1 Complex Plane

Definition 1.1 (i)**.** $i = \sqrt{-1}$

Let $z \in \mathbb{C}$ and $z = a + bi$

Definition 1.2 (Real Part of z)**.** $a = \Re z$

Definition 1.3 (Imaginary Part of z)**.** $b = \Im z$

Definition 1.4 (Modulus of z)**.** $|z| = \sqrt{a^2 + b^2}$

Definition 1.5 (Complex Conjugate of z)**.** $\overline{z} = a - bi$

Definition 1.6 (Polar Representation of z)**.** $|z| = r$ *and* $\tan\theta = \frac{b}{a}$

Definition 1.7 (Argument of z)**.** $\theta = \arg\theta$

Theorem 1.1 (Multiplication)**.** *Let* $w = s(\cos\psi + i\sin\psi)$*.* $zw = rs(\cos(\theta + \psi) + \sin(\theta + \psi))$*, so,* $|zw| = |z||w|$ *and* $\arg zw = \arg z + \arg w (mod 2\pi)$

Theorem 1.2. $\arg\overline{z} = -\arg z$

Theorem 1.3 (DeMoivre's Theorem)**.** *If* $n \in \mathbb{Z}$ *and* $n > 0$*, then* $(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$*.*

Theorem 1.4. $|z + w| \leq |z| + |w|$

Proof. $|z+w|^2 = (z+w)(\overline{z}+\overline{w}) = z\overline{z} + w\overline{w} + \overline{w}z + w\overline{z} = |z|^2 + |w|^2 + 2\left(\frac{z\overline{w} + \overline{z\overline{w}}}{2}\right)$
$= |z|^2 + |w|^2 + 2\Re(z\overline{w}) \leq |z|^2 + |w|^2 + 2|z\overline{w}| = |z|^2 + |w|^2 + 2|z||\overline{w}| = |z|^2 + |w|^2 + 2|z||w| = (|z| + |w|)^2$ ☐

Theorem 1.5. $|z| - |w| \leq |z + w|$

Proof. $|z| = |z + w - w| \leq |z + w| + |w| = |z + w| + |w| \Rightarrow |z| - |w| \leq |z + w|$ ☐

Definition 1.8 (Open Disc)**.** *The open disc of radius* r *and center* z_0 *is the set* $\{z : |z - z_0| < r\}$

Definition 1.9 (Interior Point)**.** *If* S *is a set in* $\mathbb{C}$*, then* z_0 *is an interior point if* $(\exists r > 0)$ *such that an open disc of radius* r *centered at* z_0 *is a subset of* S*.*

Definition 1.10 (Open Set)**.** S *is open if every point is an interior point.*

Definition 1.11 (Closed Set)**.** S *is closed if* $\mathbb{C} \setminus S$ *is open.*

Definition 1.12 (Boundary Point)**.** z_0 *is a boundary point of* S *if* $\forall r > 0$*, the disc of radius* r*, center* z_0 *contains both points of* S *and points not in* S*.*

Definition 1.13 (Line Segment)**.** *A line segment connecting* $p, q \in \mathbb{C}$ *is the set* $\{z : tp + (1 - t)q \text{ for } t \in [0, 1]\}$

Definition 1.14 (Convex Set)**.** *A set* S *of complex numbers is convex if* $(\forall p, q \in S)(\forall t \in [0, 1])tp + (1 - t)q \in S$*.*

Definition 1.15 (Connected Open Set)**.** *S is a connected open set if S is open and $\forall p, q \in S$ there exists a finite collection of line segments joining pairs $p = a_1, b_1, a_2, \ldots, a_n, b_n = q$ with all the line segments in S.*

Definition 1.16 (Convergent Sequence)**.** *Suppose $\{z_n\}$ is a sequence of complex numbers. We'll say $\{z_n\}$ converges to w if $(\forall \epsilon > 0)(\exists N \in \mathbb{N})$ such that if $n \geq N$ then $|z_n - w| < \epsilon$*

Theorem 1.6. *if $|r| < 1$, then*

$$\sum_{i=1}^{\infty} ar^i = \frac{a}{1-r}$$

Theorem 1.7. *If a series converges absolutely, then it converges.*

Definition 1.17 (Continuity)**.** *A function f is continuous if for every sequence $z_n \to z$ then $f(z_n) \to f(z)$.*

Theorem 1.8. *D is a connected open set. IF g is continuous and real-valued on D, and $\exists p, q \in D$ such that $f(p) > 0$ and $f(q) < 0$, then $\exists z \in D$ such that $f(z) = 0$*

Definition 1.18 (Exponential Function)**.** $e^z = e^x(\cos y + i \sin y)$

Definition 1.19 (u and v)**.** *if $f(z)$ is a complex function, then $\Re f = u(x, y)$ and $\Im f = v(x, y)$*

Definition 1.20 (Logarithm)**.** $\ln(e^{x+iy}) = x + i(y + 2\pi\mathbb{Z})$

Theorem 1.9. $x = \frac{1}{2}\ln(u^2 + v^2)$
$y = \arg(u + iv)$

Definition 1.21 (Curves)**.** *Curves are continuous functions $\gamma : [a, b] \to \mathbb{C}$, that is, $\gamma(t) = x(t) + iy(t)$ where $x(t), y(t)$ are continuous.*

Definition 1.22 (Smooth Curve)**.** *A smooth curve is a curve that is everywhere differentiable, also called a differentiable curve.*

Definition 1.23 (Sum of Curves)**.** *The sum of the curves $\sigma_1 : [a, b] \to \mathbb{C}$ and $\sigma_2 : [c, d] \to \mathbb{C}$ is $\sigma_1 + \sigma_2 = \begin{cases} \sigma_1(t) & a \leq t \leq b \\ \sigma_2(t - b + c) & b \leq t \leq b + d - c \end{cases}$*

Definition 1.24 (Simple Curve)**.** *$\sigma : [a, b] \to \mathbb{C}$ is simple when if $a < t_1 < t_2 < b$ then $\sigma(t_1) \neq \sigma(t_2) \neq \sigma(a) \neq \sigma(b)$*

Theorem 1.10. *$\int_\gamma u(z)dz$ does not depend on parametrization*

Theorem 1.11.

$$\int_{\gamma_1+\gamma_2} u(z)dz = \int_{\gamma_1} u(z)dz + \int_{\gamma_2} u(z)dz$$

Theorem 1.12.

$$\int_{-\gamma} u(z)dz = -\int_{\gamma} u(z)dz$$

Theorem 1.13.

$$\int_{\gamma} (\alpha u_1(z) + u_2(z))\, dz = \alpha \int_{\gamma} u_1(z)dz + \int_{\gamma} u_2(z)dz$$

Theorem 1.14.

$$\left| \int_C u(z)dz \right| \leq \int_C |u(z)|\, dz$$

Theorem 1.15 (ML inequality). *If M is an overestimate of the function and L is the arclength of the curve C, then*

$$\left| \int_C u(z)dz \right| \leq ML$$

Theorem 1.16.

$$i \iint_{\Box} \frac{\partial f}{\partial x} dxdy = i \int_0^1 f(1,y)dy - i \int_0^1 f(0,y)dy$$

2 Properties of Analytic Functions

Definition 2.1 (Complex Differentiable). *f is complex differentiable at z if*

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists. Is the limit exists, it is called $f'(z)$.

Definition 2.2 (Entire Function). *An entire function is a function that is differentiable for all z.*

Theorem 2.1 (Cauchy-Riemann Equations). *If $f(z) = f(x+iy) = u(x,y) + iv(x,y)$, then $u(x,y)$ and $v(x,y)$ must have partial derivatives and*

$$u_x = v_y \tag{1}$$

$$v_x = -u_y \tag{2}$$

Definition 2.3 (Harmonic Function). *$v(x,y)$ is harmonic if $v(x,y)$ is twice differentiable and $v_{xx} + v_{yy} = 0$*

Theorem 2.2. *If f is differentiable and u, v are the real and imaginary parts, then u, v are harmonic.*

Definition 2.4 (Harmonic Conjugate). *If $u + iv$ is differentiable, then v is called the harmonic conjugate of u.*

Theorem 2.3. *If $u(x,y)$ and $v(x,y)$ have continuous first partial derivatives at $z = x+iy$ and if $u_x = v_y$ and $u_y = -v_x$ at z, then $f = u+iv$ is complex differentiable at z.*

Definition 2.5 (Holomorphic Function)**.** *Suppose U is a connected open subset of $\mathbb{C}$. Then, $f : U \to \mathbb{C}$ is complex analytic, or holomorphic, if f is complex differentiable at every point of U.*

Theorem 2.4. *If f is holomorphic and if f's values are always real, then f is constant.*

Theorem 2.5. *If f is holomorphic and if u^2+v^2 is constant, then f is constant.*

Theorem 2.6. $\frac{d}{dt}f(C(t)) = f'(C(t))C'(t)$ *if f is holomorphic and C is smooth.*

Theorem 2.7. *Suppose $C(t)$, $a \le t \le b$ is piecewise smooth, START$= C(a)$ and END$= C(b)$. Suppose $f(z)$ is analytic around the curve, and that $F(t)$ is analytic and $F'(z) = f(z)$. Then,*

$$\int_{\mathscr{C}} f(z)dz = F(z)\Big]_{z=START}^{z=END}$$

Definition 2.6 (Power Series)**.** *A power series centered around z_0 is $\sum_{n=0}^{\infty} a_n(z-z_0)^n$*

Theorem 2.8. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$, if $|z| < R$, then $f(z)$ is analytic and $f'(z) = \sum_{n=1}^{\infty} na_n z^{n-1}$, $f^{(k)}(z)$ exists for all k and $a_n = \frac{f^{(n)}(0)}{n!}$*

Theorem 2.9 (Complex form of Green's Theorem)**.** *If f is continuously differentiable on a domain D which is bounded by a simple closed curve C, then*

$$\oint_C f(z)dz = i\int_D \frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}dA$$

Theorem 2.10 (Cauchy's Theorem)**.** *If $f(z)$ is analytic, $f'(z)$ is continuous, C and D are as in Green's Theorem, then $\oint_C f(z)dz = 0$*

Definition 2.7 (Simply-Connected)**.** *A domain D in $\mathbb{C}$ is simply connected if the inside of every simple closed curve in the domain is in the domain.*

Theorem 2.11. *If D is simply-connected, Γ is a closed curve of horizontal and vertical line segments, $f(z)$ is analytic on D, then $\int_\Gamma f(z)dz = 0$*

Theorem 2.12. *If f is analytic on a simply-connected domain D, then there is an analytic function F on D such that $F' = f$ on D.*

Corollary 2.13. *If f is analytic on a simply connected domain D and γ is any closed curve in D, then $\int_\gamma f(z)dz = 0$*

Theorem 2.14. *Suppose $f(z)$ is analytic on domain D and $\{z : |z - z_0| < r\} \subset D$, then*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

$$a_n = \frac{1}{2\pi i} \int_{|w - z_|=r} \frac{f(w)}{(w - z_0)^{n+1}} dW$$

and converges for $|z - z_0| < r$

Theorem 2.15. *The derivative of an analytic function is analytic.*

Theorem 2.16 (Morera's Theorem)**.** *Suppose D is simply-connected, f is continuous,*

$$\int_{\texttt{any box}} f(z) dz = 0$$

then $f(z)$ is analytic.

Theorem 2.17. *The Following Are Equivalent*

1. *$f(z)$ satisfies the Cauchy-Riemann Equations*
2. *$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$ and $f'(z)$ is continuous*
3. *Morera's Theorem*
4. *$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$, f is locally the sum of a power series*

Theorem 2.18 (Cauchy Integral Formula for derivatives)**.**

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{|w - z_0|=r} \frac{f(w)}{(w - z_0)^{n+1}} dw$$

Theorem 2.19 (Liouville's Theorem)**.** *Suppose $f(z)$ is entire and $f(z)$ is bounded. Then $f(z)$ is constant.*

Proof. $f'(z_0) = \frac{1!}{2\pi i} \int_{|z_0 - w|=r} \frac{f(w)}{(w - z_0)^{n+1}} dw$

$|f'(z_0)| \leq \frac{1}{2\pi} 2\pi r \frac{M}{r}$, where M bounds f.

Let $r \to \infty$, then $|f'(z_0)| \leq 0$. □

Theorem 2.20. *Suppose f, g are analytic on a domain D, and that $\exists z_0 \in D$ such that $\forall n \in \mathbb{N}$, $f^{(n)}(z_0) = g^{(n)}(z_0)$ then $f = g$*

Theorem 2.21. *Suppose f, g analytic on a domain D, and that there is a sequence $\{z_n\}$ in D such that $f(z_n) = g(z_n)$ and $z_n \to z \in D$. Then, $g = f$*

Theorem 2.22. *If f, g are analytic on D, $\{z_n\}$ a sequence in D, and $\lim_{n \to \infty} z_n = w \in D$ and $f(z_n) = g(z_n)$ for all n, then $f(z) = g(z)$ on all of D.*

Corollary 2.23. *Suppose* f, g *are entire. And suppose we have* $\{z_n\}$ *bounded and* $\forall n\ f(z_n) = g(z_n)$

Then $f(z) = g(z)$ *for all* z.

Corollary 2.24 (Fundamental Theorem of Algebra)**.** *Any complex polynomial has a root.*

Definition 2.8 (Order of a zero)**.** *Suppose* $f(z)$ *is analytic. If* $f(z_0) = 0$*, then the order of the zero at* z_0 *is the smallest integer* k *such that* $f^{(k)}(z_0) \neq 0$

Definition 2.9 (Isolated Singularity)**.** *A function* $f(z)$ *has an isolated singularity at* z_0 *iff* $\exists r > 0$ *such that* $f(z)$ *is analytic in* $0 < |z - z_0| < r$

Definition 2.10 (Removable Singularity)**.** *If* $\lim_{z\to z_0} f(z)$ *is finite and* $\frac{1}{f(z_0)} = 0$*, then* z_0 *is a removable singularity.*

Definition 2.11 (Pole)**.** *An isolated singularity at* z_0 *such that* $\lim_{z\to z_0} f(z) = \infty$*, then* $f(z)$ *has a pole at* z_0

Definition 2.12 (Essential Singularity)**.** *A singularity is essential if it is neither a pole nor removable.*

Theorem 2.25 (Riemann Removable Singularity Theorem)**.** *Suppose* $f(z)$ *has an isolated singularity at* z_0 *and suppose that the modulus of* $f(z)$ *is bounded near* z_0*. Then* $f(z)$ *has a removable singularity at* z_0*.*

Theorem 2.26 (Casorati-Weierstrass Theorem)**.** *If* $f(z)$ *has an essential singularity at* z_0*, then* $\forall w_0 \in \mathbb{C}$*,* $\forall a, b \in \mathbb{R}$*,* $a, b > 0$*,* $\exists z$ *with* $0 < |z - z_0| < a$ *such that* $|f(z) - w_0| < b$

Definition 2.13 (Laurent Series)**.** *The Laurent Series of a function is* $\sum_{-\infty}^{\infty} a_n(z-z_0)^n$ *for all* z *in* $0 < |z - z_0| < r$

Definition 2.14 (Residue)**.** *The residue of* $f(z)$ *at* z_0 *is the* a_{-1} *coefficient of the Laurent Series.*

Theorem 2.27 (Residue Theorem - Child's Version)**.** *If* $0 < s < r$*,* r *as above, then* $a_{-1} = \frac{1}{2\pi i}\int_{|z-z_0|=r} f(z)dz$

Theorem 2.28 (Residue Theorem)**.** *Suppose* f *is analytic on a simply connected domain except for a finite number of isolated singularities* $z_1, \ldots, z_n \in D$.

Let γ *be a positively oriented smooth simple closed curve not passing through* $z_1, \ldots, z_n$*. Then*

$$\int_\gamma f(z)dz = 2\pi i \sum_{z_i \in D} \mathrm{Res}(f; z_i)$$

3 Analytic Functions as Mappings

Theorem 3.1. *Assume* $f(z)$ *is analytic in a simply-connected domain,* $f(z)$ *is not constant, and* C *is a simple closed curve in the domain. If* $f(z) \neq 0$ *on* C *then the number of zeros inside* C*, with multiplicity, is* $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz$

Theorem 3.2 (Rouché's Theorem)**.** *Small perturbations in the function don't change the location or number of roots much. Or, formally,*

Assume $f(z)$ *is analytic in a simply connected domain,* $f(z)$ *not always zero,* C *a simple closed curve in the domain with* $f(z)$ *not zero on* C*. Then, if the perturbation* $g(z)$ *is analytic and* $|g(z)| < |f(z)|$ *on* C*, then* $f(z)$ *and* $f(z)+g(z)$ *have the same number of zeros inside* C*.*

Theorem 3.3 (Riemann Mapping Theorem)**.** *Every simply connected domain in* $\mathbb{C}$*, except* $\mathbb{C}$*, can be mapped conformally (one-to-one and analytically) onto the unit disk.*

The standard conformal mappings are the linear fractional transformations and the powers of z. Linear fractional transformations are also called Möbius Transformations.

Lemma 3.4 (Schwarz's Lemma)**.** *Suppose* f *is analytic and* f *takes* $|z| < 1 \rightarrow |z| < 1$ *and* $f(0) = 0$*. Then,* $|f(z)| \leq |z|$ *for all* z *in* $|z| \leq 1$ *and* $|f'(0)| \leq 1$