1 Lecture 1

Algebra of the Complex Numbers

Topology of Complex Numbers.

Basics.

Definition of continuous.

Definition 1.1 (Holomorphic)**.** *A function* $f : \mathbb{C} \to \mathbb{C}$ *is holomorphic at* $z \in \mathbb{C}$ *if* $\lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$ *exists. We denote it by* $f'(z)$*.*

f has holomorphic derivative at z_0 iff $f(z_0 + h) = f(z_0) + ah + h\psi(h)$ where $\lim_{h\to 0} \psi(h) = 0$.

The usual rules of differentiability: linearity, leibniz, quotient, and chain all hold.

Cauchy Riemnan Equations $u_y = -v_x$ and $u_x = v_y$.

2 Lecture 2

We are discussing equivalent notions of "holomorphic."

1. Holomorphic: f is a $\mathbb{C}$-valued function defined on $\Omega \subseteq \mathbb{C}$ such that $\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = f'(z)$ exists for all $z \in \Omega$.
2. Cauchy-Riemann Equations: assume that u and v have continuous partials, then $u_x = v_y$ and $u_y = -v_x$. That is, $\partial_{\bar{z}}(u + iv) = 0$.
3. Conformal map: f preserves angles infinitesimally. This is true if the Jacobian of f as a map $\mathbb{R}^2 \to \mathbb{R}^2$ is a constant times a rotation matrix. This is equivalent to C-R.

A power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then if $1/R = \limsup_{n\to\infty} |a_n|^{1/n}$, then the power series converges absolutely and uniformly on any disc $D_r(0)$ with $r < R$ and it diverges at any z with $|z| > R$.

Proving basic theorems about power series.

A function with a convergent complex power series $\sum a_n(z - z_0)^n$ is called an analytic function.

Analytic implies Holomorphic in the disc of convergence.

If ϕ is C^∞, then $|\phi - \sum_{j+k\leq N} a_{jk} z^j \bar{z}^k|$ is $O(|z|^{N+1})$. This is generally not convergent. If $\partial_{\bar{z}} f = 0$, then $a_{jk} = 0$ for $k \geq 1$.

For an analytic function, we have $a_{jk} = \frac{\partial_z^j \partial_{\bar{z}}^k \phi}{j!k!}(0)$.

Proposition 2.1. *If* f *is analytic in* $D_r(0)$*, then the power series for* f *about* $z_0 \in D_R(0)$ *converges in* $D_{R-|z_0|}(z_0)$*.*

Basic definitions for contour integration.

Theorem 2.2. *If* $D \subset \mathbb{C}$ *is a bounded region with piecewise smooth boundary and* ω *is a smooth* C^1 *1-form, then* $\oint_{\partial D} f dz = \int_D d(f dz) = \int_D \partial_{\bar{z}} f d\bar{z} \wedge dz$*.*

So the easy form of Cauchy's TEhorem is that if f is holomorphic on Ω and $D \subset\subset \Omega$, is bounded with piecewise C^1 boundary and $f \in C^1(\Omega)$, then $\int_{bD} f dz = 0$.

Theorem 2.3 (Goursat)**.** *Let f be holomorphic in a domain Ω and T a triangle uch that D_T, the region with $bD_T = T$ contained in Ω. Then $\int_T f(z)dz = 0$.*

See Stein-Shakarchi for all this technical babble.

3 Lecture 3

Theorem 3.1. *If f is holomorphic in a disk $D_R(z_0)$, then f has a primitive in $D_R(z_0)$.*

Corollary 3.2. *If f is holomorphic on $D_R(z_0)$ then $\int_\gamma f dz = 0$ for any closed piecewise C^1 curve $\gamma \subset D_R(z_0)$.*

Theorem 3.3. *Assume that f is holomrphic on a neighborhood of D_R where bD_R is a rectangle along with a single point $z_0 \in D_R$ and there exist an $A > 0$ and an $0 \leq \alpha < 1$ such that $|f(Z)| \leq A/|z-z_0|^\alpha$ near to z_0. Then $\int_R f(z)dz = 0$.*

Derive the Cauchy integral formula

Now that we know that f has continuous derivatives, we can apply Stoke's Theorem.

Holomorphic implies C^∞.

Theorem 3.4 (Cauchy Estimates)**.** *If f is holomorphic in the disc of radius r centered at z_0 and continuous up to the boundary with $|f(z)| \leq M$ for $|z - z_0| = r$, then $|f^{(n)}(z_0)| \leq Mn!/r^n$.*

Theorem 3.5. *If f is holomorphic in a set containing $D_R(z_0)$ then the Taylor series of f about z_0 gonverges on $D_R(z_0)$ is $f(z) = \sum_{n=0}^\infty \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$.*

That is, holomorphic iff analytic.

4 Lecture 4

We will look at $\partial_{\bar{z}} f = \varphi$. $f(z) = \frac{1}{2\pi i}\int_{bD_r(z_0)} \frac{f(w)dw}{w-z}$ when f is holomorphic on $D_r(z_0)$ and continuous up to the boundary. Sometimes we will assume that f is holomorphic on $\overline{D_r(z_0)}$, that is, on some open set containing this.

Definition 4.1 (Entire)**.** *A function holomorphic at every point of $\mathbb{C}$ is called entire.*

Theorem 4.1 (Liouville's Theorem)**.** *A bounded entire function is constant.*

Proof. $f'(z) = \frac{1}{2\pi i}\int_{|w|=R} \frac{f(w)dw}{(w-z)^2}$. $|f(w)| \leq M$ for all $w \in \mathbb{C}$. Then $|f'(z)| \leq \frac{1}{2\pi}\int_{|w|=R} \frac{|f(w)||dw|}{|w-z|^2}$. As $|w - z| \geq |w| - |z|$, we have $\leq \frac{1}{2\pi}\int_{|w|=R} \frac{M|dw|}{(R-|z|)^2} \leq \frac{MR}{(R-|z|)^2}$. Taking $R \to \infty$, $f'(z) = 0$. ☐

Corollary 4.2 (Fundamental Theorem of Algebra). *Let $p(z)$ be a polynomial $\sum_{j=0}^{n} a_j z^j$ with $a_n \neq 0$ and $n > 0$. Then there exists $z_0 \in \mathbb{C}$ where $p(z_0) = 0$.*

Proof. Assume not, then $\frac{1}{p(z)}$ is entire. $|p(z)| \geq |a_n z^n| - \sum_{j=0}^{n-1} |a_j z^j| = a_n |z|^n (1 - \sum_{j=0}^{n-1} \frac{|a_j|}{|a_n|} \frac{1}{|z|^{n-j}}$. Thus, there exists R such that $|p(z)| \geq \frac{|a_n z|^n}{2}$ for $|z| \geq R$. Thus, $1/p(z)$ is bounded, and so constant by Liouville. Thus, p was constant, which is acontradiction. □

To get the fact that p has n roots, let z_1 be a root of p. Then $p = \sum_{j=1}^{n} p^{[j]}(z_1)/j!(z - z_1)^j$. We can write this as $(z - z_1) \sum_{j=0}^{n-1} p^{[j+1]}(z_1)/(j+1)!(z - z_1)^j$. Repeating, we get this fact.

Suppose that f_n is a sequence of C^1 functions. Then we need to know that f'_n is uniformly convergent to know that $\lim f_n$ is C^1.

Theorem 4.3. *If f_n is a sequence of holomorphic functions on $\Omega \subset \mathbb{C}$ which is locally uniformly convergent to f, then f is holomorphic.*

Lemma 4.4. *If φ is a continuous function on $bD_r(z_0)$ then $F(w) = \frac{1}{2\pi i} \int_{|w - z_0| = r} \frac{\varphi(w)}{w - z} dw$ is holomrophic in $\mathbb{C} \setminus bD_r(z_0)$.*

We now prove the theorem

Proof. If $K \subset\subset \Omega$ then $f_n \to F$ uniformly on K. $\|f_n - f\|_{L^\infty(K)} \to 0$.

If $z \in \Omega$, then there exists $r > 0$ such that $D_r(z) \subset\subset \Omega$. For $\zeta \in D_r(z)$, we can write $f_n(\zeta) = \frac{1}{2\pi i} \int_{|w-z|=r} \frac{f_n(w)dw}{w - \zeta}$ for all n. Thus $f(\zeta) = \frac{1}{2\pi i} \int_{|w-z|=r} \frac{f(w)dw}{w-\zeta}$.

This same argument on the derivatives proves more. □

Theorem 4.5 (Bonel's Lemma). *Given a sequence of numbers $\{a_j\}$, there exists a C^∞ function ϕ such that $\phi^{[j]}(0) = a_j$ for all j.*

However, for holomorphic functions, we have bounds given by the Cauchy Inequalities.

Lemma 4.6. *If f is holomorphic and does not vanish in some neighborhood of z_0, then there exists k such that for $j < k$ we have $f^{[j]}(z_0) = 0$ and $f^{[k]}(z_0) \neq 0$.*

Theorem 4.7. *If f is holomorphic in Ω, then $Z_f = \{z \in \Omega | f(z) = 0\}$ is discrete.*

Proof. Let $z_n \subset Z_f$ have a limit point in Ω, call it z^*. Then $f(z^*) = 0$... □

Corollary 4.8. *If f, g are holomorphic in Ω a connected domain and $f(z_n) = g(z_n)$ on a sequence of dinstinct points with a limit in Ω, then $f \equiv g$.*

Lemma 4.9. *If f is a holomorphic function on connected Ω such that $f(z) = 0$ for $z \in U \subseteq \Omega$ an opn set, then $f \equiv 0$.*

Stated Morera's Theorem

5 Lecture 5

Let f_n be holomorphic in Ω and $f_n \to f$ locally uniformly. Then $\int_T f(z)dz = \lim_{n\to\infty} \int_T f_n(z)dz = 0$ when T is a triangle which is contained in Ω. Thus, Morera's Theorem implies f holomorphic.

A complex measure $d\mu$ is a measure fof the form $d\mu_1 + id\mu_2$ for $d\mu_1, d\mu_2$ real measures. Define $|d\mu_j| = d\mu_j^+ + d\mu_k^-$, $d\mu_j^+ \perp d\mu_j^-$, and it is of dinite total variation inf $\int_{\mathbb{C}} |d\mu| < \infty$.

Theorem 5.1. *Let $d\mu$ be a complex measure of finite total variation and compact support on $\mathbb{C}$. $\int_{\mathbb{C}} f d\mu = \int_K f d\mu$ for all $f \in C^0(\mathbb{C})$ and K is some compact subset of $\mathbb{C}$. We define $f(z) = \iint_K \frac{d\mu(w)}{w-z}$ is defined for $z \in \mathbb{C} \setminus K$.*

Then f is holomorphic in $\mathbb{C} \setminus K$.

Proof. This is a local statement, so choose a point $z \in \mathbb{C} \setminus K$ and $r > 0$ such that $\overline{D_r(0)} \subset \mathbb{C} \setminus K$.

If T is a triangle containd in $D_r(z)$, then $\int_T f(z)dz = \int_T \left(\iint_K \frac{d\mu(w)}{w-z}\right) dz = \int_T \iint_K \frac{|d\mu(w)|}{|w-z|}|dz| < \frac{|\mu|(K)|T|}{d} < \infty$.

So we can interchange the order of integration. Thus, $\iint_K \left(\int_T \frac{dz}{w-z}\right) d\mu(z)$ for $z \in D_R(z_0)$, we have $z \mapsto \frac{1}{w-z_0}$ is holomorphic, so this integral is zero. Thus, Morera's Theorem shows that $f(z)$ is analytic in $\mathbb{C} \setminus K$. □

If $d\mu$ is $\phi(w)dw$ for some $\phi \in C^0(\gamma)$ along a curve γ, then $\int \psi d\mu = \int_\gamma \psi\phi dw$ and $|d\mu| = |\phi| ds_\gamma$.

So $\int_\gamma \frac{\phi(w)dw}{w-z}$ is holomorphic in $\mathbb{C} \setminus \gamma$.

If $\phi \in C_c^\infty(\mathbb{C})$, then $f(z,\bar{z}) = \frac{1}{2\pi i} \iint \frac{\phi(w,\bar{w})dw\wedge d\bar{w}}{w-z}$ becomes $\frac{-1}{\pi} \iint \frac{\phi(a-t,b-s)dsdt}{t+is}$, where $z = a+ib$.

The key is that $\frac{1}{r}$ is an integrable singularity in the plane.

f has the same number of continuous derivatives as ϕ.

Look at $\partial_{\bar{z}} f = \phi$.

$\frac{f(a+h,b)-f(a,b)}{h} = \frac{1}{\pi} \iint \frac{\phi(a+h-t,b-s)-\phi(a-t,b-s)}{h(s+it)} dsdt$, so then $\lim_{h\to 0} \frac{1}{\pi} \iint \frac{\partial_x(\phi(a-t,b-s))dsdt}{s+it}$, and so $\partial_a^j \partial_b^k f(a,b) = \frac{1}{\pi} \iint \frac{\partial_x^j \partial_y^k \phi(a-t,b-s)dsdt}{s+it}$.

So then $\partial_{\bar{z}} f = \iint \frac{\partial_{\bar{z}}\phi(a-t,b-s)dsdt}{s+it} = \frac{1}{2\pi i} \iint \frac{\partial_{\bar{z}}\phi(w+z)d\bar{w}\wedge dw}{w} = \frac{1}{2\pi i} \iint \frac{\partial_{\bar{w}}\phi(w+z)d\bar{w}\wedge dw}{w} = -\lim_{\epsilon\to 0} \frac{1}{2\pi i} \iint_{|w|>\epsilon} d\left(\frac{\phi(w+z)dw}{w}\right) = -\lim_{\epsilon\to 0} \frac{1}{2\pi i} \oint_{|w|=\epsilon} \frac{\phi(w+z)dw}{w}$ in the clockwise direction, taking the counterclockwise, we obtain $\lim_{\epsilon\to 0} \frac{1}{2\pi i} \int_0^{2\pi} \frac{\phi(z+\epsilon e^{i\theta})\epsilon i e^{o\theta} d\theta}{\epsilon e^{i\theta}} = \phi(z)$.

Thus $d\frac{1}{2\pi i}\frac{dw}{w} = \delta_0$.

Theorem 5.2. *If $\phi \in C_c^1(\mathbb{C})$, then the equation $\partial_{\bar{z}} = \phi$ has a solution $f \in C^1(\mathbb{C})$ given by $f = \frac{1}{2\pi i} \iint \frac{\phi d\bar{w}\wedge dw}{w-z}$.*

Theorem 5.3 (Cauchy-Pompiece Formula)**.** *If ϕ is a C^1 function on $\overline{D_r(z_0)}$, then*

$$\phi(z,\bar{z}) = \frac{1}{2\pi i}\left(\int_{bD_r(z_0)} \frac{\phi dw}{w-z} + \iint_{D_r(z_0)} \frac{\partial_{\bar{z}}\phi dw \wedge d\bar{w}}{w-z}\right)$$

Let $f = u + iv$, then $u_x = v_y$, $u_y = -v_x$, $v_{yy} + u_{xx} = 0$, and $u_{xx} + u_{yy} = 0$. Thus, $\Delta u = u_{xx} + u_{yy} = 0$ so u and v will be harmonic functions.

Now let u such that $\Delta u = 0$.

Define $dv = v_x dx + v_y dy$ and $d^*u = -u_y dx + u_x dy$, and so $dv = d^*u$. $d(adx + bdy) = (b_x - a_y)dx \wedge dy$, and so we need d^*u to be d-closed, then $d(d^*u) = (u_{xx} + u_{yy})dx \wedge dy$, and so $\Delta u = 0$ if and only if $d(d^*u) = 0$.

I can define $v(x,y) = \int_{(x_0,y_0)}^{(x,y)} d^*u$. However, we might have to integrate over a noncontractible curve, for instance $D_1(0)$ for $\frac{xdy - ydx}{x^2+y^2}$. This integral will be 2π.

If u is harmonic in $\Omega \subset \mathbb{C}$ simply connected and $\Delta u = 0$, then there exists a function v defined on Ω such that $dv = d^*u$, which is equivalent to $u + iv$ is holomorphic.

If we have two such solutions, then $d(v_1 - v_2) = 0$, and so v is determined up to a constant.

v is called the conjugate harmonic function.

If u is harmonic on $D_R(0)$ and continuous to $bD_R(0)$, then $u(z,\bar{z}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-|z|^2)u(Re^{i\theta},Re^{-i\theta})d\theta}{|Re^{i\theta}-z|^2}$ for $z \in D_r(0)$.

$$\frac{1}{2}\left(\frac{Re^{i\theta}+z}{Re^{i\theta}-z} + \frac{Re^{i\theta}+\bar{z}}{Re^{i\theta}-\bar{z}}\right) = \frac{R^2 - |z|^2}{|Re^{i\theta}-z|^2}$$

$\Delta = 4\partial_z\partial_{\bar{z}} = 4\partial_{\bar{z}}\partial_z$ is why we care about this representation of the kernel.

Looking at $\int u(e^{i\theta})P(e^{i\theta},z)d\theta$, we see

1. $P(e^{i\theta},z) > 0$
2. $\int_0^{2\pi} P(e^{i\theta},z)d\theta = 1$
3. $\forall \delta > 0$, $\epsilon > 0$, there exists $r_{\delta,\epsilon}$ such that $\int_{|e^{i\theta}-e^{i\phi}|>\delta} P(e^{i\theta},re^{i\phi})d\theta < \epsilon$ if $r > r_{\delta,\epsilon}$.

$\lim_{r\to R} u(re^{i\theta},re^{-i\theta}) = \phi(Re^{i\theta},Re^{-\theta})$. Then $u(z,\bar{z}) = \int \phi P(e^{i\theta},z)d\theta$.

$|u(re^{i\phi},re^{-i\phi}) - \phi(e^{i\phi})| = |int(\phi(e^{i\theta}) - \phi(e^{i\phi}))P(e^{i\theta},re^{i\phi})d\theta| \leq \int|\phi(e^{i\theta}) - \phi(e^{i\phi})|P(e^{i\theta},re^{i\phi})d\theta$. Since ϕ is continuous on the boundary, there exists δ such that $|\phi(e^{i\theta}) - \phi(e^{i\phi})| < \epsilon$ if $|e^{i\theta} - e^{i\phi}| < \delta$, because it is a continuous function on a compact set, and thus uniformly continuous.

So $\int_{|e^{i\theta}-e^{i\phi}|<\delta} |\varphi(e^{i\theta}) - \varphi(e^{i\phi}|P(e^{i\theta},re^{i\phi})d\theta + \int_{|e^{i\theta}-e^{i\phi}|>\delta} |\varphi(e^{i\theta}) - \varphi(e^{i\phi})|P(e^{i\theta},re^{i\phi})d\theta \leq \epsilon + \int_{|e^{i\theta}-e^{i\phi}|} 2MP(e^{i\theta},re^{i\phi})d\theta$. Now, there exists $r_{\delta,\epsilon}$ such that if $r > r_{\delta,\epsilon}$, then the second term is less than ϵ, and so $|u(re^{i\phi}) - \phi(e^{i\phi})| < 2\epsilon$.

This is in fact uniform in the boundary point.

So $u(z,\bar{z}) = \frac{1}{2\pi}\int_0^{2\pi} u(z + re^{i\theta}, \bar{z} + re^{-i\theta})d\theta$. This is the mean value property of harmonic functions.

6 Lecture 6

Theorem 6.1. *Let $F(z,s)$ be defined in $\Omega \times [0,1]$ with $\Omega \subset \mathbb{C}$ such that F is continuous and F is holomorphic in z.*

Then $f(z) = \int_0^1 F(z,s)ds$ is holomorphic in Ω.

Proof. By Morera's Tehorem, it suffices to show that $\int_T f(z)dz = 0$ for all triangles $T \subset \Omega$ such that $bD_T = T$ where $\bar{D}_T \subset \Omega$. Since F is continuous in $\Omega \times [0,1]$, its continuous on $T \times [0,1]$. Since this is a compact subset of $\Omega \times [0,1]$, $|F(z,s)| \leq M$ on $T \times [0,1]$.

$\int_T f(z)dz = \int_0^1 \int_C F(z,s)dzds = \int_0^1 0ds = 0$. □

Look at $\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt$. Then define $\Gamma_n(z) = \int_{1/n}^n e^{-t}t^{z-1}dt$. Then the theorem applies to show that $\Gamma_n(z)$ is an entire function.

Claim: $\Gamma_n(t)$ is locally uniformly convergent in $Re(z) > 0$ to $\Gamma(z)$. $|t^z| = t^{Re(z)}$. If $n > m$, then $\Gamma_n(z) - \Gamma_m(z) = \int_{1/n}^{1/m} e^{-t}t^{z-1}dt + \int_m^n e^{-t}t^{z-1}dt$.

$|\Gamma_n(z) - \Gamma_m(z)| \leq \int_{1/n}^{1/m} e^{-t}t^{Rez-1}dt + \int_m^n e^{-t}t^{Rez-1}dt \leq \int_0^{1/m} t^{\delta-1}dt + \int_m^\infty e^{-t}t^{M-1}dt \leq \frac{1}{\delta}\left(\frac{1}{m}\right)^\delta + 2e^{-m}m^M$.

So then given $\epsilon > 0$, there exists m_0 with $m, n \geq m_0$ implying that $|\Gamma_n(z) - \Gamma_m(z)| < \epsilon$ on $\delta \leq Rez \leq M$.

Now, if f is holomorphic on $\bar{D}_r(z_0)$, then Cauchy'sformula implies that $f(z_0) = \frac{1}{2\pi}\int_0^\pi f(z_0 + re^{i\theta})d\theta$.

$|f(z_0)| = \frac{1}{2\pi}\left|\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta\right| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{i\theta})d\theta$.

Frequently we need that $f(z_0 + re^{i\theta}) = g_r(\theta)e^{i\phi_r}$. Suppose that $|f(z)|$ assumes a local maximum at z_0. Then $|f(z_0 + re^{i\theta})| \leq |f(z_0)| = M$ if $r < \rho$.

$M \leq \frac{1}{2\pi}\int_0^{2\pi} Md\theta$. This can only happen if $g_r(\theta) = M$ for $r < \rho$. Then $f(z_0 + re^{i\theta}) = Me^{i\phi}$ for $r < \rho$.

Theorem 6.2 (Maximum Modulus Principle)**.** *If f is holomorphic in Ω a connected open set and $|f(z)|$ has an interior local maximum, then f is constant on Ω. If $D \subset\subset \Omega$, then $|f(z)|$ assumes its maximum on bD and if it assumes this value in D, then f is constant.*

6.1 Harmonic Functions

A C^2 function u defined on $\Omega \subset \mathbb{C}$ is harmonic if $\Delta u = 0$. If u is harmonic on $D_r(0)$, then for $z \in D_r(0)$, $u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\frac{r^2-|z|^2}{|re^{i\theta}-z|^2}d\theta$. A consequence of $\Delta u = 0$ is the mean value property for harmonic functions, $\Delta(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})d\theta$.

There exists v such that $u + iv$ is holomorphic and therefor $(u + iv)(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta}) + iv(z_0 + re^{i\theta})d\theta$. Taking real parts, we get the result.

Theorem 6.3 (Maximum Principle)**.** *If u is harmonic in Ω a connected open set, then u has no interior local maximums or minimums unless u is constant.*

Proof. Suppose that $u(x, y)$ has a local maximum. $u(x, y) \leq u(x_0, y_0)$ on a small disc around (x_0, y_0) in Ω. MVP implies that $u(x, y) = u(x_0, y_0)$ in this disc. The set where $u(x, y) = \max_{(x,y)\in\Omega} u(x, y)$ is both open and closed. Either $u(x, y) < \max_\Omega u$ or it is constant.

The same argument for minimum. □

Uniqueness of the solution to Dirichlet Problem:

Dirichlet Problem: Let $\Omega \subset \mathbb{C}$ be a connected bounded set with piecewise C^1-boundary. Let $\phi \in C^0(b\Omega)$. Find a harmonic function $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ such that $u|_{b\Omega} = \phi$.

Let $\mathscr{F}_\phi = \{u \in C^2(\Omega) \cap C^0(\bar{\Omega}) | u|_{b\omega} = \phi\}$. Define $D(u) = \int_\Omega |\nabla u|^2$. Then the solution to Dirichlet's problem is the function u_0 such that $D(u_0) = \min_{u\in\mathscr{F}_\phi} D(u)$.

$\int \nabla u_0 \nabla \psi = 0$ for all $\psi \in C_c^\infty(\Omega)$. Thus, $-\int \Delta u_0 \psi = 0 = \Delta u_0 = 0$.

Problems: Why should minimum exist? And there exists ϕ such that all the Dirichlet integrals are infinite.

Uniqueness for the Dirichlet Problem: Let u be harmonic on Ω and continuous on $\bar{\Omega}$ with $u|_{b\Omega} = 0$. Then $u \equiv 0$. The maximum principle forces it.

Theorem 6.4 (Schwarz Reflection Principle)**.** *Let f be holomorphic on $D_r^+(0)$. Assume that f is continuous down to the real axis and that the imaginary part of f along the real line is zero. Then letting $F(z) = \begin{cases} f(z) & Im(z) \geq 0 \\ \bar{f}(\bar{z}) & z \in D_r^-(0) \end{cases}$ defines an analytic continuation of f to $D_r(0)$.*

Proof. F is certainly continuous in $D_r(0)$ and $\sum_{j=0}^\infty a_j(\bar{z} - z_0)^j = \sum_{j=0}^\infty \bar{a}_j(z - \bar{z}_0)^j$. So now we take $\int_T F(z)dz = 0$ for all $T \subset D_r(0)$. Thus, Morera implies that F is holo. □

But we actually get a stronger statement if we look at what the hypotheses really are.

1. $Re(f)$ is continuous up to $D_r(0) \cap \mathbb{R}$.
2. $Im(f)$ is cont and vanishes along $D_r(0) \cap \mathbb{R}$.

The first is unnecessary.

In fact, the theorem remains true without 1, and there is a reflection prinicple for harmonic functions:

Theorem 6.5. *:et v bne a real valued harmonic function on $D_r^+(0)$ with $\lim_{y\to 0^+} u(x, y) = 0$. Then define $V(x, y) = \begin{cases} u(x, y) & y \geq 0 \\ -u(x, -y) & y < 0 \end{cases}$ then V is harmonic in $D_r(0)$, in particular, it is smooth.*

Let $\phi(x, y)$ be this function. then define $V(x, y) = \frac{1}{2\pi} \int_0^{2\pi} \phi(re^{i\theta}) P(e^{i\theta}, (x, y)) d\theta$.

Look at $\int \frac{v(re^{i\theta})(r^2-|z|^2)d\theta}{|z-re^{i\theta}|^2}$. Now we note that $P(e^{i\theta},x) = P(e^{-i\theta},x)$, and so $\phi(e^{i\theta}) = -\phi(e^{-i\theta})$, so $V(x,0) = 0$.

And so V is harmonic on $D_r(0)$, and $V|_{bD_r^+(0)} = v|_{bD_r^+(0)}$. By uniqueness of the solution to the dirichlet problem, we have that $V = v$ for all of $D_r^+(0)$.

Let $-U$ be the conjugate function to V. Then $dU = -d^*V$. So then $du = -d^*v$ on $D_r^+(0)$ and $d(U-u) = 0$ on $D_r^+(0)$.

We can normalize such that $U = u$ on $D_r^+(0)$ and then $U+iV$ is holomorphic. And so we have Schwarz without the condition on the real part.

Theorem 6.6 (Runge)**.** *Suppose that f is holomorphic in $D_R(0)$. Then for all $r < R$ and $\epsilon > 0$ there exists a polynomial $p(z)$ such that $\|f-p\|_{L^\infty(D_r(0))} < \epsilon$.*

This version of Runge's Theorem is an immediate corollary of Taylor's Theorem.

7 Lecture 7

Look at $\partial_{\bar{z}} u = \phi$ for any $\phi \in C^\infty(\mathbb{R}^2)$. If $\phi \in C_c^\infty(\mathbb{R}^2)$ then $u = \frac{1}{2\pi i}\int \frac{\phi(w)d\bar{w}\wedge dw}{w-z}$.

The method is called the oil spot method. Choose a sequence of functions $\psi_j \in C_c^\infty(\mathbb{R}^2)$ such that $\psi_j(z)$ is one for $|z| \le j+1/2$ and zero for $|z| > j+2$.

Let u_1 be the solution above to $\partial_{\bar{z}} u_1 = \psi_1\phi$. Let $\tilde{u}_2$ solve $\partial_{\bar{z}}\tilde{u}_2 = \psi_2\phi$.

Then $\partial_{\bar{z}}(\tilde{u}_2 - u_1) = 0$ on $D_{j+1/2}(0)$. So now I can choose a holomorphic polynomial $p_1(z)$ such that $\|\tilde{u}_2 - u_1 - p_1\|_{L^\infty} < 1/2^2$. Let $u_2 = \tilde{u}_2 - p_1$. Then u_2 solves the equation as well. Define higher versions similarly.

It is obvious that $\lim_{n\to\infty} u_n(z) = u(z)$ exists for all $z \in \mathbb{C}$. But it is really uniform. $\sum_{n=1}^\infty |u_{n+1} - u_n| < \infty$.

So now $u(z)$ is a continuous function. Fix an R and choose $n_0 > R$. Then $u_n(z) - u_{n_0}(z) + u_{n_0}(z) = u_n(z)$. Take the first difference and call that $h_n(z)$. Then $h_n(z) \to h(z)$.

$\partial_{\bar{z}} u_{n_0} = \phi_m$ for $|z| < R$ and $n > n_0$. So for $n > n_0$, $h_n(z)$ is holomorphic in $|z| < R$.

So $h_n(z)$ is a uniformly convergent sequence of holomorphic functions of m in $|z| < R$ with limit $h(z)$. In $|z| < R-1$, $h_n \to h$ in the C^∞ topology. $\|\partial_z^j \partial_{\bar{z}}^k (h_n - h)\|_{L^\infty} \to 0$ as $n \to \infty$ for all j,k.

Hence, $u_n \to u$ locally uniformly in the C^∞ topology, and therefore $\partial_{\bar{z}} u = \phi$ as well on all of $\mathbb{C}$.

This is related to $H^1(\mathbb{C}, \mathscr{O}) = 0$.

That was a consequence of the most trivial version of the Runge Theorem.

Theorem 7.1 (Runge)**.** *Let $K \subset \mathbb{C}$ be a compact subset and suppose that f is holomorphic on U an open set containing K. For any $\epsilon > 0$, there exists a function F_ϵ of the form $F_\epsilon(z) = \sum_{j=0}^N \sum_{\ell=0}^m \frac{a_{j,\ell}}{(z-z_j)}^\ell + p(z)$ where $\{z_j\} \in K^C$, $\|f - F_\epsilon\|_{L^\infty} < \epsilon$ and if K^C is connected, then F_ϵ can be taken to be a polynomial.*

Proof. Cover $\mathbb{C}$ with a grid of side δ where $\delta \leq \frac{1}{\sqrt{2}} d(K, U^c)$. Take $\{Q_j\}$ to eb the list of squares that intersect K Then $Q_j \subset\subset U$ for all j, by the choice of δ. Choose a point $z \in K$ and consider $\sum_{j=1}^{N} \frac{1}{2\pi i} \int_{bQ_j} \frac{f(w)}{w-z} dw = f(z)$ for $z \notin \cup bQ_j$.

$f(z) = \sum_{j=1}^{M} \frac{1}{2\pi i} \int_{\gamma_j} \frac{f(w)}{w-z} dw$ for all $z \in K$. $\gamma_j \cap K$ is empty for all j. $\inf_j dist(\gamma_j, K) = d > 0$.

Bacause $\int |w - z| \geq d$, $|f(z) - \sum_{j=1}^{n} \frac{1}{2\pi i} \frac{f(w_j)}{(w_j - z)} m_j| < \epsilon$ for $z \in K$.

So then $w \mapsto f(w)/(w-z)$ is uniformly equicontinuous as a family of funtions parameterized by z and uniformly bounded. Thus, we can find a single partition of $\cup \gamma_j$ such that this gives a uniformly accurate approximation to the integrals for all $z \in K$.

$\{w_j\} \subset K^c$ and so these are functions on the desired set.

$\int_{w_j}^{w_{j+1}} \frac{f(w) dw}{w-z} - \frac{f(w_j)(w_{j+1} - w_j)}{w_j - z} = \int_{w_j}^{w_{j+1}} \frac{f(w)}{w-z} - \frac{f(w_j)}{w_j - z} dw$. Suppose that $K \subseteq D_R(0)$ and $w_j \in D_{R+\epsilon}(0)^c$. Then $1/(w_j - z) = \frac{1}{w_j} \frac{1}{1 - z/w_j} = \frac{1}{w_j} \sum (z/w_j)^n$. Let $z \in K$ then $|z/w_j| < R/(R + \epsilon)$.

Thus, $\left| \frac{1}{w_j - z} - \sum z^n / w_j^{n+1} \right| < \eta$ and so $\left| \frac{1}{(w_j - z)^\ell} - \left(\sum z^n / w_j^{n+1} \right)^\ell \right| < \eta$.

For each w_j in the unbounded component of K^c, there is a C^1 curve $\ell_j(t)$ such that $\ell_j(0) = w_j$, $\ell_j(1) = \tilde{w}_j \in D_{R+1}(0)^c$ and $\int_j d(\ell_j, K) = \rho > 0$. □

Lemma 7.2. *Suppose that $w, w' \in \ell_j([0, 1])$ with $|w - w'| < \rho/2$, then given $\eta > 0$, there exists an N such that $\left| \frac{1}{w-z} - \sum_{j=1}^{N} \frac{a_j}{(w-z)^j} \right| < \eta$ for all $z \in K$.*

Proof. $\frac{1}{z-w} = \frac{1}{z-w'} \frac{1}{1 - \frac{w-w'}{z-w'}} = \frac{1}{z-w'} \sum_{j=0}^{\infty} \left(\frac{w-w'}{z-w'} \right)^j$. The ratio is less than 1/2 if $z \in K$.

$|z - w'| > \rho$ and $|w - w'| < \rho/2$. Because of this, we can choose N such that $\left| \frac{1}{z-w} - \frac{1}{z-w} \sum_{j=0}^{N} \left(\frac{w-w'}{z-w'} \right)^j \right| < \eta$ for $z \in K$. □

Corollary 7.3. *For all ℓ and $\eta > 0$, there exists N_ℓ and $\{a_{j,\ell}\}$ such that $\left| \frac{1}{(z-w)^\ell} - \sum_{\ell=1}^{N_\ell} \frac{a_{j,\ell}}{(z-w')^\ell} \right| < \eta$ for $z \in K$.*

8 Lecture 8

In the homework, for proving the harmonic Runge theorem, asume that K^c is connected, which is stronger than K simply-connected.

Let K^c beconnected and helnce is just the unbounded component. Let$f$ be holomorphic in a nieghborhood of K. To show that given $\epsilon > 0$, there exists a polynomial $h(z)$ such that $\|f - g\|_{L^\infty(K)} < \epsilon$.

For all $z \in K$, let $f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w) dw}{w-z}$ for $\gamma \in K^c$. Define $dist(\gamma, K) = \delta > 0$ and $|w - z| > \delta$ ofr $z \in K$ and $w \in \gamma$. Then $\left| f(z) - \sum_{j=1}^{N} \frac{f(w_j)(w_{j+1} - w_j)}{(w_j - z) 2\pi i} \right| < \epsilon$ if $z \in K$.

2nd observation, if $K \subset D_R(0)$, and $w' \in D_{R+1}(0)$, then $\frac{1}{w'-z} = \frac{1}{w'}\sum_{j=0}^{\infty}\left(\frac{z}{w'}\right)^j$ converges on K. Then $|z/w'| < R/(R+1)$ if $z \in K$.

So there exists $\tilde{N}$ such that $\left|\frac{1}{w'-z} - \sum_{j=0}^{N}\frac{1}{w'}\left(\frac{z}{w'}\right)^j\right| < \epsilon$ for $z \in K$.

So now chose a C^1 path $\ell_j \subset K^c$ such that $\ell_j(0) = w_j$ and $\ell_j(1) = w'_j$ for all j.

So there exists a ρ such that $dist(\ell_j, K) \geq 2\rho$ for all j. If $w, w' \in \ell_j$, and $|w - w'| < \rho$, $\epsilon > 0$ given, then there exists $\{a_j\}$, N such that $\left|\frac{1}{w-z} - \sum_{j=0}^{N}\frac{a_j}{(w'-z)^j}\right| < \epsilon$ for $z \in K$.

Maximum principal implies that if f is holomorphic on $K \cup U$, then $\sup_{z\in U}|f(z)| \leq \sup_{z\in K}|f(z)|$. The seqeunce $(z-\zeta)p_n(z)$ actually converges to function 1 in $K \cup U$.

Because $(z-\zeta)p_n(z)|_{z=\zeta} = 0$ for all $\zeta \in U$.

8.1 Isolated Singularities

f is holomorphic in $D_r(z_0) \setminus \{z_0\}$. We call this a deleted neighborhood of z_0. There are three types of singularities: removable, poles and essential singularities.

Theorem 8.1 (Riemann Removable Singularities Theorem). *If f is holomorphic in a deleted neighborhood of z_0 and $|f(z)|$ is bounded, then f can be extended as a holomorphic function on the neighborhood.*

Proof. $f(z) = \frac{1}{2\pi i}\int_{bD_r(z_0)\setminus bD_\epsilon(z_0)} f(w)dw/(w-z)$. Eh, look this up in Stein-Shakarchi. □

Zeroes: Suppose that f is holomorphic in $D_r(z_0)$ and vanishes at z_0, but not identically. Then there exists an n such that $f^{[j]}(z_0) = 0$ for $j = 0, \ldots, n-1$ and $f^{[n]}(z_0) \neq 0$. Thus $f(z)/(z-z_0)^n$ is bounded in $D_r(z_0) \setminus z_0$, and so is holomorphic. We say that f has a zero of order n if $f(z) = (z-z_0)^n g(z)$ where $g(z_0) \neq 0$.

Let f be holomorphic in a deleted neighborhood and have infinite limit. Then there exists $\rho > 0$ such that $f(z) \neq 0$ for $z \in D_\rho(z_0) \setminus z_0$ or $g = 1/f$ is holomorphic. We have a pole, and poles have form $\frac{1}{(z-z_0)^n}\frac{1}{h(z)}$ a nonzero holomorphic functions.

Riemann Sphere basics.

Let's suppose that f is a holomorphic function at $\mathbb{C} \setminus \{z_1, \ldots, z_n\}$ and that f has at worst a pole at $z = \infty$. Then $f(z) = p(z)/q(z)$ for polynomials p and q.

9 Lecture 9

Theorem 9.1 (Residue Theorem). $\frac{n!}{2\pi i}\int_C \frac{f(w)dw}{(w-z)^{n+1}} = \frac{d^n f}{dz^n}(z)$.

Definition of residue

If the pole at z_0 is simple, then $Res_{z_0} f = \lim_{z \to z_0} (z - z_0) f(z)$.

Examples, including the fact that $\operatorname{sech}(\pi x)$ is its own Fourier transform.

Theorem 9.2 (Casorati-Weierstrass)**.** *Suppose that f has an essential singularity at $z = z_0$. Then any punctured neighborhood has image dence in $\mathbb{C}$.*

Theorem 9.3 (Picard's Great Theorem)**.** *If f has an essential singularity at z_0, then f assumes every complex value infinitely often except possibly one value.*

10 Lecture 10

Some stuff about orientation

Definition of holomorphic manifolds.

Argument Principle.

11 Lecture 11

Define winding number of $\gamma(t)$ a closed curve to be $\int_a^b \gamma'/\gamma dt$.

Proposition 11.1. *Suppose that γ_t is a family of curves not passing through zero. Then $w(\gamma_0) = w(\gamma_1)$.*

Proof. $w(\gamma_t) = \frac{1}{2\pi i} \int_a^b \frac{\gamma_t'(s)}{\gamma_t(s)} ds$. w is an integer for all $t \in [0, 1]$. Now, as $\gamma_t(s)$ is C^0 and $[a, b] \times [0, 1]$ is compact, the distance between any curve and 0 is bounded below by a positive number. Thus, $t \mapsto w(\gamma_t)$ is continuous (painful check). A continuous integer value function is constant. $\square$

The converse is also true: if γ_0 and γ_1 have the same winding number, then they are homotopy equivalent.

Argument principle

Theorem 11.2 (Rouche's Theorem)**.** *Let $f_t(z)$ be a continuous family of holomorphic functions on Ω, a bounded connected open set with piecewise smooth boundary. $(t, z) \mapsto f_t(z)$ is in $C^0([0, 1] \times \bar{\Omega})$. Also assume $f_t(z) \neq 0$ on $b\Omega$. Then $\int_{b\Omega} f_t'/f' dz$ is constant. Thus, the number of solutions of $f_t(z) = 0$ is independent of t.*

Corollary 11.3. *If $f(z)$ and $g(z)$ are holo on $\bar{\Omega}$, and $f(z) \neq 0$ for $z \in b\Omega$, and $|g(z)| < |f(z)|$ for all $z \in b\Omega$, then the number of szeros in Ω of f is the same as for $f + g$.*

The maximum modulus principle is a consequence of the open mapping theorem.

12 Lecture 12

We will review Fourier Series.

Let $f \in L^2(S^1)$, then $f(e^{i\theta}) = \sum_{n=-\infty}^{\infty} f_n e^{in\theta}/\sqrt{2\pi}$, and define $< e^{in\theta}/\sqrt{2\pi}, e^{im\theta}/\sqrt{2\pi} >= \delta_{nm}$.

Then $f_n = (f, e^{in\theta}/\sqrt{2\pi})$.

To show that the Fourier series converges to f, we will need

Proposition 12.1 (Bessel's Inequality)**.** *Define* $T_N = \{\sum_{j=-N}^{N} a_j e^{ij\theta} | a_j \in \mathbb{C}\}$. *If* $f \in L^2(S^1)$, *then* $\int_{t \in T_N} \|f - t\|_{L^2}$ *is attained only at* $S_N(f) = \sum_{j=-N}^{N} f_j e^{ij\theta}/\sqrt{2\pi}$.

$\|f - t\|_{L^2}^2 = \|f - S_N(t)\|^2 + \|T - S_N\|^2$.

Lemma 12.2. *Given* $\epsilon > 0$, *there exists* N_1 *and* $t \in T_N$ *such that* $\|t - t\|_{L^2} < \epsilon$.

Lemma 12.3. *If* $f \in L^2(S^1)$ *and* $\epsilon > 0$ *is given, then there exists* $g \in C^0(S^1)$ *such that* $\|f - g\|_{L^2} < \epsilon$.

Definition 12.1 (Good Kernel)**.** $\int_0^{2\pi} k_n(x - y)dy = 1$, $k_n(x + 2\pi) = k_n(x)$, $k_n \in C^\infty(S^1)$, $k_n \geq 0$, *and given* $\epsilon > 0$, $\delta > 0$, *there exist an* N *such that* $\int_{\delta \leq |x|} k_n(x)dx < \epsilon$ *if* $n > N$.

Convolution with a good kernel $k_n * f \to f$ in $L^2(S^1)$.

$\min f \leq k_N * f \leq \max f$.

This reduces to the problem of showing that a continuous function g can be approximated by trigonometric polynomials in the L^2-norm.

Recall the Poisson Kernel $P(r, \theta, \phi) = \sum_{n=-\infty}^{\infty} r^{n+1} e^{in(\theta - \phi)}/2\pi = \frac{1}{2\pi} \frac{1 - r^2}{|1 - re^{i(\theta - \phi)}|^2}$.

Define $g_r(\theta) = \int_0^{2\pi} P(r, \theta, \phi) g(\phi) d\phi = \sum r^{n+1} g_n e^{in\theta}$. So there exists N such that $\|g_{r_0}^N - g\|_{L^\infty} \leq 2\epsilon$.

Proposition 12.4 (Parseval Identity)**.** $\sum_{n=-\infty}^{\infty} |f_n|^2 = \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta$.

If f_n is a sequence such that $\sum |f_n|^2 < \infty$ then $\lim_{N \to \infty} \sum_{-N}^{N} f)ne^{in\theta}/\sqrt{2\pi}$ exists in $L^2(S^1)$

Take the terms with positive n or $n = 0$ to be F_+ and with negative to be F_-.

Thus, $f = F_+ + F_-$. F_+ extends to a holomorphic function on D_1.

F_- extends to a holomorphic function on $D_)1^c$ which goes to zero as it goes to infinty.

Let H_+^2 be the functions of the form $\sum_{j=0}^{\infty} a_j e^{ij\theta}$ with $\sum |a_j|^2 < \infty$, and let H_-^2 analagous.

Then $L^2(S^1) = H_+^2 \oplus H_-^2$, and $H_+^2 \perp H_-^2$.

$H^2(D_1) = \{f \in L^2(D_1) | f \text{ holomorphic on } D_1\}$.

It was an exercise to show that the L^2 functions in the unit disc which are holomorphic define a closed subspace of $L^2(D_1)$, which we will call H^2.

Definition of Schwartz Distributions.

$H^2(D_1)|_{bD_1} \neq H_+^2(S^1)$.

$H^2_+(S^1)$ is also a closed subspace of $L^2(S^1)$, there is an orthogonal projection $\pi_+ : L^2(S^1) \to H^2_+(S^1)$. $\pi_+ \left(\sum_{n=-\infty}^{\infty} f_n e^{in\theta}/\sqrt{2\pi}\right) = \sum_{n=0}^{\infty} f_n e^{in\theta}/\sqrt{2\pi}$.

Is there a "function" $s(\theta, \phi)$ such that $\pi_+(f(e^{i\theta})) = \int_0^{2\pi} f(e^{i\phi} s(\theta, \phi) d\phi$?

If f is holomorphic on D_1 and smooth up to the boundary, we know that $f(z) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(e^{i\theta}) i e^{i\theta} d\theta}{(e^{i\theta} - z)} = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{i\theta}) d\theta}{(1 - z e^{i\theta})}$, which is $\mathscr{C} f(re^{i\theta})$, the Cauchy Transform of f, where $z = re^{i\theta}$.

Proposition 12.5. *If $f \in L^2(S^1)$, then* $\lim_{r \to 1^-} \sum_{n=0}^{\infty} \frac{r^n e^{in\phi} f_n}{\sqrt{2\pi}} = \pi_+ f$

Really complicated calculation, didn't follow at all

13 Lecture 13

$\mathscr{C} : C^1(bD_1) \to H(D_1)$, holomorphic functions in the disc.

$\mathscr{C} f(z) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(e^{i\theta}) i e^{i\theta} d\theta}{e^{i\theta} - z}$.

Lemma 13.1. $\lim_{z \to bD_1} \mathscr{C} f(z) = \Pi_+ f$.

$$\lim_{r \to 1^-} \mathscr{C} f(e^{i\theta}) = \frac{f(e^{i\theta})}{2} + \frac{1]2\pi \int_0^{\pi} \frac{\sin \frac{2(\phi-\theta)}{2\pi} f(e^{i\theta} d\theta}{(1 - e^{i(\phi-\theta)})} + \frac{PV}{2\pi} \int_0^{2\pi} \frac{f}{(e^{i\theta})} \sin\left(\frac{\phi-\theta}{2}\right) \cos\left(\frac{\phi-\theta}{2}\right)}{|1 - e^{i(\phi-\theta)}|^2} d\theta$$

The last term goes to $\frac{PV}{4\pi} \int_0^{2\pi} \frac{\cos(\phi-\theta)/2}{\sin(\phi-\theta)/2} f(e^{i\theta}) d\theta$.

So then $\Pi_+ f = \frac{f}{2} + \frac{f_0}{2\sqrt{2\pi}} + \frac{i}{2} H(f)$, where $H(f) = \frac{1}{i} \sum_{n=-\infty}^{\infty} \frac{\text{sign}(n) f_n e^{in\theta}}{\sqrt{2\pi}} = \frac{1}{2\pi} PV \int_0^{2\pi} \cot\left(\frac{\phi-\theta}{2}\right) f(e^{i\theta}) d\theta$ (cot is often denoted ctg, which is insane)

For $f \in C^1(S^1)$, we have $\|Hf$

$_{L^2} \leq \|f\|_{L^2}$. Hence, as $C^1(S^1)$ is dense in $L^2(S^1)$, we have $\mathscr{H} : L^2 \to L^2$.

And so, $\Pi_+ = \frac{\text{id}}{2} + \frac{P_0}{2} + \frac{i\mathscr{H}}{2}$.

$\mathscr{H}(f) \in L^\infty(S^1)$, and so we define $T_f h = \pi_+(fh)$ is a map $L^2 \to H^2$, and often thought of as $H^2 \to H^2$.

These are called Toeplitz operators, and are Fredholm operators (and compact operators).

Let $A : \mathbb{C}^n \to \mathbb{C}^m$. When does $Ax = y$ have a solution?

Define $(z, w) = \sum z_j \bar{w}_j$ an inner product.

Then we get A^* by $(Az, w) = (z, A^* w)$.

Fredholm Alternative: The condition that $Ax = y$ has a solution iff $y \perp \ker A^*$.

A consequence is that $\dim \ker A - \dim \ker A^* = n - m$.

So now we look at $A : H \to H$, an operator on a hilbert space. In infinite dimensional, we might not have the range of A closed.

Definition 13.1 (Fredholm)**.** *A is Fredholm if RangeA is closed,* $\ker A$ *and* $\operatorname{coker} A$ *are finite dimensional.*

So then, if A is a fredholm operator, then $Ax = y$ is solvable iff $y \perp (\ker A^*)$.

We define $Ind(A) = \dim \ker A - \dim \operatorname{coker} A$. This is called the Fredholm index.

Theorem 13.2 (Toeplitz/Noether Index Theorem)**.** *The operator T_f is Fredholm provided that f is nonvanishing. Additionally, $Ind(T_f) = -w(f)$ where $w(f)$ is the winding number*

$\|T_f h$
$^2 = (\Pi_+ f \Pi_+ h, \Pi_+ f \Pi_+ h) \leq$
$f\Pi_+ h\|^2 = \int_0^{2\pi} |f|^2 |\Pi_+ h\|^2 dx \leq \|f\|^2_{L^\infty} \|\Pi_+ h\|_{L^2} \leq \|f\|^2_{L^\infty} \|h\|^2_{L^2}$.

If $f \in L^\infty(S^1)$, then $\|T_f h\|_{L^2(S^1)} \leq \|f\|_{L^\infty(S^1)} \|h\|_{L^2(S^1)}$.

If $A : X \to Y$ is 1-1 and onto and bounded, then A^{-1} is bounded.

$A : (\ker A)^\perp \to \operatorname{Im} A$ is 1-1 and onto, then the open mapping theorem says that we get an operator $B : \operatorname{Im} A \to (\ker A)^\perp$ which inversts A.

Let P_1 be the orthogonal projection onto $\operatorname{Im} A$. Set $C = BP_1 : H \to H$ and $AC = ABP_1 = P_1 = I + (P_1 - I)$, while $I - P_1$ si the orthogonal projection onto $(\operatorname{Im} A)^\perp$.

$CA = BP_1 A = BA = P_2$, and so we set $I - P_2 = K_2$ and $I - P_1 = K_1$. Then $AC = I - K_1$, and $CA = I - K_2$, where K_1 and K_2 are finite rank operators. So Fredholm operators are invertible up to a finite rank error.

Definition 13.2 (Pseudoinverse)**.** *Let $A : H \to H$ be bounded. Then C is a pseudoinverse of A if $AC = I - K_1$, $CA = I - K_2$ where K_1, K_2 are finite rank.*

Theorem 13.3. *A is fredholm iff it has a pseudoinverse.*

If A is bounded, we can define $\operatorname{tr} A = \sum_{j=1}^\infty (Af_j, f_j)$, where $\{f_j\}$ is an orthonormal basis for H.

$\operatorname{tr} K_1 = \dim \ker A^*$ and $\operatorname{tr} K_2 = \dim \ker A$, and so $Ind(A) = \operatorname{tr} K_2 - \operatorname{tr} K_1$ for any choice of pseudoinverse.

An operator is compact if there is a sequence K_n of finite rank operatos such that $|||K_n - K||| = \sup_{x \neq 0} \frac{||K_n x - Kx||}{||x||} = \Delta_n$ tends to zero as $n \to \infty$.

Proposition 13.4. *If K is compact, $\epsilon > 0$, then there is a finite dimensional subspace S such that $||Kx|| \leq \epsilon ||x||$ if $x \perp S$.*

Choose a finite rank operator $\tilde{K}$ such that $|||K - \tilde{K}||| < \epsilon$.

If $x \in \ker \tilde{K}$, then $Kx = (K - \tilde{K})x$, and so $||Kx|| \leq ||(K - \tilde{K})x|| \leq |||K - \tilde{K}||| ||x|| \leq \epsilon ||x||$.

If $|||B||| < 1$, then $(I - B)$ is invertible, and so $(I - B)^{-1} = \sum_{j=0}^\infty B^j$. This is called the Neumann series.

14 Lecture 14

$K : H \to H$ is compact if there is a sequence of linear operators of finite rank K_n such that $\|K - K_n\| \to 0$ as $n \to \infty$ uniformly convergent.

For all $h \in H$, $\lim_{n\to\infty} K_n h = Kh$ is called Strong Convergence.

Let $f \in L^2[0,1]$, then $\mathscr{F}_N(f)(k)$ gives $\hat{f}_k$ for $|k| \leq N$ and 0 else. For any fixed f, we have that $\lim_{N\to\infty} \mathscr{F}_N(f) = < \hat{f}_n >$. But it doesn't converge uniformly.

An equivalent definition of compact is that KB_1 is compact as a subset of H.

Let $\mathscr{B}$ be the bounded operators on H, $\mathscr{K} \subset \mathscr{B}$ the compact operators. Then $\mathscr{K}$ is a 2-sided ideal in $\mathscr{B}$.

So now let K_n be a finite rank sequence such that $\|K - K_n\| \to 0$ as $n \to \infty$. $\|AK - AK_n\| = \|A(K - K_n)\| \leq \|A\|\|K - K_n\|$.

Let $A : H \to H$ be Fredholm if there is a bounded operator $B : H \to H$ such that $AB = I - K_1$ and $BA = I - K_2$ where K_1, K_2 are finite rank.

It suffices to require that K_i compact.

Proposition 14.1. *If $K : H \to H$ compact and $\epsilon > 0$ given, then there exists a subspace $S \subseteq H$ such that $\dim S < \infty$ and $\|Kx\| \leq \epsilon\|x\|$ for all $x \in S^\perp$.*

Proof. CHoose a sequence K_n of finite rank operators $|||K - K_n||| \to 0$.

Then there exists N such that $|||K - K_N||| < \epsilon$. Then, if $x \in \ker K_N$, we have $Kx = (K - K_n)x$ So $\|Kx\| \leq \|(K - K_N)x\| \leq |||K - K_n|||\|x\| \leq \epsilon\|x\|$.

We can write $K_j = \tilde{K}_k + \hat{K}_j$, where $\|\hat{K}_k\| < 1/2$ and $rank(\tilde{K}_k) < \infty$. Then $AB = I - \tilde{K}_1 - \hat{K}_1$ and BA similarly.

So tehn $(I - \hat{K}_j)$ is invertible, and so $AB(I - \hat{K}_j)^{-1} = I - \tilde{K}_1(I - \hat{K}_1)^{-1}$ (and similarly for BA.)

So now we have $AB_1 = I - K_1'$ and $B_2A = 1 - K_2'$. □

$A : H \to H$ is Fredholm, then $IndA = \dim\ker A - \dim\operatorname{coker} A$.

Uf K is compact, then $A + K$ is also Fredholm and $Ind(A + K) = Int(A)$.

If A is Fredholm, then there exists $\epsilon > 0$ such that if $B : H \to H$ is bounded and $\|B\| < \epsilon$ then $A + B$ is Fredholm, and $Ind(A + B) = Ind(A)$.

If A, B are both Fredholm, so is AB and $Ind(AB) = Ind(A) + Ind(B)$.

Proposition 14.2. *If K is compact, then $I+K$ is Fredholm and $Ind(I+K) = 0$.*

Proof. Reduce to $I + \tilde{K}$ where $\tilde{K}$ is finite rank and then we reduce to a finite dimensional vector space. Suppose that B is pseudoinverse to A. Then $BA = I - K_2$ and $AB = I - K_1$. Then $Ind(AB) = Ind(A) + Ind(B) = 0$, and so $Ind(A) = -Ind(B)$.

As A^*A is self-adjoint, $\dim\ker A^*A = \dim\ker(A^*A)^* = \dim\operatorname{coker} A^*A$, and so $Ind(A^*A) = 0$, $IndA = -IndA^*$, if B is a pseudoinverse for A, then it is also a pseudoinverse for $A + K$.

Confused □

So now look at $H^2_+ \subseteq L^2(S^1)$, where the negative fourier coefficients are zero. $\Pi_+ : L^2(S^1) \to H^2_+$ the orthogonal projection of $f \in C^0(S^1)$, then we define $T_f u = \Pi_+ f \Pi_+ u$ to get $T_f : H^2_+ \to H^2_+$

Theorem 14.3 (Szego Index Theorem)**.** *If f is non-vanishing, then T_f is Fredholm and $IndT_f = -w(f)$, $f : S^1 \to \mathbb{C} \setminus \{0\}$.*

Proposition 14.4. $|||\Pi_+ f||, |||f\Pi_+||| < \|f\|_{L^\infty}$.

More confusion. What the hell is going on?

Lemma 14.5. $[\Pi_+, f] = \Pi_+ f - f\Pi_+$ *is always compact.*

Proposition 14.6. *Let $g \in \mathscr{F}_N$ (operators of rank at most N), then $[g, \Pi_+]$ is a finite rank operator.*

We still need to compute $IndT_f$.

Suppose that $t \mapsto f_t$ defines a continuous map $[0,1] \times S^1 \to \mathbb{C} \setminus 0$. Then $IndT_{f_t}$ is constant.

For fixed t, T_{f_t} is Fredholm, and so there exists ϵ_t such that if $|||T_{f_t} - T_{f_S}||| < \epsilon_t$ then $IndT_{f_t} = IndT_{f_s}$.

$|||T_{f_t} - T_{f_s}||| \leq \|f_t - f_s\|_{L^\infty}$, and so there exists $0 < r_t$ such that if $|t-s| < r_t$, then $||f_t - f_s|| < \epsilon_t$.

Thus, $IndT_{f_t} = IndT_{f_s}$ if $|s-t| < r_T$, $t \mapsto T_{f_t}$.

Recall:

Theorem 14.7. *If f and $g : S^2 \to \mathbb{C} \setminus 0$ are C^1, then $f \simeq g$ iff $w(f) = w(g)$.*

Let $n = w(f)$. $f_n(e^{i\theta}) = e^{in\theta}$ has the same index. And so $IndT_{f_n} = IndT_f$ must be a function of the winding number.

So we calculate $Ind\Pi_+ e^{in\theta}\Pi_+$. If $n = 0$, we get $Ind\Pi_+ = 0$. If $n > 0$, then ...see problem set.

Thus, $IndT_f = \frac{-1}{2\pi i}\int_0^{2\pi} \frac{f'(e^{i\theta})d\theta}{f(e^{i\theta})}$.

15 Lecture 15

We proved the following;

Theorem 15.1. *If $f : S^1 \to \mathbb{C} \setminus 0$, then $T_f = \Pi_+ f \Pi_+ : H^2_+ \to H^2_+$ is Fredholm and $IndT_f = -w(f)$.*

Theorem 15.2. *Let $f : S^1 \to \mathbb{C} - 0$.*

1. *If $w(f) > 0$, then T_f is 1-1, and so $\operatorname{codim} ImT_f = w(f)$.*
2. *IF $w(f) < 0$, then $\dim\ker T_f = n$ and T_f is onto.*
3. *If $w(f) = 0$, then T_f is an isomorphism.*

Proof. We will first prove 3. IF $w(f) = 0$, then $\log f : S^1 \to \mathbb{C}$ is cont, $\log f = g_+ + g_-$ which are in H^2_+ and $(H^2_+)^\perp$ where g_+ is the boundary values for a function holo on D_1 and g_- is on the complement.

Now take $u \in H^2_+$. Then $fu = u_+ + u_-$, so then $f = e^{g_+}e^{g_-}$, so we have $u = e^{-g_+}\Pi_+ e^{-g_-}u_+$, and so $\ker T_f = 0$, and thus, becuase index is zero, we have $\operatorname{coker} T_f = 0$ as well.

As $T_f^* = T_{\bar{f}}$, we only need to prove one of the others. Careful and straightforward computation. $\square$

we now extend our notion of "function slightly to line bundles.

Let $f_+ \in H^2_+$, $f_- \in H^2_-$ and $f_+\phi = f_-$ along the unit circle on the riemann sphere, where ϕ is a map $S^1 \to \mathbb{C} - 0$. This defined a line bundle L_ϕ and a section $F : S^2 \to L_\phi$.

Now given $\phi \in C^1(S^1)$, we want to know if there is a pair with $f_+\phi = f_-$.

$\lim_{z\to\infty} f_-(z) = 0$, then $\Pi_+(f_+\phi) = \Pi_+(f_-) = 0$.

The existence is then equivalent to $\ker T_\phi \neq 0$, that is, $w(\phi) > 0$.

So then there are holomorphic global sections iff $w(\phi) > 0$.

If $w(\phi) = 0$ then we have $\mathbb{C} \times \mathbb{P}^1$, the trivial line bundle.

Up to holomorphic equivalence, holomorphic line bundles are classified by degree (winding number)

We can generalize, taking $F : S^1 \to \mathbb{C}^n$, then $\Pi_+ F$ is done componentwise. A map $\Phi : S^1 \to gl(n, \mathbb{C})$ defines a Toeplitz operator by $T_\Phi = \Pi_+\Phi\Pi_+$.

Theorem 15.3. *T_Φ is Fredholm provided that $\phi : S^1 \to GL(n, \mathbb{C})$, that is, the determinant is nowhere zero. $[T_\Phi, \Pi_+]$ is compact.*

16 Lecture 16

Missed

17 Lecture 17

Lidski's Theorem says that the trace of K (where K is integration against $k(x, y)$ is $\int_0^{2\pi} k(x, x)dx$.

Suppose that K is diagonalizeable and there exists an orthonormal basis such that the span is $L^2(S^1)$ and $Ke_j = \lambda_j e_j$. Then $\operatorname{tr} K = \sum_{j=1}^\infty \lambda_j = \int_0^{2\pi} k(x, x)dx$.

So $Kf = \int_0^1 \phi(x - y)f(y)dy$ where ϕ is a 2π-periodic C^1-function. Computation that gives Fourier Inversion Formula.

Chosoe a function $f \in C^1(\mathbb{R})$ such that $|f(x)| \leq A/(1 + x^2)$ and $|f'(x)| \leq A/(1 + x^2)$. Then set $\phi(x) = \sum_{n=-\infty}^\infty f(x + n)$. Standard stuff that is in Stein-Shakarchi.

Theorem 17.1 (Paley-Wiener). *A function f with moderate decrease along $\mathbb{R}$, has an extension to $\mathbb{C}$ as an entire function satisfying $|f(z)| \leq Ce^{2\pi M|z|}$ if and only if $\hat{f}(\xi)$ is supported in $[-M, M]$.*

We showed last time that if the support of the Fourier transform is compact, then there is an analytic extension as desired. Now we suppose $f(z)$ entire and $|f(x + iy)| \leq Ce^{2\pi M|y|}$ and $|f(x + iy)| \leq A/(1 + x^2)$.

We can compute $\hat{f}(\xi) = \int_{-\infty}^\infty f(x+iy)e^{-2\pi i(x+iy)\xi}dx = \int_{-\infty}^\infty f(x+iy)e^{-2\pi ix\xi}e^{2\pi iy\xi}dx$ for $\xi < -M$.

Thus $|\hat{f}(\xi)| \leq \int_{-\infty}^\infty \frac{Ce^{2\pi iM|y|}e^{2\pi y\xi}}{1+x^2}dx \leq \tilde{C}e^{2\pi(M-\xi)|y|}$ goes to zero as $y \to \pm\infty$ and so $|\hat{f}(\xi)| = 0$ for $|\xi| > M$.

We start removing hypotheses.

Given $|f(z)| \leq Ae^{2\pi M|z|}$ with $|f(x)| \leq B/(1+x^2)$, we need to have $|f(x+iy)| \leq \tilde{A}e^{2\pi M|y|}$.

We use that Phragman-Lindelof Principle: f is holo in S and continuous up to $\bar{S}$ with $|f(z)| \leq 1$ for $z \in bS$. If for some constants c, C, we have $|f(z)| \leq Ce^{c|z|}$, then we have $|f(z)| \leq 1$ for all $z \in S$.

Define $F_\epsilon(z) = f(z)e^{-\epsilon z^{3/2}}$, $z = re^{i\theta}$. So through some calculations, we have $|F_\epsilon(z)| \leq Ce^{c|z|(1-\alpha\epsilon|z|^1/2}$ So the result follows.

We in fact don't need $Ce^{c|z|}$, we can use $Ce^{c|z|^\gamma}$ for any $\gamma < 2$.

So now we want to show that $f(z) \leq A/(1+x^2)$ and $|f(z)| \leq Ce^{2\pi M|z|}$ implies that $|f(z)| \leq Ce^{2\pi M|y|}$.

Look at $F(z) = f(z)e^{2\pi iMz}$. Then $F(x) = f(r)e^{2\pi iMy}$, and so $|F(x)| \leq A/(1+x^2)$ and $F(iy) = f(iy)e^{-2\pi My}$. Then $|F(iy)| \leq Ce^{e^{2\pi My - 2\pi My}} \leq C$

So then Phragman-Lindelof says that $|F(z)| \leq C$ in the quadrant, and so $|F(x+iy)| \leq C$ got $y \geq 0$. By looking at $f(z)e^{-2\pi Mz}$ we get the result on the bottom half plane.

So what did we use? f is entire and bounded on $\mathbb{R}$, and $|f(z)| \leq Ce^{2\pi M|z|}$ which gives us that $|f(x+iy)| \leq tildeCe^{2\pi M|y|}$.

Theorem 17.2 (Hadamard Three Lines Theorem)**.** *Let f be defined in a strip S and continuous up to $\bar{S}$ with $|f(z)| \leq 1$ on each boundary component with $|f(z)| \leq Ce^{c|x|^\gamma}$ for $\gamma < 2$, then $|f(z)| \leq 1$ in $\bar{S}$.*

Proof. Look at $f(z)e^{-\epsilon z^2}$. As $z^2 = x^2 - y^2 + 2ixy$, we have $|e^{-\epsilon z^2}| = e^{-\epsilon(x^2-y^2)} \leq Ce^{-\epsilon x^2}$.

$|f(z)e^{-\epsilon z^2}| \leq \tilde{C}e^{c|z|^\gamma - \epsilon x^2}$, and thus we have shown that the limit as $|z| \to \infty$ of this is zero for all $\epsilon > 0$. The maximum principle says that there exists w with $Im(w) = 0$ on 1 and $|f(w)|e^\epsilon \geq |f(w)e^{-\epsilon w^2}| \geq |f(z)e^{-\epsilon z^2}|$ for all $z \in S$. □

18 Lecture 18

Hadamard needed his three lines theorem to prove the Prime Number Theorem. He used it to get some estimates.

Theorem 18.1 (Hadamard)**.** *Let f be a bounded analytic function in $0 \leq Res \leq 1$ and let $N(a) = \sup_{y\in\mathbb{R}} |f(a+iy)|$, then $N(a) \leq N(0)^{1-a}N(1)^a$.*

Corollary 18.2. *$\log N(a)$ is a convex function.*

If (X, dm) and (Y, dn) are measure spaces, look at $L^p(X, dm) = \{f$ measurable with $(\int_X |f|^p dm)^{1/p} < \infty\}$. Let F be a linear operator and carry measurable functions on X to measurable functions on Y. Then $F(f) = \int_0^{2\pi} f(\theta)e^{-in\theta}d\theta = \hat{f}(n)$ takes measurable functions on S^1 to measurable functions on $\mathbb{Z}$.

Hadamard implies that for $1 < p < 2$, we have $L^p(S^1) \to \ell_{p/(p-1)}$, because $L^1(S^1) \to \ell_\infty$ and $L^2(S^1) \to \ell_2$.

To show this, we define $F : L^p(X, dm) \to L^q(Y, dn)$ is bounded if $\sup_{x\neq 0} \|F(x)\|_{L^q(dn)}/\|x\|_{L^p(dm)} = M(p, q) < \infty$.

So then $\|F(x)\|_{L^q(dn)} \leq M(p,q)\|x\|_{L^p(dm)}$.

Theorem 18.3 (F. Riesz-Markov). *$(L^p(X,dm))^\vee \cong L^{p'}(X,dm)$ for $1/p + 1/p' = 1$, and also $\left|\int_X fg dm\right| \leq \left(\int |f|^p dm\right)^{1/p} \left(\int |g|^{p'} dm\right)^{1/p'}$.*

A consequence is that $f \mapsto (f,g) = \int fg$ gives us $\|f\|_{L^p(X,dm)} = \sup_{g\neq 0}(f,g)/\|g\|_{L^{p'}(X,dm)}$.

Theorem 18.4 (N. Riesz Interpolation Theorem). *Suppose that $M : L^{p_0}(X,dm) \to L^{q_0}(Y,dn)$ and $M : L^{p_1}(X,dm) \to L^{q_1}(Y,dn)$ are bounded. Then $M : L^{p(a)}(X,dm) \to L^{q(a)}(Y,dn)$ for $(1/p(a), 1/q(a)) = (1-a)(1/p_0, 1/q_0) + a(1/p_1, 1/q_1)$ satisfies $M(p(a), q(a)) \leq M^{1-a}(p_0,q_0)M(p_1,q_1)^a$.*

This would apply, for instance, to the Fourier transform from $L^2(S^1)$ to ℓ_2 and $L^1(S^1)$ to ℓ_∞.

Proof. For $a = 0$ or $a = 1$, the estimate is trivial, so choose $0 < a < 1$.

Choose a function $f \in L^p(X,dm)$ and $h \in L^{q'}(Y,dn)$ where $p = p(a)$ and $q = q(a)$ such that Mf is defined and belongs to L^q.

$f = |f|e^{i\mu}$ and $h = |h|e^{i\nu}$. Then $f(\zeta) = |f|^{p(a)/p(\zeta)}e^{i\mu}$ and $h(\zeta) = h^{q'(a)/q'(\zeta)}e^{i\nu}$, where $1/p(\zeta) = (1-\zeta)1/p_0 + \zeta/p_1$.

Define $\phi(\zeta) = (Mf(\zeta), h(\zeta)) = \int_Y Mf(\zeta)h(\zeta)dn$.

Then $\phi(\zeta)$ is analytic in the strip. Suppose that $\|f\|_{p(a)} = 1$ and $\|h\|_{q'(a)} = 1$. Then if $N(a) = \sup_{\eta\in\mathbb{R}} |\phi(a+i\eta)|$ then $N(0) \leq M(p_0,q_0)$ and $M(1) \leq M(p_1,q_1)$.

After a nasty computation, we see that $\|Mf(i\eta)|_{L^{q_0}} \leq M(p_0,q_0)\|f\|_{L^{p_0}} \leq M(p_0,q_0)$.

We conclude that $|\phi(i\eta)| = |(Mf(i\eta), h(i\eta))| \leq \|Mf(i\eta)\|_{p_0}\|h(i\eta)\|_{q_0'} \leq M(p_0,q_0)$.

$|phi(1+i\eta)| \leq M(p_1,q_1)$. Now, $\phi(\zeta)$ satisfies the hypotheses of Hadamard's theorem, and so $|\phi(a)| \leq M(p_0,q_0)^{1-a}M(p_1,q_1)^a$.

We recall that $\phi(a) = (Mf,h)$. We note that $\|Mf\|_{L^q} = \sup|(Mf,h)|$ over h with $\|h\| = 1$.

A bit more computation proves the theorem. □

Now that we can interpolate between $L^1(S^1)$ and $L^2(S^1)$, we will return to Paley-Wiener type theorems.

If we assume that $|f(x)| \leq \frac{C}{(1+|x|)^N}$, then this estimate allows us to differentiate the integral $\hat{f}$ up to $N-(1+\epsilon)$ times. (we can only just barely not do it $N-1$ times)

gibberish...

Theorem 18.5. *Let $f \in L^2(\mathbb{R})$, then teh following two conditions are equivalent:*

1. *There exists a holomorphic $F(x+iy)$ for $y > 0$ with $\sup_{y\geq 0}\int_{-\infty}^{\infty} |F(x+iy)|^2 dx < C$ and $\lim_{y\to 0}\int_{-\infty}^{\infty} |F(x+iy) - f(x)|^2 dx = 0$*

2. *$\hat{f}(\xi) = 0$ for $\xi < 0$.*

19 Lecture 19

We will prove the theorem from last time. 2 implies 1 is easy. See Stein-Shakarchi

Corollary 19.1. *$L^2(\mathbb{R}) = L^2_+(\mathbb{R}) \oplus L^2_-(\mathbb{R})$ where $L^2_+(\mathbb{R})$ are the functions which satisfy the first part of the previous theorem.*

To see that $L^2_+ \cap L^2_- = 0$, we note that the Fourier transform takes the first to $L^2((0,\infty))$ and the second to $L^2((-\infty,0))$.

For $x \in [0,\infty)$, we define the Wiener-Hopf equation to be $\int_0^\infty k(x-y)f(y)dy + \lambda f(x) = g(x)$.

Theorem 19.2. *Suppose that u is Hölder continuous of order $\alpha > 0$ and that $u \in L_\infty \cap L_p$ for $1 \le p < \infty$. Define $U(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} u(\xi)d\xi/(\xi - z)$. Then U is analytic in $\mathbb{C} \setminus \mathbb{R}$.*

To use this to solve the Wiener-Hopf equation, replace g by $g + h$, and then extend f, g by zero to $(-\infty, 0)$, and set $h(x)$ to be zero for $x > 0$ and $\int_0^x k(x-y)f(y)dy$ if $x < 0$.

Now we take Fourier transforms and obtain $\hat{k}(\xi)\hat{f}(\xi) + \lambda\hat{f}(\xi) = \hat{g}(\xi) + \hat{h}(\xi)$.

Now $\hat{f}$ and $\hat{g}$ have analytic extensions to H_- and $\hat{h}$ has an analytic extension to H_+.

Gibberish

20 Lecture 20

Very confused about integral equations.

Mittag-Leffler Problem: I specify points $\{a_j\} \subset \mathbb{C}$ and polynomials $p_j(z)$ without constant term. Does there exist a function f holomorphic in $\mathbb{C} \setminus \{a_j\}$ with $f - p_j(1/(z-a_j))$ is holomorphic near to a_j with $\lim_{j\to\infty} |a_j| = \infty$?

Recall that we've shown that if $\phi \in C^\infty(\mathbb{C})$ then there is a function u such that $\partial_{\bar{z}} u = \phi$.

Choose $0 < r_j$ such that $D_{2r_j}(a_j) \cap D_{2r_j}(a_k) = \emptyset$ for all $j \neq k$.

$\sum_{j=1}^\infty \psi\left(\frac{|z-a_j|}{r_j}\right) p_j\left(\frac{1}{z-a_j}\right)$ with $\psi\left(\frac{|z-a_j|}{r_j}\right)$ supported on $D_{r_j}(a_j)$.

Define ϕ to be the partial with respect to $\bar{z}$ of this function.

Use teh existence theorem to solve $\partial_{\bar{z}} u = \phi$.

So there exists a C^∞ function u which solves this equation.

Then we define $f(z)$ to be the original function minus u.

Clearly, $\partial_{\bar{z}} f = 0$ in $\mathbb{C} \setminus \{a_j\}$, and near to $z = a_j$ we have $\partial_{\bar{z}} u = 0$, so near to a_j, we have $f(z) = p(1/(z-a_j)) - u(z)$, so we have $\sum_{j=1}^N p_j(1/(z-a_j))$.

Let $p(z)$ be degree d and $N(r) = \sup_{0\le\theta\le 2\pi} |p(re^{i\theta})|$ and $\lim_{r\to\infty} \log N(r)/\log r = d$, where d is the number of solutions to $p(z) = 0$.

Theorem 20.1 (Jensen's Formula)**.** *Let Ω be an open set containig $\bar{D}_R$ and let f be holomorphic in Ω such that $f(0) \neq 0$, f does not vanish on bD_R and $\{z_1, \ldots, z_N\}$ are the zeroes of f in D_R, listed with multiplicity. Then*

$$\log|f(0)| = \sum_{k=1}^{N} \log\left(\frac{|z_k|}{R}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|d\theta$$

Proof. Suppose that g does not vanish on $\bar{D}_R$. Then this reduces to the mean value theorem.

Let $f(z) = \prod_{j=1}^{N}(z-z_j)g(z)$. Then $\int_0^{2\pi} \log|f(Re^{i\theta})|d\theta/2\pi = \sum_{j=1}^{N} \int \log|Re^{i\theta} - z_j|d\theta/2\pi + \log|g(0)|$...lots of stuff. See book. □

We define $n_f(r) = n(r)$ to be the number of zeroes of f less than r.

Then $\int_0^R n(R)/rdr$ will be a step function counting the zeroes.

Definition 20.1 (Finite Order)**.** *An entire function with finite order of growth ρ satisfies the estimate $|f(z)| \leq Ae^{B|z|^\rho}$.*

Gibberish which is in the book.

21 Lecture 21

If f is entire and $|f(z)| \leq Ae^{B|z|^\rho}$, and $n(r)$ is the number of zeroes of modulus less than r with multiplicity, then $n(r) \leq Cr^\rho$ for large r.

Given (z_j, n_j), with $z_j \in \mathbb{C}$ and $n_j \in \mathbb{N}$, with $z_j \to \infty$ as $j \to \infty$, does there exist a holomorphic function f defined in $\mathbb{C}$ such that f vanishes only at the z_j and the order of the zero at z_j is n_j?

Read the book. Seriously, everything's there.

22 Lecture 22

Missed some asymptotic stuff

23 Lecture 23

Riemann Mapping Theorem

Theorem 23.1 (Riemann Mapping Theorem)**.** *If $D \subsetneq \mathbb{C}$ is a simply connected domain, then there exists a holomorphic map $f : \mathbb{D} \to D$ which is 1-1 and onto.*