1 Space Curves

Definition 1.1 (Curve)**.** *A continuous function* $\alpha : [a,b] \to \mathbb{R}^3$ *is a curve* $\alpha(t) = (x(t), y(t), z(t))$*, and each component is continuous.*

Definition 1.2 (Differentiable Curve)**.** α *is a differentiable curve if* x, y, z *are differentiable.*

Definition 1.3 (Plane Curve)**.** *Curves* $\alpha : [a,b] \to \mathbb{R}^2$ *are called plane curves.*

Definition 1.4 (Regular Curve)**.** *A curve is regular if the vector* $\alpha'(t) \neq \vec{0}$*, that is,* $|\alpha'(t)| \neq 0$

Theorem 1.1. *The arclength along a curve is* $s = \int_a^b |\alpha'(t)| dt$

Proof. $\sum |\alpha(t_i) - \alpha(t_{i-1})|$ is a first approximation.

By the mean value theorem, this is equal to $\sum |\alpha'(t_c)||t_i - t_{i-1}|$.

If we take the limit as $(t_i - t_{i-1}) \to 0$, then we get $\int_a^b |\alpha'(t)| dt$ □

Definition 1.5 (Unit Tangent Vector)**.** *Let* $\alpha(t)$ *be a regular curve. Then the unit tangent vector is* $T(t) = \frac{\alpha'(t)}{|\alpha'(t)|}$

Definition 1.6 (Curvature)**.** *Let* $T(s)$ *be the unit tangent vector parametrized by arclength.*

Then $\frac{dT}{ds} = k(s)\vec{n(s)}$ *with* $k(s) > 0$

$k(s) = \left|\frac{dT}{ds}\right|$ *is called the curvature, with* $\vec{n}(s)$ *as the unit normal vector.*

Definition 1.7 (Dot Product)**.** *Let* $\vec{u}, \vec{v} \in \mathbb{R}^3$*.*

$\vec{u} \cdot \vec{v} = x_1 x_2 + y_1 y_2 + z_1 z_2$

Lemma 1.2. $\frac{d}{dt}(\vec{u} \cdot \vec{v}) = u' \cdot v + u \cdot v'$

Corollary 1.3. *Given a unit vector, ie* $|\vec{u}| = 1$ *for* $t \in I$*, then* $\frac{d\vec{u}}{dt}$ *is orthogonal to* $\vec{u}(t)$

Theorem 1.4. *Let* $\alpha(s)$ *be parametrized by arclength. Then* $|\alpha'(s)| = 1$*.*

Corollary 1.5. α' *and* α'' *are orthogonal.*

Definition 1.8 (Singular Points of the First Kind)**.** *Places where* $k(s) = 0$ *are singular points of the first kind.*

Definition 1.9 (Bi-normal Vector)**.** *The bi-normal vector* $b(s)$ *is defined as* $b(s) = T(s) \wedge n(s)$ *where* $\wedge$ *signifies the cross product.*

By this definition, the binormal vector is orthogonal to both the unit tangent vector and the unit normal vector, and is itself a unit vector.

Definition 1.10 (Frenet Trihedron)**.** *The Frenet Trihedron is the set of orthonormal vectors* $b(s), T(s), n(s)$*.*

Lemma 1.6. $b'(s) = \tau(s) n(s)$

Proof. $b'(s) = \frac{d}{ds} T(s) \wedge n(s) = T'(s) \wedge n(s) + T(s) \wedge n'(s)$

$T(s) = k(s)n(s)$, so $b'(s) = T(s) \wedge n'(s)$, and so $b'(s)$ is perpendicular to both $T(s)$ and $b(s)$ (because $b(s)$ is a unit vector) so $b'(s) = \tau(s)n(s)$. □

Definition 1.11 (Torsion)**.** *$\tau(s)$ is called the torsion of the curve.*

Lemma 1.7. *If $\tau(s) \equiv 0$, then α lies in a plane.*

Proof. First, fix the plane Π.

Pick a point $\alpha(s_0)$ on the curve and let that be the origin and let b_0 be the binormal vector, which is constant because $\tau \equiv 0$.

Our curve lies in the plane Π.

Let $f(S) = \alpha(s) \cdot b_0$

$f(0) = 0$, so if the function has a zero derivative, then we are done.

$f'(s) = \alpha'(s) \cdot b_0 + \alpha(s) \cdot 0 = \alpha'(s) \cdot b_0$, but, as $\alpha'(s) = T(s)$, $f'(s) = 0$. □

Theorem 1.8 (Frenet Relations)**.** *The Frenet Relations are*

1. $\frac{dT}{ds} = k(s)n(s)$
2. $\frac{db}{ds} = \tau(s)n(s)$
3. $\frac{dn}{ds} = -k(s)T(s) - \tau(s)b(s)$

Proof. The first two Frenet Relations are either previously defined or proved.

As $\frac{dn}{ds}$ is perpendicular to $n(s)$, it is $\frac{dn}{ds} = a_1(s)T(s) + a_2(s)b(s)$.

$n' \cdot T = \alpha_1 \Rightarrow (T \cdot n)' - T' \cdot n = a_1 \Rightarrow -T' \cdot n = a_1 \Rightarrow -kn \cdot n = a_1 \Rightarrow a_1(s) = -k(s)$

$n' \cdot b = \alpha_2 \Rightarrow (b \cdot n)' - b' \cdot n = a_2 \Rightarrow -b' \cdot n = a_2 \Rightarrow -\tau n \cdot n = a_2 \Rightarrow a_2(s) = -\tau(s)$ □

Theorem 1.9 (Fenchel-Milner Theorem)**.** *Take a simple, closed, space curve c.*

$\int_c k(s)ds \geq 2\pi$ and if $\int_c k(s)ds = 2\pi$ then the curve is a circle. Also, if c is knotted, then $\int_c k(s)ds \geq 4\pi$

Theorem 1.10 (Fundamental Theorem for Curves)**.** *Given $k(s) > 0$ and $\tau(s)$, then $\exists! \alpha(s)$ such that it has curvature $k(s)$ and torsion $\tau(s)$, up to a rigid motion, that is, $Ax + b$ where A is a matrix such that $A^tA = AA^t = I$, b is a vector, and x is an arbitrary point in space.*

Proof. We are given $k(s) > 0$ and $\tau(s)$, some function which is continuous. We will construct a curve $\alpha(s)$ such that $\alpha(s)$ passes through p and whose unit tangent vector points in the direction v_0.

We have the Frenet Relations, and if we let $W(s) = \begin{pmatrix} T(s) \\ n(s) \\ b(s) \end{pmatrix}$, and with this we can rewrite the Frenet Relations, with an initial condition, as:

$$\frac{dW}{dt} = \begin{pmatrix} 0 & k(s) & 0 \\ -k(s) & 0 & -\tau(s) \\ 0 & \tau(s) & 0 \end{pmatrix} W \qquad W(0) = \begin{pmatrix} v_0 \\ v_0' \\ v_0 \wedge v_0' \end{pmatrix}$$

This is a system of linear ordinary differential equations, and so a solution exists, and $\alpha(s) = \int_0^s T(u)du + p$

Now, we attempt to prove that this solution is unique.

We will translate the starting points of two such curves so that they coincide, rotate them so that their initial Frenet Trihedrons coincide, and use the fact that under rigid motions, $k(s), \tau(s)$ and arclength are invariant.

Let M be a rigid motion such that

$$Mx = M \begin{pmatrix} x_1 \\ x_2 \\ x_2 \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \\ x_2 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \\ c_2 \end{pmatrix}$$

Then, consider $M\alpha(s) = \beta(s)$. Then, α, β have the same $k(s), \tau(s)$.

$\beta'(s) = A\alpha'(s)$, so $\beta' \cdot \beta' = |\beta'|^2 = A\alpha \cdot A\alpha = A^t A\alpha' \cdot \alpha' = I\alpha' \cdot \alpha' = |\alpha'|^2$

Consider (T, n, b) for α_1 and $(\overline{T}, \overline{n}, \overline{b})$ for α_2, with $T(0) = \overline{T}(0)$, $n(0) = \overline{n}(0)$, $b(0) = \overline{b}(0)$

We claim that having the same T, n, b at $s = 0$ implies that they are the same for all s.

Let $f(s) = |T - \overline{T}|^2 + |n - \overline{n}|^2 + |b - \overline{b}|^2$

Note, $f(0) = 0$, so we must show that $f(s) \equiv 0$.

So, if $f'(s) = 0$, then $f(s) = c$, but as $f(0) = 0$, we must have $c = 0$.

$f'(s) = 2\left(\left\langle T - \overline{T}, T' - \overline{T}'\right\rangle + \langle n - \overline{n}, n' - \overline{n}'\rangle + \left\langle b - \overline{b}, b' - \overline{b}'\right\rangle\right)$

And, by the Frenet Relations, this is 0.

Thus, $\alpha_1'(s) = T(s) = \overline{T}(s) = \alpha_2'(s)$, so, by integration, $\alpha_1 = \alpha_2 + c$, but $\alpha_1(0) = \alpha_2(0)$, so $\alpha_1 = \alpha_2$ □

So, now we have a global result.

Theorem 1.11 (Isoperimetric Inequality)**.** *Let Γ be a simple, closed, plane curve. Let Γ enclose a region D with finite area A. Let length of Γ be denoted $|\Gamma| = L$. Then, $4\pi A \leq L^2$, and equality holds if and only if Γ is a circle.*

Lemma 1.12 (Geometric-Arithmetic Mean Inequality)**.** $\sqrt{ab} \leq \frac{a+b}{2}$

Lemma 1.13 (Cauchy-Schwartz Inequality)**.** $|\langle u, v\rangle| \leq |u||v|$

Theorem 1.14 (Green's Theorem in the Plane)**.** *Given $P(x, y)$ and $Q(x, y)$ and a simple closed curve Γ enclosing a region D, then*

$$\oint_\Gamma P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx dy$$

Now, we are ready to prove the Isoperimetric Inequality.

Proof. Step 1 is to slide two parallel lines towards D until the fixed point of contact.

Step 2 is to set up coordinates and a circle of radius r.

Step 3 is to Parametrize Γ by arclength s so $\Gamma = (x(s), y(s)) = \alpha(s)$ so $|\alpha'(s)| = 1$

The circle C is also parametrized as $(x(s), \overline{y}(s))$

Step 4 is to apply Green's Theorem

$A = \oint_\Gamma x \frac{dy}{ds} ds = \oint_\Gamma x ds$ and $\pi r^2 = -\oint_C \overline{y} \frac{dx}{ds} ds$

So, $A + \pi r^2 = \int_0^C \left(x\frac{dy}{ds} - \overline{y}\frac{dx}{ds}\right) ds \leq \int_0^L \left|x\frac{dy}{ds} - \overline{y}\frac{dx}{ds}\right| ds$

By the Cauchy-Schwartz Inequality, this is equal to

$$\int_0^L (x^2+\overline{y}^2)^{1/2}\left(\frac{dx}{ds}^2+\frac{dy}{ds}^2\right)^{1/2} ds = r\int_0^L ds = rL$$

By the Geometric-Arithmetic Mean Inequality,we have that

$$\sqrt{A\pi r^2} \leq \frac{A+\pi r^2}{2} \leq \frac{rL}{2} \tag{1}$$

, and so $A\pi r^2 \leq \frac{r^2L^2}{4} \Rightarrow 4A\pi \leq L^2$.

Assume that $4A\pi = L^2$. Then, we need that Γ is a circle. So, we show that if $4\pi A = L^2$, then $r = f(L, A)$. From (1), we have that if $L^2 = 4A\pi$, then $\sqrt{A\pi r^2} = \frac{rL}{2}$, then $r = \frac{L\pm\sqrt{L^2-4\pi A}}{2\pi} = \frac{L}{2\pi}$

And so,

$$A+\pi r^2 = \int_0^L (xy' - \overline{y}x')\, ds = \int_0^L \left(x^2+\overline{y}^2\right)^{1/2}\left(x'^2+y'^2\right)^{1/2} ds$$

Thus, $(xy' - \overline{y}x')^2 = (x^2+\overline{y}^2)(x'^2+y'^2) \Rightarrow (xx' + \overline{y}y')^2 = 0$

So, $xx' + \overline{y}y' = 0$, then we divide by $x'y'$ to get $\frac{x}{y'} = \frac{-\overline{y}}{x'} = a$.

So, $-\overline{y} = ax'$ and $x = ay'$. This means that $\overline{y}^2 = a^2x'^2$ and $x^2 = a^2y'^2$. So if we add the two equations together, we get that $x^2+\overline{y}^2 = a^2(x'^2+y'^2) = a^2$

Thus, $\sqrt{x^2+\overline{y}^2} = \pm a$, meaning that $a = \pm r$, so $\frac{x}{y'} = \frac{-\overline{y}}{x'} = \pm r$

So, $x = \pm ry'$, and my changing the directions of the parallel lines, we can just switch x and y, so $y - y_0 = \pm rx'$.

Therefore, $(y-y_0)^2 + x^2 = r^2$ □

2 Regular Surfaces

We want to be able to do multivariable calculus on a curved surface. But first we need to discuss defining coordinates on them. Because there are no natural coordinates, we need to find the interesting geometric quantities that are coordinate invariant. One such property is Gauss Curvature.

Definition 2.1 (Regular Surfaces). *A set S in $\mathbb{R}^3$ such that for every point in S we can find a ball, V, in $\mathbb{R}^3$ and a map Φ from an open set U in $\mathbb{R}^2$ to $V \cap S$ such that*

1. $\Phi : U \to V \cap S$ *is in* C^1
2. $\Phi^{-1} : V \cap S \to U$ *is in* C^0
3. $d\Phi$ *is one to one, that is, it has full rank*

   (in the case of surfaces, $\operatorname{rank} d\Phi = 2$*)*

It is clear that $d\Phi = \begin{pmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{pmatrix}$

Regular Surfaces generalize to $n-$Manifolds, and are the special case where $n = 2$

Theorem 2.1. *A surface is regular if* $(x_u, y_u, z_u) \wedge (x_v, y_v, z_v) \neq \vec{0}$*, that is,* $\Phi_u \wedge \Phi_v \neq \vec{0}$

Theorem 2.2. *A graph is a regular surface, that is,* $z = f(x, y)$ $\quad y = g(x, z)$ *and* $x = h(y, z)$

Proof. Assume the graph is of the form $z = f(x, y)$.

$f \in C^1$, so $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are continuous.

U is the domain of f. So, let $\Phi = (u, v, f(u, v))$. Then $x(u, v) = u$, $y(u, v) = v$ and $z(u, v) = f(u, v)$. As these are all in C^1, their partials exist, so Φ_u and Φ_v exist and are continuous. By simple calculation, we have $\Phi_u \wedge \Phi_v \neq \vec{0}$, so z is regular. □

Corollary 2.3. *The Sphere,* S^2*, is a regular surface.*

Consider $F : W \subset \mathbb{R}^3 \to \mathbb{R}$ and $t = F(x, y, z)$ for $(x, y, z) \in W$.

Definition 2.2 (Critical Point)**.** *Any point* (x_0, y_0, z_0) *with* $\nabla F(x_0, y_0, z_0) = 0$ *is called a critical point.*

Definition 2.3 (Critical Value)**.** *The value at a critical point is a critical value.*

Definition 2.4 (Regular Point)**.** *A noncritical point is a regular point.*

Definition 2.5 (Regular Value)**.** *The value at a regular point is a regular value.*

Theorem 2.4. *Consider the level at* $S_a = \{(x, y, z) : F(x, y, z) = a\}$*. If* a *is a regular value, then* S_a *is a regular surface.*

Corollary 2.5. $\mathbb{T}^2$*, ie the two dimensional genus one torus, is a regular surface.*

Theorem 2.6 (Inverse Function Theorem)**.** *Let* $F : U \subset \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (f(x, y), g(x, y))$*. If* $\begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$ *is invertible, then* F *has a local inverse.*

Theorem 2.7. *Let* S *be a regular surface.*

Let Φ *be a parametrization for* $p \in S$.

$\Phi : U \to V \cap S$, $p \in V \cap S$, *and with the first and third conditions of* Φ *verified, then if* Φ^{-1} *exists, it is continuous.*

Theorem 2.8. *Given a regular surface* S *and any point* $p \in S$*, there is a neighborhood* V*,* $p \in V \subset S$ *such that the piece of the surface in* V *may be viewed as a graph of the type* $z = f(x, y)$ *or* $y = g(x, z)$ *or* $x = h(y, z)$

Proof. Our surface is regular, so we can assign local coordinates $(u, v) \in U$.

$\Phi(u, v) = (x(u, v), y(u, v), z(u, v))$

We know $\Phi_u \wedge \Phi_v \neq 0$, so let us assume it is the z component that is nonzero.

Then, $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \neq 0$

Consider $\Psi : (u, v) \mapsto (x(u, v), y(y, v))$ with $U \to \mathbb{R}^2$

By the inverse function theorem, Ψ^{-1} exists. □

Definition 2.6 (Tangent Plane)**.** T_pS *=tangent plane=* $\{$*all tangent vectors of* S *at* $p\}$

Theorem 2.9. *Let T be a tangent vector to S. Then, there exists an a in $\mathbb{R}^2$ such that*

$$\begin{bmatrix} \frac{\partial x}{\partial u}(u_0,v_0) & \frac{\partial x}{\partial v}(u_0,v_0) \\ \frac{\partial y}{\partial u}(u_0,v_0) & \frac{\partial y}{\partial v}(u_0,v_0) \end{bmatrix} a = T$$

Proof. Consider $(u_0, v_0) + t\vec{v} = \alpha(t)$, $\alpha'(0) = \vec{v}$

Consider $\Phi(\alpha(t)) = \beta(t)$

$\Phi(\alpha(0)) = \Phi(u_0, v_0) = p$

$\beta'(0) = d\Phi(\alpha'(0)) = d\Phi(\vec{v})$

So, the set of tangent vectors $T_pS \supseteq \{d\Phi(p)\begin{pmatrix} a \\ b \end{pmatrix}\}$

Let $w \in T_pS$.

$w = \beta'(0)$, where $\beta(t)$ is a C^1 curve on S with $\beta(0) = 0$

Consider $\alpha(t) = \Phi^{-1}(\beta(t)) \Rightarrow \frac{d}{dt}\Phi(\alpha(t)) = \frac{d}{dt}\beta(t)$

So, $d\Phi\begin{pmatrix} a \\ b \end{pmatrix} = d\Phi(\alpha'(0)) = \beta'(0) = w$ □

Remark - T_pS is a $2-$dimensional vector space.

Definition 2.7 (Coordinate Curves)**.** *Let $p \in S$ and (u_0, v_0) with $\Phi(u_0, v_0) = p$. Call this the center point. Consider $\alpha(t) = (u_0, v_0 + t)$ and $\beta(t) = (u_0 + t, v_0)$*

Then, $\Phi(\alpha(t)), \Phi(\beta(t))$ are the coordinate curves through p.

Remark 2.1. *So, $\Phi(\alpha(t)) = (x(u_0, v_0 + y), y(u_0, v_0 + y), z(u_0, v_0 + y))$, then $\frac{d}{dt}\Phi(\alpha(t))|_{t=0} = (x_v(u_0, v_0), y_v(u_0, v_0), z_v(u_0, v_0)) = \Phi_v(u_0, v_0)$*

And similarly with β and u.

So, a basis for T_pS is Φ_v, Φ_u at (u_0, v_0).

Definition 2.8 (Diffeomorphism)**.** *Given 2 regular surfaces S_1, S_2, assume there is a map $F : S_1 \to S_2$.*

If F is a bijection and, given $p \in S$, let u, v be local coordinates etcetera, and $\Psi^{-1} \circ F \circ \Phi(u, v)$ is differentiable and the differential is invertible, then F is a diffeomorphism.

$dF = \begin{pmatrix} \frac{\partial G_1}{\partial u} & \frac{\partial G_2}{\partial u} \\ \frac{\partial G_1}{\partial v} & \frac{\partial G_2}{\partial v} \end{pmatrix}$

So, we can say that if F is a diffeomorphism, we have $dF : T_p(S_1) \to T_{F(p)}(S_2)$

Definition 2.9 (First Fundamental Form)**.** *Let S be a regular surface. Let $E = \Phi_u \cdot \Phi u$, $F = \Phi_u \cdot \Phi_v$, $G = \Phi_v \cdot \Phi_v$.*

Then, $\frac{ds}{dt}^2 = E\frac{du}{dt}^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G\frac{dv}{dt}^2$, and so we define $ds^2 = Edu^2 + 2Fdudv + Gdv^2$ as the first fundamental form.

Definition 2.10 (Orthogonal Parametrization)**.** *A parametrization is orthogonal if $\langle \Phi_u, \Phi_v \rangle = 0$*

Definition 2.11 (Isothermal Coordinates)**.** *Coordinates are called isothermal if $ds^2 = E(u, v)\left(du^2 + dv^2\right)$*

Theorem 2.10. *On a regular surface, isothermal coordinates always exist.*

Theorem 2.11. *$dA = \sqrt{EG - F^2}dudv$ and does not depend on the coordinates.*

Remark 2.2. *For a graph $z \in f(x,y)$, $f \in C^1$,*

$$ds^2 = (1+f_x^2)dx^2 + 2f_xf_ydxdy + (1+f_y^2)dy^2 = \left[1+|\nabla f|^2\right]^{1/2} dxdy$$

2.1 Rhomb Lines, or, Loxodromes

(Theory of discontinuous groups, loxodromic/automorphic forms.)

Definition 2.12 (Hyperbolic Metric)**.** *The metric on the Poincare half plane is $ds^2 = \frac{dx^2+dy^2}{y^2}$, and is called the hyperbolic metric.*

Theorem 2.12 (Hilbert, 1960s Efimov)**.** *If $K \le \delta < 0$ an the 2-Manifold is complete, then one cannot embed it in $\mathbb{R}^3$*

Definition 2.13 (Rhomb Line/Loxodrome)**.** *A curve in S^2 is called a rhomb line, or loxodrome, if and only if it intersects every meridian at a constant angle β.*

3 The Gauss Map

Definition 3.1 (Unit Normal of a surface)**.** *Let S be a regular surface. We define the unit normal as $N(u,v) = \frac{\Phi_u \wedge \Phi_v}{|\Phi_u \wedge \Phi_v|}$*

Definition 3.2 (Gauss Map)**.** *The Gauss Map is $N : S \to S^2 : p \mapsto N(p)$ and $dN : T_pS \to T_{N(p)}S^2 \approx T_pS$*

Definition 3.3 (Endomorphism)**.** *A homomorphism from a group into itself is an endomorphism.*

Theorem 3.1. *The differential of the Gauss Map is an endomorphism.*

Definition 3.4 (Ends)**.** *On the punctured sphere, the punctures are called ends.*

Theorem 3.2. *The map $dN : T_pS \to T_pS$ is a self-adjoint linear map.*

Proof. It is clearly linear, so all we need is $\langle dN(v), w)\rangle = \langle v, dN(w)\rangle$

We have the equations

$$\langle dN(\Phi_u), \Phi_u\rangle = \langle \Phi_u, dN(\Phi_u)\rangle$$

and

$$\langle dN(\Phi_v), \Phi_v\rangle = \langle \Phi_v, dN(\Phi_v)\rangle$$

Thus, we only need to show that $\langle dN(\Phi_u), \Phi_v\rangle = \langle \Phi_u, dN(\Phi_v)\rangle$, that is $\langle \frac{\partial N}{\partial u}, \frac{\partial \Phi}{\partial v}\rangle = \langle \frac{\partial \Phi}{\partial u}, \frac{\partial N}{\partial v}\rangle$

We not that $f(u,v) = \langle N, \Phi_u\rangle = 0$, so $\langle N_v, \Phi_u\rangle + \langle N, \Phi_{uv}\rangle = 0$ and $\langle N_u, \Phi_v\rangle + \langle N, \Phi_{uv}\rangle = 0$. Subtracting one from the other gives the result. □

Definition 3.5 (Quadratic Form)**.** *If A, is a matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then we say $\langle Av, v\rangle = Q(v,v)$ is a quadratic form.*

Definition 3.6 (The Second Fundamental Form)**.** *$II_p(w) = -\langle dN(w), w\rangle$ is called the second fundamental form.*

If $w = x\Phi_u + y\Phi_v$ then $II_p(w) = ex^2 + 2fxy + gy^2$ where $e = -\langle N_u, \Phi_u\rangle$, $f = -\langle N_u, \Phi_v\rangle$, $g = -\langle N_v, \Phi_v\rangle$.

Definition 3.7 (Normal Curvature). *If $|w| = 1$, $p \in S$ and $w \in T_pS$, then we call $II_p(w)$ the normal curvature.*

So, $II_p(w) = k\cos\theta$ where k is the curvature of a curve on the surface and θ hte angle between $N(u,v)$ and the normal to the curve at p.

Theorem 3.3 (Meusnier). *Curves passing through p and having the same tangent vector have the same normal curvature.*

Proof. Let $\alpha(s)$ be a curve $\alpha : I \to S$ in arclength parametrization, with $\alpha(0) = p$, $\alpha'(0) = w$.

Note, that $\langle N(s), \alpha'(s)\rangle = 0$. Taking the derivative, we get $\langle \frac{dN}{ds}, \alpha'(s)\rangle + \langle N(s), \alpha''(s)\rangle = 0$, so $k\cos\theta = -\langle \frac{dN(s)}{ds}, \alpha'(s)\rangle = -\langle dN(w), w\rangle = II_p(w)$ □

Lemma 3.4. *Let $w = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$. Consider $Q(\cos\theta, \sin\theta)$.*

Then $f(\theta) = a\cos^2\theta + 2b\cos\theta\sin\theta + c\sin^2\theta$, if $\max_{x\in[0,2\pi]} f(\theta) = f(0)$ then $b = 0$.

Proof. $f(\theta)$ is periodic, so we look at it on $[-\pi, \pi]$. Then $f'(0) = 0$, so $f'(0) = 2b\cos 2\theta|_{\theta=0} = 2b = 0$ □

If this is the case, then $Q(x,y) = ax^2 + cy^2$, so $f(\theta) = a\cos^2\theta + c\sin^2\theta$ and $f(0) = a$. Note $f(\pi/2) = c \le a$.

Notice now that $f(\theta) = a\cos^2\theta + c\sin^2\theta \ge c\cos^2\theta + c\sin^2\theta = c$, so c is the minimum value, and is at a right angle to a.

Application

Consider $w \in T_pS$. $w = \cos\theta\Phi_u + \sin\theta\Phi_v$.

$-\langle dN(w), w\rangle = e\cos^2\theta + 2f\cos\theta\sin\theta + g\sin^2\theta = f(\theta)$, where

$$e = -\langle dN(\Phi_u), \Phi_u\rangle = -\langle \frac{\partial N}{\partial u}, \frac{\partial \Phi}{\partial u}\rangle, f = -\langle \frac{\partial N}{\partial u}, \frac{\partial \Phi}{\partial v}\rangle \text{ and } g = -\langle \frac{\partial N}{\partial v}, \frac{\partial \Phi}{\partial v}\rangle$$

.

Assume that $\max_{\theta\in[0,2\pi]} f(\theta) = f(\theta_0)$. Rotate the axes so that $\theta_0 = 0$.

So $\begin{pmatrix} \cos\theta_0 & \sin\theta_0 \\ -\sin\theta_0 & \cos\theta_0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u' \\ v' \end{pmatrix}$, so $w = \cos\theta_0\Phi_u + \sin\theta_0\Phi_v$. That is, $w = \Phi_u$ if $\theta_0 = 0$.

$-\langle dN(w), w\rangle = e$, so we let $\tilde{w}$ be orthogonal to w in T_pS. Now we claim that w and $\tilde{w}$ are principal directions. That is, $-dN(w) = k_1 w$ and $-dN(\tilde{w}) = k_2\tilde{w}$. k_1, k_2 are called the principal curvatures.

Proof. Note that $\{w, \tilde{w}\}$ is a basis for T_pS. So we consider $\langle dN(w), \tilde{w}\rangle$. Recall that $w = \Phi_u$ and $\tilde{w} = \Phi_v$. We claim that $\langle dN(w), \tilde{w}\rangle = 0$.

$b = 0 = f$ so $f = \langle dN(\Phi_u), \Phi_v\rangle = 0$.

$dN(\Phi_u) = c_1\Phi_u + c_2\Phi_v$, so $0 = \langle dN(\Phi_u), \Phi_v\rangle = c_1\langle \Phi_u, \Phi_v\rangle + c_2\langle \Phi_v, \Phi_v\rangle$.

We will now check that $w, \tilde{w}$ are the principal directions:

Step 1: Let us call $\Phi_u = e_1, \Phi_v = e_2$. Choose the direction in the tangent plane such that the quadratic form $f(\theta) = e\cos^2\theta + 2f\cos\theta\sin\theta + g\sin^2\theta$ has a maximum in the direction $\theta = 0$. Then $w(\theta) = e_1\cos\theta + e_2\sin\theta$ gives $-\langle dN(w), w\rangle = e\cos^2\theta + 2f\sin\theta\cos\theta + g\sin^2\theta$.

Step 2: At p consider $\tilde{w}$, the orthogonal direction. What is f for the pair $\{w, \tilde{w}\}$? $f = 0\langle dN(\Phi_u), \Phi_v\rangle = -\langle dN(e_1), e_2\rangle = -\langle dN(w), \tilde{w}\rangle = 0$.

Step 3: Note that $dN(w) = c_1 w + c_2\tilde{w}$, thus $0 = \langle dN(w), \tilde{w}\rangle = c_1\langle w, \tilde{w}\rangle + c_2\langle \tilde{w}, \tilde{w}\rangle = c_2$. □

Definition 3.8 (Line of Curvature)**.** *A line of curvature C is a curve on S such that for all points on C, the tangent direction is principal.*

Theorem 3.5 (O. Rodriguez)**.** *A curve $\alpha(t)$ is a line of curvature iff $\frac{dN}{dt} = k(t)\alpha'(t)$, where $N(t) = N(\alpha(t))$.*

Definition 3.9 (Gauss Curvature)**.** *The Guass curvature at $p \in S$ is $K = k_1 k_2$.*

Definition 3.10 (Mean Curvature)**.** *The mean curvature if $H = \frac{k_1+k_2}{2}$.*

Definition 3.11 (Classification of Points)**.** *A point $p \in S$ is said to be elliptic if $K(p) > 0$, hyperbolic if $K(p) < 0$, parabolic if $K(p) = 0$ but either k_1 or k_2 is not 0, and planar if $k_1 = k_2 = 0$.*

3.1 Euler Formula

Let e_1, e_2 be principal directions, and pick $v \in T_pS$ such that $|v| = 1$. Then $v = \cos\theta e_1 + \sin\theta e_2$. That is, $\{e_1, e_2\}$ is an orthonormal basis of T_pS.

Recall: $-\langle dN(v), v\rangle$ =normal curvature $k_n = K\cos\phi$. So we take

$$-\langle dN(e_1\cos\theta + e_2\sin\theta), e_1\cos\theta + e_2\sin\theta\rangle =$$

$$-\langle dN(e_1)\cos\theta + dN(e_2)\sin\theta, e_1\cos\theta + e_2\sin\theta\rangle =$$

$$\langle k_1\cos\theta e_1 + k_2\sin\theta e_2, e_1\cos\theta + e_2\sin\theta\rangle = k_1\cos^2\theta + k_2\sin^2\theta = k_n$$

.

Definition 3.12 (Umbilic Points)**.** *A point $p \in S$ is said to be umbilic iff $k_1(p) = k_2(p)$*

At an umbilic point, every direction is principal.

Theorem 3.6. *Let S be a connected surface with every point umbilic. Then S is either a piece of a sphere or a flat plane.*

Proof. $-\frac{\partial N}{\partial u} = -dN(\Phi_u) = k(u,v)\Phi_u$, $-\frac{\partial N}{\partial v} = -dN(\Phi_v) = k(u,v)\Phi_v$.

Now we look at $\frac{\partial - dN(\Phi_u)}{\partial v} - \frac{\partial - dN(\Phi_v)}{\partial u}$.

So $0 = k_v\Phi_u - k_u\Phi_v$, so $k_v\Phi_u = k_u\Phi_v$. Thus, $k_u = k_v = 0$, so $k = c$. By connectedness, we have $k = c$ everywhere.

Now, if $k = 0$, then $\frac{\partial N}{\partial v} = \frac{\partial N}{\partial u} = 0$, so $N(u,v) = N_0$, so S is flat.

If $k = c \neq 0$, then $f(u,v) = \Phi(u,v) + \frac{1}{c}N(u,v)$. Thus $\frac{\partial f}{\partial u} = \Phi_u + \frac{1}{c}\frac{\partial N}{\partial u} = 0$ and $\frac{\partial f}{\partial v} = \Phi_v + \frac{1}{c}\frac{\partial N}{\partial v} = 0$. So $f(u,v) = \vec{v_0}$, adn $v_0 = \Phi(u,v) + \frac{1}{c}N \Rightarrow v_0 - \Phi(u,v) = \frac{1}{c}N$, so $|v_0 - \Phi(u,v)|^2 = \frac{1}{c^2}$, so S is a part of a sphere. □

Theorem 3.7 (Liebman)**.** *If S is a surface for which $K(p) = c > 0$, then S is a part of a sphere.*

Definition 3.13 (Asymptotic Direction)**.** *A direction $v \in T_pS$ is an Asymptotic Direction iff $\langle dN(v), v\rangle = 0$.*

Definition 3.14 (Asymptotic Curve)**.** *A curve $\gamma(t) \in S$ is called an asymptotic curve iff $\gamma'(t)$ is an asymptotic direction for all $t \in I$.*

Remark: If a point is elliptic, then there are no asymptotic directions.

Definition 3.15 (Minimal Surface)**.** *A Minimal Surface is a surface satisfying* $\frac{k_1+k_2}{2} = 0$ *for all* $p \in S$.

Theorem 3.8 (Osserman)**.** *If* S *is minimal and complete, and* $S^2 \setminus N(S)$ *is open, then* S *is the plane.*

Definition 3.16 (Conjugate directions)**.** *Let* $w_1, w_2 \in T_pS$. *We say* w_1, w_2 *are conjugate directions iff* $\langle dN(w_1), w_2\rangle = \langle dN(w_2), w_1\rangle = 0$.

Theorem 3.9. *If* w *is an asymptotic direction, then* w *is conjugate to itself.*

Theorem 3.10. *If* e_1, e_2 *are principal, then they are conjugate.*

3.2 Dupin Indicatrix

Given $p \in S$, $\mathscr{D} = \{w \in T_pS : II_p(w) = \pm 1\}$.

Theorem 3.11. $\mathscr{D}$ *is the union of conics.*

Proof. Let e_1, e_2 be principal directions, then $w = r\cos\theta e_1 + r\sin\theta e_2$.

$II_p(w) = -\langle dN(w), w\rangle = \langle r\cos\theta k_1 e_2 + r\sin\theta k_2 e_2, r\cos\theta e_1 + r\sin\theta e_2\rangle = r^2\cos^2\theta k_1 + r^2\sin^2\theta k_2 = \pm 1$

Case 1: $k_1 > k_2 > 0$. Then $r^2\cos^2\theta k_1 + r^2\sin^2\theta k_2 = 1$. Let $\xi = r\cos\theta$ and $\eta = r\sin\theta$. Then $\xi^2 k_1 + \eta^2 k_2 = 1$, which is the equation of an ellipse.

Case 2: $k_1 > 0 > k_2$. Then $k_1\xi^2 - |k_1|\eta^2 = \pm 1$, so we get $k_1\xi^2 - |k_1|\eta^2 = 1$ and $k_1\xi^2 - |k_2|\eta^2 = -1$, a pair of hyperbolas. ☐

3.3 Gauss Map in Local Coordinates

Note: $dN(\Phi_u) = \frac{\partial N}{\partial u} = a_{11}\Phi_u + a_{12}\Phi_v \in T_pS$, $dN(\Phi_v) = \frac{\partial N}{\partial v} = a_{21}\Phi_u + a_{22}\Phi_v \in T_pS$.

This gives a matrix $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ with eigenvalues k_1, k_2 and eigenvectors the principal directsions.

Note $\det A = k_1k_2$ the Gauss curvature, and $\operatorname{tr} A/2 = \frac{k_1+k_2}{2}$, the Mean Curvature.

$$\begin{aligned} \langle N_u, \Phi_u\rangle &= -e = a_{11}E + a_{12}F \\ \langle N_u, \Phi_v\rangle &= -f = a_{11}F + a_{12}G \\ \langle N_v, \Phi_u\rangle &= -f = a_{21}E + a_{22}F \\ \langle N_v, \Phi_v\rangle &= -g = a_{21}F + a_{22}G \end{aligned}$$

The above equations tell us that $-\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} E & F \\ F & G \end{pmatrix}$, so we get the following theorem:

Theorem 3.12.

$$-\begin{pmatrix} e & f \\ f & g \end{pmatrix}\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

The Gauss Curvature is then $K(p) = \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. Thus $K(p) = \frac{eg-f^2}{EG-F^2}$.

Now, recall that $-dN(v) = k_1 v = k_1 I v$, and $-dN(w) = k_2 w$. So $(dN + kI)v = 0$, and so, in the basis $\{\Phi_u, \Phi_v\}$, we have $dN = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $dN+kI = \begin{pmatrix} a_{11}+k & a_{12} \\ a_{21} & a_{22}+k \end{pmatrix}$, and so $\det(dN + kI) = (a_{11} + k)(a_{22} + k) - a_{12}a_{21} = k^2 - 2kH + K$ where H is the mean curvature and K is the Gaussian curvature. Thus, $k = H \pm \sqrt{H^2 - K}$, and so, $\sqrt{|k_1||k_2|} \leq \frac{|k_1|+|k_2|}{2}$.

3.4 Equations of Weingarten

$\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} = \frac{1}{EG-F^2} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix}$

And so, we have

$$\begin{aligned} a_{11} &= \frac{fF - eG}{EG - F^2} \\ a_{12} &= \frac{eF - fE}{EG - F^2} \\ a_{21} &= \frac{gF - fG}{EG - F^2} \\ a_{22} &= \frac{fF - gE}{EG - F^2} \end{aligned}$$

Thus, $H = \frac{eG-2fF+gE}{2(EG-F^2)}$.

Theorem 3.13. *If (u, v) is a parametrization, $f = F = 0$ and p is not an umbilic point, then the coordinate curves are lines of curvature.*

Proof. Recall the above matrix equation.

$\Phi(u(t), v(t)) = \gamma(t)$, $\gamma' = u'\Phi_u + v'\Phi_v$. $dN(\gamma') = \lambda(t)\gamma'$ along a line of curvature.

$$dN(\gamma') = dN(u'\Phi_u + v'\Phi_v) = u'dN(\Phi_u) + v'dN(\Phi_v) = \lambda u'\Phi_u + \lambda v'\Phi_v =$$

$$u'\left(a_{11}\Phi_u + a_{12}\Phi_v\right) + v'\left(a_{21}\Phi_u + a_{22}\Phi_v\right),$$

so we must have $a_{11}u' + a_{12}v' = \lambda u'$ and $a_{21}u' + a_{22}v' = \lambda v'$.

That is, $\frac{fF-eG}{EG-F^2}u' + \frac{gF-fG}{EG-F^2}v' = \lambda(t)u'$ and $\frac{eF-fE}{EG-F^2}u' + \frac{fF-gE}{EG-F^2}v' = \lambda(t)v'$ are the equations of lines of curvature in general.

Assume that $f = F = 0$, $v(t) = c$, then $-\frac{e}{E} = \lambda(t)$.

Now we assume that the coordinate curves are lines of curvature. As we are at a nonumbilic point, we have $\langle \Phi_u, \Phi_v \rangle = F = 0$, and so the lines of curvature are $-e/E = \lambda_1(t)$ and $-f/G = 0$. so $f = 0$. □

Theorem 3.14. *A necessary and sufficient condition for coordinate curves to be asymptotic curves in a neighborhood of a hyperbolic point if $e = g = 0$.*

Proof. $II_p(w) = -\langle dN(\gamma'), \gamma' \rangle$, $\gamma'(t) = w = u'\Phi_u + v'\Phi_v$, so $II_p(w) = e(u')^2 + 2fu'v' + g(v')^2$, and we need this to be 0.

Assume $u(t) = t$ and $v(t) = c$. Then $e = g = 0$.

Assume $e = g = 0$, then we have $2fu'v' = 0$. Since $eg - f^2 < 0$, we have $f \neq 0$, so $u'v' = 0$, giving us the conclusion. □

Theorem 3.15 (S. Bernstein 1916)**.** *If S is a minimal surface such that it is a graph and is defined $\forall x, y \in \mathbb{R}$, then S is a plane.*

This theorem was extended in 1966 by F. Almgren and E. DiGiorgi to three dimensions, J. Simon in 1969 to $n \leq 7$ and E. Bombieri, E. DiGiorgi and Ginsti proved, in 1970, that it is false for $n \geq 8$.

Theorem 3.16. *If $K > 0$ at $p \in S$, then the surface sits to one side of the tangent plane at p.*

If $K < 0$ at $p \in S$, then the surface sits on both sides.

Proof. $\Phi(0,0) = p$. $(\Phi_u(u,v) - \Phi(0,0)) \cdot N$ at $(0,0)$ is

$$\big(u\Phi_u + v\Phi_v + u^2\Phi_{uu} + 2uv\Phi_{uv} + v^2\Phi_{vv} + \ldots\big) \cdot N$$

$= u^2\langle \Phi_{uu}, N\rangle + 2uv\langle \Phi_{uv}, N\rangle + v^2\langle \Phi_{vv}, N\rangle = eu^2 + 2fuv + gv^2$+error It is positive at p, so all is on one side.

The other part goes similarly. □

Theorem 3.17. *Let M be a compact surface, $\partial M = \emptyset$. Then $\exists$ at least one elliptic point.*

Remark: If M is compact and not homeomorphic to S^2, then M will in fact have both elliptic and hyperbolic points.

Proof. Pick $x_0 \notin M$ and consider a sphere $S_R(x_0)$. Claim: the first points touched as the sphere is contracted are elliptic.

Let $f(q) = |q - x_0|^2$ for $q \in M$. $f(q)$ achieves maxima at $x_1, \ldots$. Let $\alpha(s) \in M$ be any curve such that $\alpha(0) = x_1$. Then consider $g(s) = |\alpha(s) - x_0|^2 = \langle \alpha(s) - x_0, \alpha(s) - x_0\rangle$.

Note $g'(0) = 0$ and $g''(0) \leq 0$.

$g'(0) = 2\langle \alpha'(0), \alpha(0) - x_0\rangle = 2\langle \alpha'(0), x_1 - x_0\rangle = 0$, so $x_1 - x_0$ is normal to M at x_1.

$g'(s) = 2\langle \alpha'(s), \alpha(s) - x_0\rangle$, $g''(s) = 2\langle \alpha''(s), \alpha(s) - x_0\rangle + 2\langle \alpha'(s), \alpha'(s)\rangle$.

$g''(0) = 2\langle \alpha''(0), x_1 - x_0\rangle + 2 \leq 0$, so $\langle \alpha''(0), N(0)\rangle \leq -1$. We recall that $II_p(\alpha'(s)) = -\langle \frac{\partial N}{\partial s}, \alpha'(s)\rangle$, where $\frac{\partial N}{\partial s} = \frac{\partial}{\partial s}(N(\alpha(s)))$.

But $-\langle \frac{\partial N}{\partial s}, \alpha'\rangle = \langle N, \alpha''\rangle$, so $II_p(x_1) \leq -1$, so the principal curvatures are less than zero, so elliptic. □

4 Intrinsic Geometry of Surfaces

Let S_1, S_2 be surfaces, $\varphi : S_1 \to S_2$ a diffeomorphism. Take $p \in S_1$ and a coordinate neighborhood of p.

Look agt $\Phi_2^{-1} \circ \varphi \circ \Phi_1 : V \subset \mathbb{R}^2 \to W \subset \mathbb{R}^2 : (u,v) \mapsto (f(u,v), g(u,v))$ where f_u, f_v, g_u, g_v exists and are continuous, with $\begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}$ invertible for all $(u,v) \in V$.

Then φ is a diffeomorphism and if $f, g \in C^\infty$ then it is a smooth diffeomorphism.

Definition 4.1 (Isometry)**.** *An isometry is a map* $\varphi : S_1 \to S_2$ *such that* φ *is a diffeomorphism which preserves the inner product, that is,* $\langle v, w\rangle_{T_pS_1} = \langle d\varphi(v), d\varphi(w)\rangle_{T_{\varphi(p)}S_2}$*.*

Remark: if $\langle d\varphi(w), d\varphi(w)\rangle = |w|^2\ \forall w \in T_pS$, then $\langle d\varphi(w_1), d\varphi(w_2)\rangle = \langle w_1, w_2\rangle$. This is becuase $|d\varphi(w)|^2 = |w|^2$ gives $\langle d\varphi(w_1 + w_2), d\varphi(w_1 + w_2)\rangle = \langle w_1 + w_2, w_1 + w_2\rangle$.

Thus, $\langle d\varphi(w_1), d\varphi(w_1)\rangle + \langle d\varphi(w_2), d\varphi(w_2)\rangle + 2\langle d\varphi(w_1), d\varphi(w_2)\rangle = |w_1|^2 + |w_2|^2 + 2\langle w_1, w_2\rangle$.

Definition 4.2 (Local Isometry)**.** *Given two regular surfaces and a map* $\varphi : S_1 \to S_2$ *such that* φ *is a smooth map,* $\mathrm{Jac}(\varphi) = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}$ *is invertible for all* $(u, v) \in V$*, and* $\forall p \in S_1$*,* $\exists V_1 \subset S_2$ *a neighborhood of* p *and a corresponding neighborhood* V_2 *of* $\varphi(p)$ *in* S_2 *such that* $\varphi : V_1 \to V_2$ *is an isometry.*

Theorem 4.1. *Let* $\Phi : U \to S_1$*,* $\tilde{\Phi} : U \to S_2$*. Then* $E = \tilde{E}$*,* $F = \tilde{F}$*, and* $G = \tilde{G}$ *iff* $\tilde{\Phi}^{-1} \circ \Phi : U \to U$ *is a local isometry.*

Proof. Let $v \in T_{p_0}S_1$ and $w = a\Phi_u + b\Phi_v$. Let $\alpha(t) \subset S_1$, with $\frac{d}{dt}|_{t=0}\Phi(\alpha(t)) = w$.

$$\frac{d}{dt}|_{t=0}\tilde{\Phi}(\alpha(t)) = a\tilde{\Phi}_u + b\tilde{\Phi}_v = d\Psi(w)$$

$|d\Psi(w)|^2 = \langle d\Psi(w), d\Psi(w)\rangle = |a\tilde{\Phi}_u + b\tilde{\Phi}_v|^2 = a^2(\tilde{\Phi}_u \cdot \tilde{\Phi_u}) + 2ab(\tilde{\Phi}_u \cdot \tilde{\Phi}_v) + b^2(\tilde{\Phi}_v \cdot \tilde{\Phi}_v) = a^2\tilde{E} + 2ab\tilde{F} + b^2\tilde{G}$.

As $\tilde{E} = E$, $\tilde{F} = F$ and $\tilde{G} = G$, we have $a^2E + 2abF + b^2G = |w|^2$.

The converse is trivial. □

Definition 4.3 (Conformal Mapping)**.** *Let* $\varphi : S_1 \to S_2$ *be a diffeomorphism. If* $\langle d\varphi(w_1), d\varphi(w_2)\rangle_{T_{\varphi(p)}S_2} = \lambda^2(p)\langle w_1, w_2\rangle_{T_pS_1}$*, then* φ *is conformal.*

Theorem 4.2. *Two regular surfaces are locally conformal.*

Theorem 4.3 (Chow)**.** *Assume* $H \subset \mathbb{CP}^n$ *a complex analytic manifold. Then* H *is an algebraic variety.*

4.1 Christoffel Symbols

Consider a regular surface S, Φ a local chart and $\Phi_{uu} = a\Phi_u + b\Phi_v + cN = (\Phi_{uu} - \langle\Phi_{uu}, N\rangle N) + \langle\Phi_{uu}, N\rangle N$.

So $c = e$ from the second fundamental form, and a and b are called Christoffel Symbols.

Definition 4.4 (Christoffel Symbols)**.** *We write* $\Phi_{uu} = \Gamma^1_{11}\Phi_u + \Gamma^2_{11}\Phi_v + eN$*,* $\Phi_{uv} = \Gamma^1_{12}\Phi_u + \Gamma^2_{12}\Phi_v + fN$*, and,* $\Phi_{vv} = \Gamma^1_{22}\Phi_u + \Gamma^2_{22}\Phi_v + gN$ *and we call* Γ^k_{ij} *the Christoffel symbols.*

As $\Phi_{uv} = \Phi_{vu}$, we have $\Gamma^k_{ij} = \Gamma^k_{ji}$, and so there are six Christoffel symbols.

Lemma 4.4. *There are formulas for the Christoffel symbols in terms of E, F and G and their derivatives.*

Proof. By taking inner products, we get equations of the form $\langle \Phi_{uu}, \Phi_u \rangle = \frac{1}{2}E_u = \Gamma^1_{11}E + \Gamma^2_{11}F$, etcetera. By Cramer's Rule, we can solve for the Γ^k_{ij}. $\square$

Now, Gauss's equation implies the Theorema Egregium, and we will also demonstrate Codazzi's Equation.

Theorem 4.5 (Gauss's Equation)**.** *Assume that $E \neq 0$. Then we have*

$$-EK = (\Gamma^2_{11})_u - (\Gamma^2_{11})_v + \Gamma^1_{12}\Gamma^2_{11} + (\Gamma^2_{12})^2 - \Gamma^2_{11}\Gamma^2_{22} - \Gamma^2_{11}\Gamma^2_{12}$$

Proof. Observe that $(\Phi_{uu})_v = (\Phi_{uv})_u$, $(\Phi_{vv})_u = (\Phi_{uv})_v$ and $N_{uv} = N_{vu}$.

So we have $(\Phi_{uu})_v - (\Phi_{uv})_u = 0$, we can write this as $A_1\Phi_u + B_1\Phi_v + C_1 N = 0$, and since Φ_u, Φ_v, N are linearly independent, $A_1 = B_1 = C_1 = 0$. So we start with $\Phi_{uu} = \Gamma^1_{11}\Phi_u + \Gamma^2_{11}\Phi_v + eN$ and $\Phi_{uv} = \Gamma^1_{12}\Phi_u + \Gamma^2_{12}\Phi_v + fN$, and so $(\Phi_{uu})_v = (\Gamma^1_{11})_u\Phi_u + (\Gamma^2_{11})_v\Phi_v + e_vN + \Gamma^1_{11}\Phi_{uv} + \Gamma^2_{11}\Phi_{vv} + eN_v$, and $(\Phi_{uv})_u = (\Gamma^1_{12})_u\Phi_u + (\Gamma^2_{12})_v\Phi_v + f_uN + \Gamma^1_{12}\Phi_{uu} + \Gamma^2_{12}\Phi_{uv} + fN_v$.

We focus on B_1, the coefficient of Φ_v, and it is

$$\left((\Gamma^2_{11})_v - (\Gamma^2_{12})_u + \Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{12}\Gamma^2_{11} - \Gamma^2_{12}\Gamma^2_{12}\right)\Phi_v = 0$$

$\square$

Corollary 4.6 (Theorema Egregium)**.** *The Gauss Curvature of a surface is invariant under a local isometry.*

Proposition 4.7 (Codazzi-Mainardi Equations)**.** $e_v - f_u = e\Gamma^1_{12} + f(\Gamma^2_{12} - \Gamma^1_{11}) - g\Gamma^2_{11}$ *and* $f_v - g_u = e\Gamma^1_{22} + f(\Gamma^2_{22} - \Gamma^1_{12}) - g\Gamma^2_{12}$

Theorem 4.8 (Bonnet's Theorem)**.** *Given E, F, G, e, f, g and $EG - F^2 > 0$ which satisfy the Gauss and Codazzi-Mainardi Equations, then there exists a map $\Phi : (u, v) \in U \subset \mathbb{R}^2 \to \mathbb{R}^3$ such that Φ defines a regular surface with first fundamental form $Edu^2 + 2Fdudv + Gdv^2$ and second fundamental form $edu^2 + 2fdudv + gdv^2$.*

Additionally, this surface is unique up to rigid motion.

Definition 4.5 (Orthogonal Parametrization)**.** *A parametrization is orthogonal iff $F = 0$.*

4.2 Parallel Transport

Given a tangent vector field on $U \subset S$ for each point, $p \in U$, there exists $w(p) \in T_p(S)$.

Let $\alpha : [-\epsilon, \epsilon] \to U$ be a regular curve such that $\alpha(0) = p$ and $\alpha'(0) = v \in T_pS$. Consider $w(\alpha(t))$, and also dw/dt.

The covariant derivative $\frac{Dw}{dt}$ (or $\nabla_v w$, or $D_v w$) is the orthogonal projection of $\frac{dw}{dt}$ back to the tangent plane.

Note that this defines a map $D_v : T_pS \to T_pS$

Let $\Phi(u(t), v(t)) = \alpha(t)$. Then $w(u, v) = a(u, v)\Phi_u + b(u, v)\Phi_v$, and $\frac{dw}{dt} = (a_uu' + a_vv')\Phi_u + a(\Phi_{uu}u' + \Phi_{uv}v') + (b_uu' + b_vv')\Phi_v + b(\Phi_{uv}u' + \Phi_{vv}v')$. This, after some manipulation, is equal to

$$(a_uu' + a_vv' + a\Gamma^1_{11}u' + a\Gamma^1_{12}v' + bu'\Gamma^1_{12} + bv'\Gamma^1_{22})\Phi_u +$$

$$(b_u u' + b_v v' + a\Gamma_{11}^2 u' + a\Gamma_{12}^2 v' + bu'\Gamma_{12}^2 + bv'\Gamma_{22}^2)\Phi_v + cN)$$

The Levi-Civita Connection is our covariant derivative, and we set $D_Y X - D_X Y = [X, Y]$.

So $D_v w$ doesn't depend on u or v, just the tangent directions.

From this formula, all that matters is the value at zero of u, v, u', v'. The curve $\alpha(t)$ is irrelevant, and things only depend on the first fundamental form $D_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \Gamma_{ij}^k \Phi_{u_k}$.

$$D_{u'\Phi_u + v'\Phi_v} w = D_{u'\Phi_u} w + D_{v'\Phi_v} w = u' D_{\Phi_u} w + v' D_{\Phi_v} w$$

Definition 4.6 (Parallel)**.** *the tangent vector field $w(t)$ is said to be parallel along $\alpha(t)$ iff $D_{\alpha'(t)} w = 0$*

Theorem 4.9. *Let $X(t), Y(t)$ be two parallel vector fields along $\alpha(t)$. Then $|X(t)|, |Y(t)|, \langle X(t), Y(t)\rangle$ are all constants.*

Proof. $\frac{d}{dt}\langle X(t), Y(t)\rangle = \langle \frac{dX}{dt}, Y\rangle + \langle X, \frac{dY}{dt}\rangle$. As Y is tangent, we have $\langle \frac{dX}{dt}, Y\rangle = \langle D_{\alpha'} X, Y\rangle$ and so $\langle D_{\alpha'} X, Y\rangle + \langle X, D_{\alpha'} Y\rangle = \frac{d}{dt}\langle X, Y\rangle$. By hypothesis, this is zero, and so $\langle X, Y\rangle$ is constant. □

Definition 4.7 (Geodesic)**.** *A curve $\alpha(t)$ in S is a geodesic iff $D_{\alpha'(t)}\alpha'(t) = 0$.*

Remark 4.1. *As $D_{\alpha'(t)}\alpha'(t) = 0$, we have $|\alpha'(t)| = c$, and so $t/c = s$ is the arclength, so $\alpha(t/c) = \beta(t)$ is the arc length parameterization.*

$$\beta'(t) = \alpha'(t)/c \Rightarrow |\beta'| = \frac{1}{c}|\alpha'| = c/c = 1$$

If $X(0) = w$ and $D_{\alpha'} X = 0$, then this differential equation is solved by the solution to

$$\begin{aligned} a' + au'\Gamma_{11}^2 + av'\Gamma_{12}^1 + bu'\Gamma_{12}^1 + bv'\Gamma_{22}^1 &= 0 \\ b' + au'\Gamma_{11}^2 + av'\Gamma_{12}^2 + bu'\Gamma_{12}^2 + bv'\Gamma_{22}^2 &= 0 \end{aligned}$$

Set $a = u', b = v'$. Then $X = a(t)\Phi_u + b(t)\Phi_v$, and we have a system of equations called the geodesic equations:

$$\begin{aligned} u'' + (u')^2\Gamma_{11}^1 + 2u'v'\Gamma_{12}^1 + (v')^2\Gamma_{22}^1 &= 0 \\ v'' + (u')^2\Gamma_{11}^2 + 2u'v'\Gamma_{12}^2 + (v')^2\Gamma_{22}^2 &= 0 \end{aligned}$$

For a surface of revolution, this simplifies to $u'' + 2u'v'f'/f = 0$ and $v'' - ff'(u')^2 = 0$.

Theorem 4.10. *If Ψ is a lcoal isometry between S_1 and S_2, then geodesics are mapped to geodesics.*

Proof. Consider the geodesic equations for $(u(t), v(t))$. A local isometry will preserve the first fundamental form. Thus, a local isometry will preserve the geodesic equations. That is, the curve has the same image in $\mathbb{R}^2$ and the Christoffel symbols are the same. □

Theorem 4.11 (Picard)**.** *Given a point $p_0 \in S$ and $v \in T_{p_0}S$, there exists a unique geodesic $\alpha(t)$, $\alpha(0) = p_0$ and $\alpha'(0) = v$ for $t \in (-\epsilon, \epsilon)$.*

Proof. This follows from the existence and uniqueness theorem for 2nd order nonlinear ODEs. □

Theorem 4.12. *If S_1, S_2 are tangent along Γ, then covariant derivative along Γ is the same for S_1 and S_2.*

Theorem 4.13. *Let $F : S \to S$ and assume that F is an isometry. Assume that $Fix(F) = \{p|F(p) = p\}$ is a regular curve. Then $Fix(F)$ is a geodesic.*

Proof. Let $p_0 \in Fix(F)$ and $\gamma'(p_0) = v$.

By previous theorem there is a geodesic $\alpha(s)$ such that $\alpha(0) = p_0$, $\alpha'(0) = v$. $\sigma = F \circ \alpha$ is also a geodesic as F is an isometry. Moreover, $\sigma(0) = p_0$ and $\sigma'(0) = dF(\alpha'(0))$. So $F(\gamma) = \gamma$ and so $dF(\alpha'(0)) = \alpha'(0) = v$. Thus, $\alpha = \sigma$, and so $\alpha \subset Fix(F)$, so $\alpha = \gamma$. □

We are working towards the local Gauss-Bonnet Theorem, and Clairant's Theorem/Relation.

4.3 Algebraic Value of the Covariant Derivative

Assume that we fix an orientation for $U \subset S$, we have determined N. Let $w(t) \in TS$ a tangent vector field with $|w| = 1$. We define $\left[\frac{Dw}{dt}\right] = [D_v w] = \lambda(t)$, where $\frac{Dw}{dt} = \lambda(t) N \wedge w$.

Note that $N \wedge w$ is always a unit tangent vector.

Definition 4.8 (Geodesic Curvature)**.** *Let $\alpha(s)$ be a regular curve parameterized by arclength. Geodesic curvature is $k_g = \left[\frac{D\alpha'}{ds}\right]$.*

Remark 4.2. *$k_g = 0$ iff $\alpha(s)$ is a geodesic.*

So given a regular curve $\alpha(t)$, and $|v(t)| = |w(t)| = 1$, $v, w \in TS$ then $\left[\frac{Dw}{dt}\right] - \left[\frac{Dv}{dt}\right] = \frac{d\phi}{dt}$ with ϕ the angle between v and w.

$k^2 = k_n^2 + k_g^2$.

Lemma 4.14. *Let $\alpha : I \to S$ be a regular curve. Let $v(t), w(t)$ be a fmaily of unit tangent vectors along α. Then $\left[\frac{Dw}{dt}\right] - \left[\frac{Dv}{dt}\right] = \frac{d\phi}{dt}$ where $\phi(t)$ is one determination of the angle from v to w.*

Proof. Construct $\bar{v} = N(t) \wedge v(t)$. Now $w(t) = \cos\phi(t) v(t) + \sin\phi(t)\bar{v}(t)$. Also $w'(t) = -\sin\phi\phi' v + \cos\phi v' + \cos\phi\phi'\bar{v} + \sin\phi\bar{v}'$.

Note that $\langle w', w\rangle = 0$ as $|w| = 1$.

Form $\bar{w} = N \wedge w$ and $\left[\frac{Dw}{dt}\right] = \lambda(t)$ where $\frac{Dw}{dt} = \lambda(t) N \wedge w$.

So $\lambda(t) = \langle \frac{dw}{dt}, N \wedge w\rangle = \langle \frac{dw}{dt}, \bar{w}\rangle$ and $N \wedge w = \bar{w} = \cos\phi N \wedge v + \sin\phi N \wedge \bar{v} = \cos\phi\bar{v} - \sin\phi v$.

So we have

$$\begin{aligned} \langle -\sin\phi\phi' v + \cos\phi v' + \cos\phi\phi'\bar{v} + \sin\phi\bar{v}', \cos\phi\bar{v} - \sin\phi v\rangle &= \\ \cos^2\phi\langle v', \bar{v}\rangle + \cos^2\phi\phi' + \sin^2\phi\phi' - \sin^2\phi\langle\bar{v}', v\rangle &= \langle v', \bar{V}\rangle + \phi' \end{aligned}$$

And so $\left[\frac{Dw}{dt}\right] = \langle v', \bar{v}\rangle + \phi' = \left[\frac{Dv}{dt}\right] + \phi'$. □

Lemma 4.15. *Assume (u, v) are orthogonal parameters for S. Let $w(t)$ be a unit tangent vector field along $\Phi(u(t), v(t)) = \alpha(t)$. Then*

$$\left[\frac{Dw}{dt}\right] = \frac{1}{2\sqrt{EG}}\left[G_u\frac{dv}{dt} - E_v\frac{du}{dt}\right] + \frac{d\phi}{dt}$$

where $\phi(t)$ is the angle between Φ_u and $w(t)$.

Proof. Choose $v(t) = \Phi_u/\sqrt{E}$. We compute $\frac{dv}{dt}$, then find $\langle \frac{dv}{dt}, N \wedge v\rangle = \left[\frac{Dv}{dt}\right]$.

$$\begin{aligned}
\frac{dv}{dt} &= \frac{\Phi_{uu}\frac{du}{dt} + \Phi_{uv}\frac{dv}{dt}}{\sqrt{E}} + \Phi_u\frac{d}{dt}\frac{1}{\sqrt{E}} \\
\left\langle \frac{dv}{dt}, N \wedge v \right\rangle &= \frac{1}{E}\left\langle \Phi_{uu}u' + \Phi_{uv}v', N \wedge \Phi_u \right\rangle \\
&= \frac{1}{E}\left\langle \Phi_{uu}, N \wedge \Phi_u \right\rangle u' + \frac{1}{E}\left\langle \Phi_{uv}, N \wedge \Phi_v \right\rangle v' \\
&= \sqrt{\frac{G}{E}}\Gamma^2_{11}\left\langle \frac{\Phi_v}{\sqrt{G}}, N \wedge \frac{\Phi_u}{\sqrt{E}} \right\rangle u' + \sqrt{\frac{G}{E}}\Gamma^2_{12}\left\langle \frac{\Phi_v}{\sqrt{G}}, N \wedge \frac{\Phi_u}{\sqrt{E}} \right\rangle v' \\
&= \frac{\sqrt{G}}{\sqrt{E}}\left[\Gamma^2_{11}u' + \Gamma^2_{12}v'\right]
\end{aligned}$$

And so the result holds. □

Definition 4.9 (Triangle)**.** *A triangle $T \subset S$ is a region such that*

1. *T is homeomorphic to a disc*
2. *∂T must be a piecewise regular simple closed curve having three vertices*

Definition 4.10 (Triangulable)**.** *A surface is said to be triangulable iff there exists a family $\mathscr{F}$ of triangeles $\{T_i\}$ such that*

1. *$S = \cup \bar{T_i}$*
2. *If $T_i \cap T_j = \emptyset$, then they only share a common vertex or edge.*

Theorem 4.16 (Radó)**.** *A regular surface is traingulable.*

One can define $F - E + V$ for a triangulation to be the Euler characteristic. This is a topological invariant.

Fact: In the traingulation, you can always arrange it so that each triangle is in a local chart.

We define the genus g to be the number of holes or handles attached to a sphere to get the surface, and $\chi(S) = 2 - 2g$.

Theorem 4.17. *For a surface of revolution*

1. *Meridians are geodesics*
2. *Parallels of lattitude with $f'(u_2) = 0$ are geodesics.*
3. *Other geodesics intersect parallels with angel θ, and $c = f(u_2)\theta$ is constant.*

Proof. The geodesic equations are $u_1'' + \frac{2f'}{f}u_1'u_2' = 0$ and $u_2'' - ff'(u_1')^2 = 0$.

1. On a meridian, $u_1 = a$ is constant, and $u_2 = t$. So then we have $a'' + \frac{2f'}{f}a't' = 0$, and $0 - ff'0 = 0$, which are both zero, and so the meridians are geodesics.

2. On a parallel, $u_1 = t, u_2 = a$, and so we need to have $ff' = 0$, and $f \neq 0$, so $f' = 0$.

3. Let $(u(t), v(t))$ be a geodesic with arclength parameterization. That is, $\alpha(t) = \Phi(u(t), v(t))$ and $|\alpha'(t)| = 1 = |\Phi_u u' + \Phi_v v'|$. On a circle of latitude, $\beta(t) = (f(a)\cos t, f(a)\sin t, g(a))$ and $\beta'(t) = (-f(a)\sin t, f(a)\cos t, 0)$. So $|\beta'(a)| = f(a)$, and so $\cos\theta = \alpha'(t) \cdot \frac{\beta'(t)}{|\beta'(t)|}$. Thus, $\cos\theta = (\Phi_u u' + \Phi_v v') \cdot (\sin t, \cos t, 0)$.

   So $\cos\theta = (\Phi_{u_1}u_1' + \Phi_{u_2}u_2') \cdot \frac{\Phi_{u_1}}{f(a)} = \frac{\Phi_{u_1} \cdot \Phi_{u_1}}{f(a)}u_1' = fu_1'$, and so $\cos\theta = fu_1'$, which menas we have $f\cos\theta = f^2u_1'$. By the geodesic equations, we have that $u_1'' + 2\frac{f'(u_2)}{f(u_2)}u_1'u_2' = 0$, so $f^2u_1'' + 2f'fu_1'u_2' = 0 = (f^2u_1')'$.

□

Lemma 4.18. *If a geodesic has been parameterized by arclength, then* $du_1 = \frac{cdu_2}{f(u_2)(f^2(u_2)-c^2)^{1/2}}$

Definition 4.11 (Simple Region)**.** *A simple region $R \subset S$ is a region R which is homeomorphic to a disc in $\mathbb{R}^2$ and ∂R is piecewise regular and simple.*

Theorem 4.19 (Local Gauss-Bonnet)**.** *Consider a simple region $R \subset S$. Assume a positive orientation for $\partial R = \Gamma_1 + \Gamma_2 + \ldots + \Gamma_n$. Then*

$$\sum_{i=1}^{n} \int_{\Gamma_i} k_g(s)ds + \iint_R K d\sigma + \sum_{i=1}^{n} \theta_i = 2\pi$$

where θ_k is the exterior angle between Γ_k and Γ_{k+1}.

Theorem 4.20 (Global Gauss-Bonnet)**.** *Given any region R with boundary ∂R (which may be empty), then*

$$\sum_{i=1}^{n} \int_{\Gamma_i} k_g(s)ds + \iint_R K d\sigma + \sum_{i=1}^{n} \theta_i = 2\pi\chi(R)$$

Corollary 4.21. *If S is a compact surface, then*

$$\iint_S K d\sigma = 2\pi\chi(S)$$

We will prove the local result first, and we will need the following:

Theorem 4.22 (Turning Tangents)**.**

$$\sum(\phi(t_{i+1}) - \phi(t_i)) + \sum \theta_i = 2\pi$$

Now we prove Local G-B

Proof. First we recall that we can choose an orthogonal coordinate system $ds^2 = Edu^2 + Gdv^2$. So now $K = \frac{-1}{2\sqrt{EG}}\left[\left(\frac{G_u}{\sqrt{EG}}\right)_u + \left(\frac{E_v}{\sqrt{EG}}\right)_v\right]$

Now we assume that R is in our orthogonal coordinate chart.

So then

$$\iint_R K d\sigma = \iint_R K(u,v)\sqrt{EG}dudv = \iint_R -\frac{1}{2}\left(\frac{\partial}{\partial u}\left(\frac{G_u}{\sqrt{EG}}\right) + \frac{\partial}{\partial v}\left(\frac{E_v}{\sqrt{EG}}\right)\right) dudv$$

Then, by Green's Theorem, we have that $-\frac{1}{2}\int_{\partial R}\left(\frac{G_u}{\sqrt{EG}}\frac{dv}{ds} + \frac{E_v}{\sqrt{EG}}\frac{du}{ds}\right) ds$

Recall that $\left[\frac{Dw}{ds}\right] = \frac{1}{2\sqrt{EG}}\left(G_u\frac{dv}{ds} + E_v\frac{du}{ds}\right) + \frac{d\phi}{ds}$ where ϕ is the angle from Φ_u to $w(s)$ along $\alpha(s)$. Apply this to $w = \alpha'$, then

$$\begin{aligned}
-\iint_R K d\sigma &= \frac{1}{2}\int_{\partial R}\left(\frac{G_u}{\sqrt{EG}}\frac{dv}{ds} - \frac{G_v}{\sqrt{EG}}\frac{du}{ds}\right) ds \\
&= \frac{1}{2}\sum\int_{\Gamma_i} k_g(s)ds - \sum\int_{\Gamma_i}\frac{d\phi}{ds} \\
\sum_{i=1}^{n}\int_{\Gamma_i}\frac{\partial\phi}{\partial s}ds &= \iint_R k d\sigma + \sum_{i=1}^{n}\int_{\Gamma_i} k_g(s)ds
\end{aligned}$$

And so $\sum \phi(t_{i+1}) - \phi(t_i) = \iint_R K d\sigma + \sum_{i=1}^n \int_{\Gamma_i} k_g(s)ds$, and so $\sum\phi(t_{i+1}) - \phi(t_i) + \sum\theta_i = 2\pi = \int_R K\sigma + \sum_{i=1}^n k_g(s)ds + \sum\theta_i$. □

Theorem 4.23. *Let $R \subset S$, and let ∂R be bounded by $C_1, \ldots, C_n$ piecewise regular curves. Let θ_i be the external angles at the vertices v_i on C_i. Then*

$$\sum_{i=1}^{n}\oint_{C_i} k_g(s)ds + \iint_R K d\sigma + \sum\theta_i = 2\pi\chi(R)$$

where $\chi(R) = F - E + V$ for any triangulation and C_i has positive orientation.

Proof. We traingulate R into T_i with $\cup T_i = R$. For each T_i we have

$$\sum_{k=1}^{3}\int_{C_{ik}} k_g(s)ds + \iint_{T_i} R d\sigma + \sum_{i_k=1}^{3}\theta_{ik} = 2\pi \tag{1}$$

If we sum (1) over all the triangles, we get $\sum_{i=1}^n \int_{C_i} k_g(s)ds + \iint_R K d\sigma + \sum_{i=1}^F\sum_{k=1}^3 \theta_{ik} = 2\pi F$.

We now must analyze $\sum_{i=1}^F\sum_{k=1}^3\theta_{ik}$. We rewrite as interior angles, $\theta = \pi - \phi_{ik}$ with ϕ_{ik} the interior angle, and so $\sum_{i=1}^F\sum_{k=1}^3(\pi - \phi_{ik}) = \sum_{i=1}^F(2\pi - \sum_{k=1}^3\phi_{ik})$.

This is equal to $3\pi F - \sum_{i=1}^F\sum_{k=1}^3\phi_{ik}$. Call E_e the number of exterior edges and E_I the number of interior edges, and define V_e and V_I similarly.

Fact: $3F = 2E_I + E_e$. And so $3\pi F = 2\pi E_I + \pi E_e$, which means we have $2\pi E_I + \pi E_e - \sum_{i=1}^F\sum_{k=1}^3\phi_{ik} = 2\pi E - \pi E_e - \sum\sum\phi_{ik}$, and as $E_e = V_e$, we have $2\pi E - \pi V_e - \sum\sum\phi_{ik} = 2\pi E - \pi V_e - 2\pi V - \sum_{ext}\phi_{ik}$.

The exterior triangulation splits into two pieces. So we note that $\sum_{ext}\phi_{ik} = \sum_{\text{vert of } T_i} + \sum_{\text{vert of } C_i} = \pi V_{eT} + \sum\pi - \theta_i = \pi V_{eT} + \pi V_{ec} - \sum\theta_i = \pi V_e - \sum\theta_i$, and so the formula gives $2\pi E - 2\pi V + \sum\theta_i$, and so we establish the theorem. □

Corollary 4.24. *If S is compact and $\partial S = \emptyset$, then* $\iint_S K d\sigma = 2\pi\chi(S)$

Corollary 4.25. *A compact surface with positive curvature is homeomorphic to S^2.*

Proof. $\iint_S K d\sigma > 0 \Rightarrow \chi(S) = 2$ □

Corollary 4.26. *Let $K_S \leq 0$, S compact and $\partial S = \emptyset$, then* $\iint_S K d\sigma$.