to this process: lattice L^σ which cannot be embedded in $U^2 \oplus (-E8)^2$.

Gross-Wilson: For BV's, we can fully justify the SYZ prediction: there exists a recipe for completing the singular fibers so $\chi(\check{X}) = -\chi(X)$.