1 Point Set Topology

Definition 1.1 (Metric Space). *(X, d) is a metric space iff $d : X \times X \to \mathbb{R}_{\geq 0}$ a metric such that:*

1. $d(x, y) = d(y, x) \geq 0$
2. $d(x, y) = 0$ *iff* $x = y$
3. $d(x, y) + d(y, z) \leq d(x, z)$.

e.g. $\mathbb{R}^n$, $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ and $(x, y) = \sum x_i y_i$ is the dot product. $d(x, y) = \sqrt{\sum (x_i - y_i)^2} = (x - y, x - y)^{1/2}$.
$\ell^2 = \{x = (x_1, \ldots)\}$ with $\sum x_i^2 < \infty$ by the same metric.
$\ell^1 = \{x = (x_1, \ldots)\}$ with $\sum |x_i| < \infty$ and $d(x, y) = \sum_i |x_i - y_i|$.

Lemma 1.1 (Cauchy Inequality). *Let $x, y \in \mathbb{R}^n$. Then $|(x, y)| \leq \|x\|\|y\|$ and $\|x\| = (x, x)^{1/2}$.*

Proof. Take $t \in \mathbb{R}$. Then $(tx + y, tx + y) = t^2\|x\|^2 + 2t(x, y) + \|y\|^2 \geq 0$ for all t. Assume, without loss of generality, that $x \neq 0$.

We call the left hand side $q(t)$ because it is a quadratic polynomial in t. $q(t) = 0$ cannot have two distinct real roots iff the discriminant of $q(x)$ is ≤ 0.

So $4(x, y)^2 - 4\|x\|^2\|y\|^2 \leq 0$, done. □

Definition 1.2 (Open Ball). *Let (X, d) a metric space. $\forall x \in X, \gamma \in \mathbb{R}_{>0}$ we have the open ball $B_\gamma(x) = \{y \in X : d(x, y) < \gamma\}$.*

Definition 1.3 (Open Set). *An open set in (X, d) is a union of open balls.*

Definition 1.4 (Cauchy Sequence). *A sequence $\{x_n\}$ in (X, d) is Cauchy if $\forall \epsilon > 0$, $\exists N \in \mathbb{Z}_{\geq 0}$ such that when $\forall n, m \geq N$ then $d(x_n, x_m) < \epsilon$.*

Definition 1.5 (Complete). *A metric space (X, d) is complete if every Cauchy sequence converges.*

Lemma 1.2. *If $\lim b_n = q$ in X then $\lim f(b_n) = f(q)$ for f a contracting mapping.*

Proof. $d(f(b_n), f(q)) \leq \lambda d(b_n, q) \to 0$ □

Theorem 1.3 (Contracting Mapping, CMT). *Suppose $f : X \to X$ and (X, d) is a complete metric space such that $\exists \lambda \in [0, 1)$ with $d(f(x), f(y)) \leq \lambda d(x, y)$ for all $x, y \in X$. Then there is a unique $p \in X$ such that $f(p) = p$.*

Proof. Uniqueness is clear, because if $f(p) = p$ and $f(q) = q$ then $d(p, q) = d(f(p), f(q)) \leq \lambda d(p, q)$ so $d(p, q) = 0$.

Pick any point $a_1 \in X$. Define $a_{n+1} = f(a_n)$. Claim: This sequence is Cauchy. Look at $d(a_{n+1}, a_n) = d(f(a_n), f(a_{n-1})) \leq \lambda d(a_n, a_{n-1})$. Define $\alpha_n = d(a_n, a_{n-1})$

By the ratio test for series, $\sum \alpha_n < \infty$. So $\forall \epsilon > 0, \exists N$ such that $m \geq n \geq N$ implies $\sum_{i=n}^{m-1} \alpha_i < \epsilon$ and it is equal to $d(a_n, a_{n+1}) + \ldots + d(a_{m-1}, a_m) \geq d(a_n, a_m)$ by the triangle inequality.

So for (X, d) complete, $\lim_{n\to\infty} a_n = p$.

By Lemma 2, we have $f(p) = \lim f(a_n) = \lim a_{n+1} = p$. □

e.g. If $f : [a, b] \to \mathbb{R}$ satisfies $|f'(x)| \leq \lambda$ for all $x \in [a, b]$ then $|f(x) - f(y)| \leq \lambda|x - y|$ for all $x, y \in [a, b]$

Theorem 1.4 (Newton's Method)**.** *Suppose $f : [a, b] \to \mathbb{R}$ is C^2-smooth (that is, f'' exists and is continuous) and $f(c) = 0$, $f'(c) \neq 0$ for some $c \in (a, b)$. Then, $\exists \epsilon > 0$ such that $\forall x_1 \in (c - \epsilon, c + \epsilon)$, and $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ then $\lim x_n = c$.*

Proof. Let $g(x) = x - \frac{f(x)}{f'(x)}$.

Choose $\epsilon > 0$ small such that $f'(x) \neq 0$ in $[c - \epsilon, c + \epsilon]$ and $\left|\frac{f(x)f''(x)}{f'(x)^2}\right| \leq \frac{1}{2}$ in $[c - \epsilon, c + \epsilon]$.

Now in $[c - \epsilon, c + \epsilon]$, $g'(x) = 1 - \frac{f'(x)^2 - f(x)f''(x)}{f'(x)^2} = \frac{f(x)f''(x)}{f'(x)^2}$ so $|g'(x) \leq \frac{1}{2}$ in $[c - \epsilon, c + \epsilon]$. So $|g(x) - g(y)| \leq \frac{1}{2}|x - y|$ so contracting, and $g(c) = c$ so $g[c - \epsilon, c + \epsilon] \subset [c - \epsilon, c + \epsilon]$.

Using the closed interval, by the CMT for $x_1 \in [c-\epsilon, c+\epsilon]$ and $x_{n+1} = g(x_n)$ converges to c. □

Topological Spaces

Definition 1.6 (Topological Space)**.** *A topological space $(X, \mathscr{T})$ where $\mathscr{T}$ is a collection of subsets of X called open sets such that*

1. *$\emptyset$ and X are open*
2. *if $U_\alpha \in \mathscr{T}$ for $\alpha \in I$ then $\bigcup_{\alpha\in I} U_\alpha \in \mathscr{T}$*
3. *if $U_1, \ldots, U_n \in \mathscr{T}$ then $\bigcap_{i=1}^n U_i \in \mathscr{T}$.*

(X, d) is a topological space with open sets the union of open balls.

Example: Quotient Topology: Suppose X is a topological space and R is an equivalence relation on X. Let the set $X/R = \{[x] : x \in X\}$ be the set of equivalence classes, and $q : X \to X/R$ be the quotient map of sets.

Definition 1.7 (Quotient Topology on X/R)**.** *$U \subset X/R$ is open iff $q^{-1}(U)$ is open in X.*

Definition 1.8 (Compactness)**.** *$(X, \mathscr{T})$ a topological space is compact if every open cover $\mathscr{U} = \{U_\alpha : \alpha \in I\} \subset \mathscr{T}$ where $\bigcup U_\alpha = X$ has a finite subcover.*

Theorem 1.5. *If $f : X \to Y$ a continuous onto map and X is compact then $Y = f(X)$ is compact.*

Proof. This follows trivially from the definition. □

Theorem 1.6. *Suppose X is a compact space and $Y \subseteq X$ is closed. Then Y is compact.*

Proof. Take any open cover of Y, $V = \{V_\alpha\}$. Let $U = V \cup \{X \setminus Y\}$ be an open cover of X. X is compact, so it has a finite subcover, $\{V_1, \dots, V_n\} \cup (X \setminus Y)$ and $\{V_1, \dots, V_n\}$ covers Y. □

Theorem 1.7. *X is compact Hausdorff and $Y \subseteq X$ a compact subspace. Then Y is closed.*

Proof. Goal: $X \setminus Y$ is open, i.e.,

For all $x \in X \setminus Y$ there is an open set $U \subset X$ such that $x \in U \subset X \setminus Y$.

Fix x, $\forall y \in Y$, Hausdorff implies that $\exists$ disjoint open sets U_y, V_y such that $y \in U_y$ and $x \in V_y$. $\{U_y : y \in Y\}$ open cover of Y.

As Y is compact, we can find a finite subcover, $U_{y_1}, \dots, U_{y_n}$.

$U = \cap_{i=1}^n V_{y_i}$ is an open set containing x and missing all U_{y_i}'s. So $U \subseteq X \setminus Y$. □

Theorem 1.8. *Suppose $f : X \to Y$ is continuous, injective and surjective with X compact and Y Hausdorff. Then f is a homeomorphism, that is, f^{-1} is continuous as well.*

Proof. $f^{-1} : Y \to X$ is well defined, and it is continuous if and only if $\forall C \subset X$ closed, $(f^{-1})^{-1}(C)$ is closed in Y.

Take closed $C \subset X$. As X is compact, C is compact. f continuous means that $f(C)$ is a compact subset of Y, and as Y is Hausdorff, then $f(C)$ is closed in Y. □

Applications

Notation: X is a topological space, $A \subset X$, then $X/A = \{[x] : x \in X\}$ with the quotient topology. $q : X \to X/A$ is the quotient map, $q(x) = [x]$.

Quotient Topology: $U \subset X/A$ is open iff $q^{-1}(U)$ is open in X.

1. Show that $[0,1]/0 \sim 1 \simeq S^1$.

$[0,1]$ compact, so $[0,1]/0 \sim 1$ is compact, and S^1 is Hausdorff.

Consider $h : [0,1] \to S^1 : t \mapsto e^{2\pi i t}$ H is onto and continuous such that $|h^{-1}(p)| = 1$ for $p \neq 1$ and for $p = 1$ it is 2.

In particular, h induces a one-to-one, onto map $[0,1]/0 \sim 1 \to S^1$ $f([t]) = h(t)$. Claim: f is continuous.

$$\begin{array}{ccc} [0,1] & \xrightarrow{q} & [0,1]/0 \sim 1 \\ & \searrow^{h} & \downarrow f \\ & & S^1 \end{array}$$

Take $U \subset S^1$ open. $h^{-1}(U) = (f \circ q)^{-1}(U) = q^{-1}(f^{-1}(U))$. Now h continuous implies $h^{-1}(U)$ is open so $q^{-1}(f^{-1}(U))$ is open, and as we are using the quotient topology, $f^{-1}(U)$ is open.

2. $\mathbb{B}^n/\partial\mathbb{B}^n \simeq S^n$. Prove for homework.

Lemma 1.9 (Lebesgue Lemma). *(X, d) is a sequentially compact metric space and $\mathscr{U} = \{U_\alpha : \alpha \in I\}$ is an open cover of X. Then $\exists \epsilon > 0$, sometimes called the Lebesgue number of $\mathscr{U}$, such that $\forall x \in X$, $B_\epsilon(x) \subset U_\alpha$ for some α.*

Proof. Suppose otherwise. $\forall n \in \mathbb{Z}_{>0}$, $\epsilon = \frac{1}{n}$, then $\exists x_n \in X$ such that $B_{\frac{1}{n}}(x_n) \not\subset U_\alpha$ for all $\alpha \in I$.

X sequentially compact implies that $\exists x_{n_i} \to p \in X$

$\mathscr{U}$ is an open cover $\Rightarrow \exists \beta \in I$ such that $p \in U_\beta$. Also, U_β open $\Rightarrow \exists \delta > 0$ such that $B_\delta(p) \subset U_\beta$.

Now take n_i large such that $\frac{1}{n_i} < \frac{\delta}{2}$. Thus, $B_{\frac{1}{n_i}} \subset B_{\frac{\delta}{2}}(x_{n_i}) \subset B_\delta(p) \subset U_\beta$. Contradiction. □

Theorem 1.10. *Suppose (X, d) is a metric space. Then (X, d) is compact if and only if every sequence in X contains a convergent subsequence (ie, is sequentially compact).*

Proof. $\Rightarrow$: Suppose otherwise. Then $\exists$ a sequence $\{x_n\}$ in X without any convergent subsequence.

Claim 1: $\forall x \in X$, $\exists \epsilon_x > 0$ such that $B_{\epsilon_x}(x)$ contains only finitely many of x_n's. Then, $\{B_{\epsilon_x}(x) : x \in X\}$ is an open cover of X, and X compact implies it has a finite subcover, $B_{\epsilon_{x_1}}(x_1), \ldots, B_{\epsilon_{x_n}}(x_n)$. So there are only finitely many x_i's in each of finitely many balls, so there are only finitely many in the union. However, the balls form an open cover, and so their union is X, thus, there are only finitely many x_i's, contradiction.

$\Leftarrow$: We will use the Lebesgue lemma. Suppose otherwise, then $\exists$ open cover $\mathscr{U} = \{U_\alpha : \alpha \in I\}$ of X without any finite subcover.

Let $\epsilon > 0$ be the Lebesgue number of $\mathscr{U}$. Construct the sequence $\{x_n\}$ as follows: Take $x_1 \in X$. Then $\exists U_{\alpha_1}$ such that $B_\epsilon(x_1) \subset U_{\alpha_1}$. $U_{\alpha_1} \neq X$ by assumption, so we choose $x_n \notin U_{\alpha_1}$. Then $\exists U_{\alpha_2}$ such that $B_\epsilon(x_2) \subset U_{\alpha_2}$. Inductively, we find $x_{n+1} \notin U_{\alpha_1} \cup \ldots \cup U_{\alpha_n}$.

Claim: $d(x_n, x_m) \geq \epsilon$ for $n > m$. Otherwise, $d(x_n, x_m) < \epsilon$ if and only if $x_n \in B_\epsilon(x_m) \subset U_{\alpha_m}$, but $x_n \notin U_{\alpha_1} \cup \ldots \cup U_{\alpha_{n-1}}$. Contradiction. □

Theorem 1.11 (Tychonoff). *The product of compact spaces is compact.*

e.g. X compact in $\mathbb{R}^n$ if and only if X is closed and bounded if and only if $X \subset [-a, a]^n$ for some a, which implies the first by Theorem 5 and Tychonoff.

Definition 1.9 (Disconnected). *A topological space X is disconnected if $X = A \cup B$ where $A, B \neq \emptyset$, $A \cap B = \emptyset$ and both are open.*

We say $\{A, B\}$ is a separation.

Otherwise, X is connected.

Theorem 1.12. *If $f : X \to Y$ onto and continuous and X is connected then Y is connected.*

Proof. If not, then $Y = A \cup B$ is a separation. Then $f^{-1}(A) \cup f^{-1}(B) = X$ is a separation of X. Contradiction. □

Theorem 1.13. $[0, 1]$ *is connected.*

Proof. If not, then $[0, 1] = A \cup B$ is a separation with $0 \in A$. $B = A^c$ implies that B is closed, and A must also be closed by the same argument.

Consider $t = \inf\{x \in B\}$. As B is closed, $t \in B$.

B is open, so $t > 0$ and $\exists \epsilon > 0$ such that $t - \epsilon \in B$. Contradiction. □

Definition 1.10 (Path Connected)**.** *A topological space X is path connected if $\forall a, b \in X$, $\exists$ continuous map $f : [0, 1] \to X$ such that $f(0) = a$ and $f(1) = b$.*

Theorem 1.14. *If $f : X \to Y$ is continuous and onto and X is path connected, then Y is path connected.*

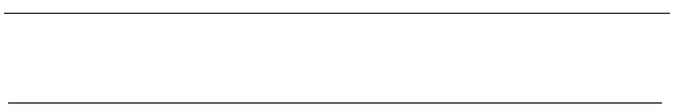
Theorem 1.15. *Path connected implies connected.*

e.g. $\mathbb{R}^2 \setminus \{0\}$ is path connected. In fact, path-connected for $R^n \setminus \{0\}$, $n \geq 2$.

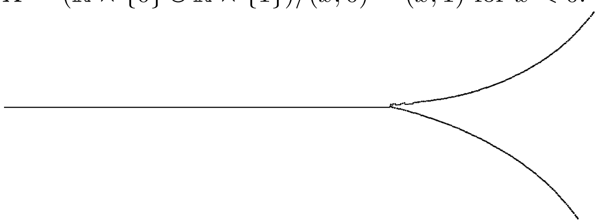
e.g. S^n for $n \geq 1$ is path connected.

2 Manifolds

Definition 2.1 (Topological Manifold)**.** *M^n is a topological n-manifold if M^n is a Hausdorff space with a countable basis such that $\forall x \in M^n$, $\exists$ an open set U containing x and a homeomorphism $\phi : U \to V \overrightarrow{\text{open}} \mathbb{R}^n$. We call (U, ϕ) a topological chart.*



$X = (\mathbb{R} \times \{0\} \cup \mathbb{R} \times \{1\})/(x, 0) \sim (x, 1)$ for $x < 0$.



This is not a manifold.

e.g. Open set $\Omega \subset \mathbb{R}^n$ with chart (Ω, id).

$\mathbb{R}^{n \times m}$ is the space of all $n \times m$ matrices.

$GL(n, \mathbb{R})$ is all $n \times n$ real matrices A with $\det A \neq 0$. $\det : \mathbb{R}^{n \times n} \to \mathbb{R}$ is a polynomial, and so is continuous.

$GL(n, \mathbb{C})$ is similar, except it is connected.

Definition 2.2 (Smooth Manifold)**.** *A smooth manifold M is a topological manifold with a special collection of charts (smooth charts). $\mathscr{F} = \{(U_\alpha, \varphi_\alpha) : \alpha \in I\}$ such that*

1. $\cup_\alpha U_\alpha = M$

2. *If $U_\alpha \cap U_\beta \neq \emptyset$ then the transition function $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is C^0 (smooth).*

We call (U_α, ϕ_α) smooth charts for M.

examples

1. $U \subset \mathbb{R}^n$ is open, then there is one chart (U, id).
2. If M^n is a smooth manifold and $U \subset M^n$ open then U is smooth with smooth chars $\{(U_\alpha \cap U, \varphi_\alpha|_U : \alpha \in I\}$
3. General linear groups on $\mathbb{R}$ and $\mathbb{C}$.
4. $C_n = \{(z_1, \ldots, z_n) : z_i \neq z_j$ for $i \neq j\} \subseteq \mathbb{C}$ open
5. The space of all circles in $\mathbb{R}^2$.
6. Space of all lines in $\mathbb{R}^2$.
7. Graph of a continuous function $f : \Omega \to \mathbb{R}^1$ with $\Omega \subset \mathbb{R}^2$ open with one chart, (Ω, π).
8. S^n with two charts, let $N = (0, \ldots, 0, 1)$ and $S = (0, \ldots, 0, -1)$. $U_N = S^n \setminus \{N\}$ and $U_S = S^n \setminus \{S\}$. $\varphi_N : U_N \to \mathbb{R}^n$ by stereographic projection. South pole similarly. $\varphi_N \circ \varphi_S^{-1} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ is $x \mapsto \frac{x}{\|x\|^2}$ is smooth, in fact, real analytic.

   Remark: $\det D(\varphi_N \circ \varphi_S^{-1}) < 0$, that it, the transition map reverses orientation.

Definition 2.3 (Smooth Structure). *A smooth structure on a smooth manifold maximal collection $\mathscr{F}$ satisfying the smooth manifold conditions.*

Definition 2.4 (Smooth Function). *If M, N smooth manifolds, and $f : M \to N$ continuous, we say f is smooth if $\forall$ smooth charts (U, φ) of M and (V, ψ) of N, we have $\psi \circ f \circ \varphi^{-1}$ is smooth.*

Definition 2.5 (Diffeomorphism). *We say that f is a diffeomorphism if f is a smooth homeomorphism such that f^{-1} is smooth as well.*

$f : \mathbb{R} \to \mathbb{R} : x \mapsto x^3$ is a homeomorphism, but not a diffeomorphism.

Take $M = GL(n, \mathbb{R})$ and $f : GL(n, \mathbb{R}) \to GL(n, \mathbb{R}) : A \mapsto A^{-1}$, $g : GL(n, \mathbb{R}) \times GL(n, \mathbb{R}) \to GL(n, \mathbb{R}) : (A, B) \mapsto AB$, $h(A) = \det A$ are all smooth maps. (in fact, all rational functions in the coordinates are)

Inverse Function Theorem

Recall the following:

Definition 2.6 (Differentiable Function). *Suppose U is open in $\mathbb{R}^n$ and $F : U \to \mathbb{R}^m$ is a map. We say F is differentiable at $a \in U$ if $\exists$ linear map $A : \mathbb{R}^n \to \mathbb{R}^m$ such that $F(a + t) = F(a) + At + e(t)$ for t small where $\lim_{t \to 0} \frac{\|e(t)\|}{\|t\|} = 0$. We say $A = DF(a)$.*

eg: $f(A) = \det A : \mathbb{R}^{n\times n} \to \mathbb{R}$. What is $DF(A)$. (hint: $ad(A)$)
eg: Define $F : GL(n, \mathbb{R}) \to \{A : A = A^T\} : X \mapsto X \cdot X^t$.
Show that $DF(A) : \mathbb{R}^{n\times n} \to \mathbb{R}^{(n\times(n+1))/2}$ is always onto.

Proof. Let $T \in \mathbb{R}^{n\times n}$ small. Then $F(A+T) = (A+T)(A+T)^t = AA^t + TA^t + AT^t + TT^t$, so $DF(A) : \mathbb{R}^{n\times n} \to \mathbb{R}^{(n\times(n+1))/2} : T \mapsto TA^t + AT^t$. For any $B \in \mathbb{R}^{n\times(n+1)/2}$ with $B = B^t$, $\exists X \in \mathbb{R}^{n\times n}$ such that $B = XA^t + AX^t$. As $B = B/2 + B^t/2$, we can solve $B/2 = XA^t$ and find $X = \frac{1}{2}B(A^t)^{-1}$. Note, this solution is not unique. □

Recall the following definition:

Definition 2.7 (Norm of a Matrix)**.** *If $A : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation, then $\|A\| = \max\{\|Ax\| : \|x\| = 1\}$.*

Lemma 2.1. *If f : some ball in $\mathbb{R}^n \to \mathbb{R}^n$ a C^1 function so that $\|Df'(x)\| \le M$ in the ball, then $\|f(x_1) - f(x_2)\| \le M\|x_1 - x_2\|$ for x_1, x_2 in the ball.*

Proof. Let $g(t) = f((1-t)x_1 + tx_2)$, then $f(x_1) - f(x_2) = g(0) - g(1) = g'(c)(0-1) = -g'(c)$ for some $c \in [0,1]$.

By the chain rule, we know $g'(c) = Df((1-t)x_1 + tx_2) \cdot (x_2 - x_1)$ so $\|f(x_1) - f(x_2)\| = |g'(c)| = \|Df(\xi)(x_2 - x_1)\| \le \|Df(\xi)\|\|x_2 - x_1\| \le M\|x_2 - x_1\|$. □

Theorem 2.2 (Inverse Function Theorem (IFT))**.** *Suppose U is open in $\mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ is C^1 smooth and $Df(a)$ is invertible for some $a \in U$. Then $\exists$ neighborhood V of a and W of $f(a)$ in $\mathbb{R}^n$ such that $f|_V : V \to W$ is a one-to-one onto map and $g = (f_V)^{-1}$ is C^1.*

Proof. For simplicity, we assume that $a = 0$, $f(a) = 0$ (after translation).

We may assume $Df(a) = I$ after replacing f by $Df(a)^{-1} \circ f$.

Choose $\gamma > 0$ small so that on the ball $D = \{x \in \mathbb{R}^n : \|x\| < \gamma\}$ such that $\|Df(x) - I\| < \frac{1}{2}$, due to $Df(0) = I, f \in C^1$.

Put $\omega(x) = f(x) - x$, $D\omega = Df(x) - I$, so $\|D\omega(x)\| \le \frac{1}{2}$ for $x \in D$.

Lemma 1$\Rightarrow$ $\|\omega(x+h) - \omega(x)\| \le \frac{1}{2}\|h\|$, $\forall x, x+h \in D$.

$\|f(x+h) - f(x) - h\| \le \frac{1}{2}\|h\|$, $\forall x, x+h \in D$.

Take $W = \{x \in \mathbb{R}^n : \|x\| < \frac{\gamma}{2}\}$

Claim 1: $W \subseteq f(D)$

Claim 2: $f|_D : D \to \mathbb{R}^n$ is one-to-one.

To see claim 2, we take $x, x+h \in D$ such that $f(x+h) = f(x)$. This gives $\|h\| < \frac{1}{2}\|h\|$, so $h = 0$, thus $x = x+h$.

To prove 1, Take $y \in W$. Consider $u(x) = x - f(x) + y : D \to \mathbb{R}^n$. The fixed point of $x = u(x)$ solves $f(x) = y$. $\|u(x_1) - u(x_2)\| = \|x_1 - x_2 + f(x_2) - f(x_1)\| = \|f(x_1 + (x_2 - x_1)) - f(x_1) - (x_2 - x_1)\| \le \frac{1}{2}\|h\| = \frac{1}{2}\|x_1 - x_2\|$.

Furthermore, $u(D) \subseteq D$. $\|u(x)\| = \|x - f(x) + y\| \le \|x - f(x)\| + \|y\|$. Taking $x = 0$, we get $\le \|f(h) - h\| \le \frac{1}{2}\|h\| \le \frac{1}{2}\|x\| + \|y\| < \gamma$.

Now, by the contracting mapping theorem, $u : D \to D$ has a fixed point which satisfies $f(x) = y$.

We now define $V = (f|^{-1}_{\text{int})(D)}(W)$, so $f|_V : V \to W$ is 1-1, onto and continuous.

Claim 3: $g = (f|_V)^{-1} : W \to V$ is continuous.

Take $y \in W$ and $y + k \in W$. Let $x = g(y)$, $x + h - g(y + k)$.

Goal: $\lim_{k\to 0} h = 0$. We have $f(x) = y$ and $f(x + h) = y + k$.

So, $\|f(x + h) - f(x) - h\| \le \frac{1}{2}\|h\|$, ie, $\|y + k - y - h$
$\le \frac{1}{2}\|h\| \iff \|h\| - \|k\| \le \|k - h\| \le \frac{1}{2}\|h\|$

Thus, $\|h\| \le 2\|k\|$.

Claim 4: g is differentiable with $Dg(y) = Df(x)^{-1}$ for $x = g(y)$.

f differentiable at x with $Df(x) = A$. Thus, $f(x + h) - f(x) = Ah + e(h)$ where $\lim_{h\to 0} \frac{\|e(h)\|}{\|h\|} = 0$.

A is invertible by assumption, and so $y + k - u = k = Ah + e(h)$. Applying A^{-1}, $h = A^{-1}k - A^{-1}e(h)$, so $\lim_{k\to 0} \frac{\|A^{-1}e(h)\|}{\|k\|} = 0$, then we have $Dg(h)$ exists and is A^{-1}.

$$\frac{\|A^{-1}e(h)\|}{\|k\|} \le \|A^{-1}\| \frac{\|e(h)\|}{\|h\|} \frac{\|h\|}{\|k\|} \le 2\|A^{-1}\| \frac{\|e(h)\|}{\|h\|} \to 0$$

□

Corollary 2.3. *If f is C^∞ then so is f^{-1} in IFT.*

Proof. Indeed, for $y = f(x)$, we have $D(f^{-1})(y) = (Df(x))^{-1} = \text{Inv} \cdot Df(x)$ where $\text{Inv} : GL(n, \mathbb{R}) \to GL(n, \mathbb{R})$ sending a matrix A to A^{-1}. $\text{Inv}(A) = \frac{1}{\det A}\text{adj}(A)$, so each entry is a rational function in the entries of A, and the composition of real analytic functions with C^∞ functions is C^∞. □

Corollary 2.4. *If f is analytic (ie, has convergent power series expansion) and Df^{-1} exists, then f^{-1} is analytic.*

Corollary 2.5. *If U open in $\mathbb{C}^n$ and $f : U \to \mathbb{C}$ complex analytic, with Df^{-1} exists. then f^{-1} is complex analytic.*

Remarks: $(M, \mathscr{F})$ is real analytic if $\varphi_\alpha \circ \varphi_\beta^{-1}$ is real analytic $\forall \alpha, \beta$.

$(M, \mathscr{F})$ complex manifolds if $\varphi_\alpha \circ \varphi_\beta^{-1}$ with charts to $\mathbb{C}^n$ are complex analytic.

$(M, \mathscr{F})$ is an affine manifold if $\varphi_\alpha \circ \varphi_\beta^{-1} = Ax + b$, $A \in GL(n, \mathbb{R})$.

Conjecture 2.1. *If M^n a closed (ie, compact) affine manifold, then $\chi(M) = 0$ (χ the Euler characteristic) (ie iff M supports a vector field nowhere 0).*

$(M, \mathscr{F})$ is a polynomial manifold if $\varphi_\alpha \circ \varphi_\beta^{-1}$ is a polynomial.

S^n has no polynomial structure, but $\mathbb{T}^2 = S^1 \times S^1$ does.

Conjecture 2.2. *For all surfaces of genus≥ 2, there are no polynomial structures.*

Note: Smooth means C^∞ for the remainder of course.

Theorem 2.6 (Implicit Function Theorem)**.** *If $U \subseteq \mathbb{R}^n, V \subseteq \mathbb{R}^m$ is open and $f : U \to \mathbb{R}^m$ smooth such that $X = f^{-1}(p)$ satisfies the condition $\forall a \in X$, $Df(a) : \mathbb{R}^n \to \mathbb{R}^m$ is onto (we call a a regular value), then X is a smooth manifold of $\dim = n - m$ such that the inclusion map $\iota : X \to \mathbb{R}^n$ is smooth.*

eg 1: $f(x) = \|x\|^2 : \mathbb{R}^n \to \mathbb{R}^1$. Then 1 is a regular value, so $f^{-1}(1) = S^{n-1}$ is a smooth $(n-1)$-manifold.

Indeed, $a = (a_1, \ldots, a_n) \in f^{-1}(1)$, then $a \neq 0$, $Df(a) = \nabla f(a) = (2a_1, \ldots, 2a_n) \neq 0$.

eg 2: $U = \mathbb{R}^{n \times n}$, $f(A) = \det A : \mathbb{R}^{n\times n} \to \mathbb{R}$. Then 1 is a regular value, so $f^{-1}(A) = SL(n, \mathbb{R})$ is a manifold of dimension $n^2 - 1$.

eg 3: $U = \mathbb{R}^{n\times n}$, V is the space of all symmetric matrices, $f(A) = AA^T$. The identity is a regular value for f, so $f^{-1}(I) = \{A \in \mathbb{R}^{n\times n} : AA^T = I\} = SO(n, \mathbb{R})$ is a smooth manifold of complex dimension $n(n-1)/2$.

eg 4: $U = \mathbb{C}^{n\times n}$, $f(A) = \det A : \mathbb{C}^n \to \mathbb{C}$. 1 is a regular value. $f^{-1}(A) = SL(n, \mathbb{C})$ is a complex manifold of dimension $n^2 - 1$.

We will now prove the Implicit Function Theorem

Proof. Assume without loss of generality that $p = 0$. $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$, $f = (f_1, \ldots, f_n)$. Take $a \in X$, i.e. $f(a) = 0$ if any only if rank $\left[\frac{\partial f_i}{\partial t_j}(a)\right]$ has rank m if and only if Df contains a submatrix with is nonsingular.

Consider $G : U \to \mathbb{R}^n = \mathbb{R}^{n-m} \times \mathbb{R}^m$. $G(x, y) = (x, f(x, y))$. $DG(a) = \begin{bmatrix} 1 & * \\ 0 & D_y f(a) \end{bmatrix}$, $\det DG(a) \neq 0$.

By the inverse function theorem, $\exists$ neighborhood W of a in $\mathbb{R}^n$ and $\tilde{W}$ of $G(a)$ in $\mathbb{R}^n$, $G| : W \to \tilde{W}$ is a diffeomorphism, $(G|)^{-1}(x, y) = (\psi(x, y), \phi(x, y))$, $\psi \in \mathbb{R}^{n-m}, \phi \in \mathbb{R}^m$.

Claim: $X \cap W = \{(x, y) \in W : f(x, y) = 0\} = \{(x, \phi(x, 0)) : x \in W \cap \mathbb{R}^{n-m} \times 0\}$

Indeed, $G\circ(G^{-1})(u, v) = (u, v) = G(\psi(u, v), \phi(u, v)) = (\psi(u, v), f(\psi(u, v), \phi(u, v))) \Rightarrow f(u, \phi(u, v)) = v$ for all $(u, v) \in \tilde{W}$.

Suppose $(x, y) \in W$. $f(x, y) = 0$. Then $(x, y) = G^{-1}(u, v) = (u, \phi(u, v))$. $x = u$, $y = \phi(u, v)$.

Thus, $0 = f(x, y) = f(u, \phi(u, v)) = v$, so $v = 0$, $x = u$, ie $y = \phi(x, 0)$. If we define a smooth $\Phi(x) = \phi(x, 0)$, then $X \cap W$ =graph of $\Phi = \{(x, \Phi(x)) : x \in W \cap \mathbb{R}^{n-m} \times 0\}$.

Smooth chart $(X \cap W, \pi_1)$, $\pi_1(x, \Phi(x)) = x$.

Suppose (W, π_2) is another chart produced in this way. The transition function $\pi_2 \circ \pi_1^{-1} = \pi_2(x, \Phi(x)) = \pi_2(t_1, \ldots, t_n) = (t_{i_1}, \ldots, t_{i_n})$, this is clearly smooth. to see $\iota : X \to \mathbb{R}^n$ is smooth, we notice that it is true if and only if $\iota \circ \pi_1^{-1}(x) = (x, \Phi(x)) \subseteq \mathbb{R}^n$ is smooth, which is clear. □

Corollary 2.7. *Under the same assumption, $\forall a \in X$, $\exists$ smooth chart (W, g) of $\mathbb{R}^n$ such that $g(W \cap X) \subseteq \mathbb{R}^{n-m} \times 0$*

Proof. $g = G$. remark: the last condition is called "smooth submanifold". □

Corollary 2.8. *If W open in $\mathbb{R}^k$ and $h : W \to \mathbb{R}^n$ smooth so that $h(W) \subset X$, then $h : W \to X$ is smooth.*

Proof. By previous corollary, we may assume $X = \mathbb{R}^{n-m} \times 0 \subseteq \mathbb{R}^n$. $h(x) = (h_1(x), \dots, h_{n-m}(x), 0, \dots, 0)$ smooth iff smooth $\forall i$. □

Definition 2.8 (Lie Group)**.** *A Lie Group G is a group with multiplication $m : G \times G \to G$ and inverse $i : G \to G$ so that G is also a smooth manifold and both m and i are smooth.*

Example: $GL(n, \mathbb{R}), SL(n, \mathbb{R}), O(n), GL(n, \mathbb{C}), SL(n, \mathbb{C}), U(n)$ are Lie groups. $SU(n)$ is a Lie Group

Definition 2.9 (Complex Analytic Manifolds)**.** *M^n is a complex manifold if it has smooth charts $\{(U_\alpha, \phi_\alpha) : \alpha \in I\}$ such that $\varphi_\alpha(U_\alpha)$ is open in $\mathbb{C}^n$, $\cup_{\alpha \in I} = M^n$ and $\phi_\alpha \circ \phi_\beta^{-1}$ is complex analytic.*

S^2 is a complex manifold with stereographic projection $\{(U_N, \phi_N), (U_S, \phi_S),$ but it is not complex analytic with these charts. However, it is with the charts $\{(U_N, \overline{\phi}_N), (U_n, \phi_S)\}$.

Definition 2.10 (Riemann Surface)**.** *A Riemann Surface is a one dimensional complex analytic manifold.*

Conjecture 2.3. *Does S^6 have a complex analytic structure? Conjecture is no.*

Known: S^{2n}, $n \neq 1, 3$ does not have complex structure.

Theorem 2.9 (Inverse Function Theorem)**.** *U open in $\mathbb{C}^n$ and $f : U \to \mathbb{C}^n$ is complex analytic such that $Df(a) : \mathbb{C}^n \to \mathbb{C}^n$ is nonsingular, then $\exists$ a neighborhood W of a and $\tilde{W}$ of $f(a)$ such that $f : W \to \tilde{W}$ is a diffeomorphism with f^{-1} complex analytic.*

This implies the implicit function theorem for complex analytic maps.
$D(f^{-1})(y) = (Df(x))^{-1} = inv \circ Df \circ f^{-1}$, $D(f^{-1}) = inv \circ Df \circ f^{-1}$.
Suppose U open in $\mathbb{C}^n$, $f : U \to \mathbb{C}$ continuous, f is complex analytic iff $\forall a \in U$, $f(z_1, \dots, z_n) = \sum C_{k_1, \dots, k_n} (z - a_1)^{k_1} \dots (z - a_n)^{k_n}$ is convergent in a neighborhood of a

Theorem 2.10 (Osgood)**.** *Suppose U open in $\mathbb{C}^n$ and $F : U \to \mathbb{C}^m$ continuous. Then F is complex analytic (componentwise) iff $\forall a \in U$, $DF(a)$ exists and $DF(a) : \mathbb{C}^n \to \mathbb{C}^m$ is complex linear.*

This implies the inverse function theorem, since $inv(A) \in GL(n, \mathbb{C})$ if $A \in GL(n, \mathbb{C})$.

Proof. $\Rightarrow$ is trivial
$\Leftarrow$: The condition implies that $F(z_1, \dots, z_n)$ is complex analytic in z_i when all other coordinates $z_j \neq z_i$ are fixed.

So $F(z_1+h, z_2, \ldots, z_n) = F(z_1, \ldots, z_n) + A(h, 0, \ldots, 0) + e(h, 0, \ldots, 0)$ and $A = Df(z)$.

$= F(z) + h\alpha + e(h)$, $\lim_{h \to 0} \frac{|e(h)|}{|h|} = 0$, so $\frac{\partial F}{\partial z_1} = \alpha$ exists.

Take $a \in U$ choose $\delta > 0$ small so that $P = \{z = (z_1, \ldots, z_n) : |z_i - a_i| \leq \delta, \forall i\} \subseteq U$.

In P, consider $f(z_1, \ldots, z_n)$, analytic in z_1, and so by the Cauchy Integral, $f(z_1, \ldots, z_n) =$

$$\frac{1}{2\pi i} \int_{|w_1 - a_1| = \delta} \frac{f(w_1, z_2, \ldots, z_n)}{w_1 - z_1} dw_1$$

$f(w_1, z_2, \ldots, z_n) = \frac{1}{2\pi i} \int_{|w_2 - a_2| = \delta} \frac{f(w_1, w_2, z_3, \ldots, z_n)}{w_2 - z_2} dw_2$. Iterating, we get

$$f(z_1, \ldots, z_n) = \left(\frac{1}{2\pi i}\right)^n \int_{|w_1 - a_1| = \delta} \cdots \int_{|w_n - a_n| = \delta} \frac{f(w_1, \ldots, w_n)}{(w_1 - z_1) \ldots (w_n - z_n)} dw_1 \ldots dw_n$$

Each term above can be transformed into power series for δ small enough, and so we obtain out result. □

Example: U open in $\mathbb{C}$. $f : U \to \mathbb{C}$, $Df : \mathbb{C} \to \mathbb{C}$ is $\mathbb{R}$-linear if $Df = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

But if $L : \mathbb{C} \to \mathbb{C}$ is complex liner, then L must be $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$.

So we let $f(z) = (u(x, y), v(x, y))$ where $z = x + iy$. Then $Df = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$, which gives the Cauchy-Riemann Equations.

So Osgood's Theorem gives us Cauchy-Riemann equations in higher dimensions.

Definition 2.11 (Holomorphic Map)**.** *If M, N are complex analytic and $F : M \to N$ continuous, then F is holomorphic if $\forall$ complex charts (U_α, ϕ_α) for M and (V_β, ψ_β) for N we have $\psi_\beta \circ F \circ \phi_\alpha^{-1}$ is holomorphic $\forall \alpha, \beta$.*

Example: The Hyper-Elliptic Riemann Surface: Take $a_1, \ldots, a_n \in \mathbb{C}$, $a_i \neq a_j$. Then $\Sigma = \{(z, w) \in \mathbb{C}^2 : z^2 = \prod_{n=1}^n (w - a_i)\}$ is a Riemann Surface. So $P(z, w) = z^2 - \prod_{i=1}^n (w - a_i)$, $\Sigma = p^{-1}(0)$. Use the Implicit Function Theorem, we must show that 0 is a regular value. If not, then $(z, w) \in p^{-1}(0)$, such that $\frac{\partial P}{\partial z} = \frac{\partial P}{\partial w} = 0$, that is, $2z = 0$ and $z^2 = \prod_{i=1}^n (w - a_i)$ and $\sum_{j=1}^n \left(\prod_{i \neq j} (w - a_i)\right) = 0$. By the second one, $w = a_k$ for some k, say $w = a_1$. Put into the third and we get $\prod_{i=1}^n (a_1 - a_i) = 0$, which is impossible due to $a_i \neq a_j$.

Example: Projective Space $\mathbb{RP}^n$. This is the space of all lines in $\mathbb{R}^{n+1}$ through the origin. ($\mathbb{CP}^n$ is similarly defined). We can define charts (U_i, ϕ_i) for $\mathbb{RP}^n$. U_i will be the set of all points whose i^{th} homogeneous coordinate is nonzero, and define $\phi_i([a_1, \ldots, a_{n+1}]) = \left(\frac{a_1}{a_i}, \ldots, \frac{a_{i-1}}{a_i}, \frac{a_{i+1}}{a_i}, \ldots, \frac{a_{n+1}}{a_i}\right)$.

The transition maps $\phi_2 \circ \phi^{-1}(x_1, \ldots, x_n) = \left(\frac{1}{x_1}, \ldots, \frac{x_2}{x_1}, \ldots, \frac{x_n}{x_1}\right)$.

But what is the topology? It is the quotient topology of $\mathbb{R}^{n+1}/\mathbb{R}^*$.

Example: $\mathbb{CP}^1 = \mathbb{C}^2/\mathbb{C}^*$ has two charts, $(U_1, \phi_1), (U_2, \phi_2)$. The transition map is $\frac{1}{z}$, and each map is a homeomorphism onto $\mathbb{C}$. So $\mathbb{CP}^1 = S^2$ complex analytically.

Definition 2.12 (Biholomorphic)**.** *M, N are complex analytic. Then $f : M \to N$ is biholomorphic if f is a homeomorphism such that f, f^{-1} holomorphic.*

Proposition 2.11. *$\mathbb{RP}^n \simeq S^n/x \sim -x$, $\forall \in S^n$.*

$\mathbb{CP}^n \simeq S^{2n+1}/z \sim \lambda z$, $\lambda \in S^1$.

Proof. Consider $\phi : S^n \to \mathbb{RP}^n : x \mapsto$ the 1-dimensional subspace containing x. ϕ is onto, and ϕ is continuous, so ϕ induces a $1-1$ and onto map $\tilde{\Phi} : S^n/x \sim -x \to \mathbb{RP}^n$, with the domain compact and the codomain Hausdorff, so it is a homeomorphism.

The other is similar. □

Fact: U open in $\mathbb{C}$, and $f : U \to \mathbb{C}$ complex analytic and non-constant, then f is an open mapping.

Corollary 2.12. *If M, N are Riemann Surfaces and $f : M \to N$ is a nonconstant holomorphic map, then f sends open sets to open sets.*

Example: Suppose $p(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_0$ is a polynomial from $\mathbb{C}$ to $\mathbb{C}$. $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$. We define $\tilde{p} : \mathbb{CP}^1 \to \mathbb{CP}^1$ by $\tilde{p}([z, w]) = [\sum_{j=1}^n a_j z^j w^{n-j}, w^n]$, $\tilde{p}(z) = z$ for $z \in \mathbb{C}$ and $\tilde{p}(\infty) = \infty$. Claim: $\tilde{p}$ is analytic. With this, we obtain the fundamental theorem of algebra.

Tangent Spaces

Warm up: U is open in $\mathbb{R}^n$ and $\gamma : (-\epsilon, \epsilon) \to U$ a smooth path, $\gamma(0) = p$. Then the derivative $\frac{d}{dt}\gamma(t)|_{t=0} = v \in \mathbb{R}^n$ is a tangent vector of U at p. Question, $\gamma(t) = [t, t^2 + 1, t], t \in [-1, 1]$, $\gamma : (-1, 1) \to \mathbb{RP}^2$ is smooth. What is $\frac{d}{dt}\gamma(t)|_{t=0}$? Does this make sense?

The intrinsic definition of $\frac{d}{dt}|_{t=0}\gamma(t) = u$. If u acts on smooth functions on U. If $f : U \to \mathbb{R}^1$ smooth, then $u(f) = \frac{d}{dt}f(\gamma(t))|_{t=0}$ is a directional derivative and is equal to $Df|_{\gamma(0)} \cdot v$.

It satisfies $u(f + kg) = u(f) + ku(g)$ an $u(fg) = u(f)g + fu(g)$. evaluated at p.

U open in $\mathbb{R}^n$ and a smooth path $\gamma(t) : (-\epsilon, \epsilon) \to U$, $\gamma(0) = p$.

Tangent vector $V = \frac{d}{dt}|_{t=0}\gamma(t) = \gamma'(0) \in \mathbb{R}^n$

Notation: M^n a smooth manifold, $C^\infty(M)$ is the space of all smooth functions on M. If $f \in C^\infty(V)$ where $p \in V \overset{\subset}{open} U$, then $v(f) = \frac{d}{dt}|_{t=0}f(\gamma(t)) = D(f)_p(0)$ directional derivative.

It satisfies the Leibniz rule $v(fg)v(f)g(p) + f(p)v(g)$, linearity $v(f + kg) = v(f) + kv(g)$ and locality if f, g are the same on an open neighborhood of p, then $v(f) = v(g)$.

M^n a smooth manifold, $p \in M^n$. $C_p^\infty(M) = \{(f,U) : f \in C^\infty(U), p \in U \underset{\text{open}}{\subset} M$

$(f,U) + (g,V) = (f+g, U \cap V)$ gives this a vector space structure, and $(f,U)(g,V) = (fg, U \cap V)$ makes it an algebra.

Definition 2.13 (Tangent Vector)**.** *A tangent vector v to M at p is a map $v : C_p^\infty(M) \to \mathbb{R}$ satisfying the Leibnitz condition, linearity and locality.*

Let T_pM be the space of all tangent vectors to M at p. We claim that T_pM is a vector space. Let $u, v \in T_pM$, then $(u+v)f = u(f) + v(f)$ and $(ku)(v) = ku(v)$ gives it this structure.

eg: M^n open in $\mathbb{R}^n$, $v = \frac{d}{dt}|_{t=0}\gamma(t)$ with γ a smooth path in M^n, then $v(f)$ =tangent vector at $\gamma(0)$. Question: Are these all?

Lemma 2.13. *If $A \in T_pM^n$, $M^n \subset \mathbb{R}^n$ open, then $A = \frac{d}{dt}|_{t=0}\gamma(t)$ for some smooth path γ with $\gamma(0) = p$.*

In particular, $T_pM^n \simeq \mathbb{R}^n$ with basis $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ where $\frac{\partial}{\partial x_i} = \frac{d}{dt}(p + t(0, \ldots, 1, \ldots, 0))|_{t=0}$.

Proof. Without loss of generality, assume $p = 0$. $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $x_i : \mathbb{R}^n \to \mathbb{R}$ smooth. $x_i|_M \in C^\infty(M)$.

Let $a_i = A(x_i) \in \mathbb{R}$, and let $\gamma(t) = t(a_1, \ldots, a_n)$. $\gamma(0) = 0 = p$.

Claim: $\forall f \in C^\infty(M)$, $0 \in U \subset M^n$, then $A(f) = \frac{d}{dt}|_{t=0} f(\gamma(t))$.

Choose $\delta > 0$ such that $B_\delta(0) \subset U$ for $x \in B_\delta(0)$, $g(t) = f(tx)$, $t \in (-1,1)$, $f(x) - f(0) = g(1) - g(0) = \int_0^1 f'(t)dt$. By the chain rule, $g'(t) = \sum_{i=1}^n x_i \frac{\partial f}{\partial x_i}$.

Thus, $\int_0^1 f'(t)dt = \int_0^1 \sum_i x_i \frac{\partial f}{\partial x_i}(tx)dt = \sum_{i=1}^n x_i \int_0^1 \frac{\partial f}{\partial x_i}(tx)dt = \sum_{i=1}^n x_i h_i(x)$, for all $x \in B_\delta(0)$, where $h_i \in C^\infty(B_\delta(0))$ and so $h_i(0) = \frac{\partial f}{\partial x_i}(0)$.

So $A(f(x)) = A(f(x) - f(0)) = A(\sum x_i h_i(x)) = \sum A(x_i h_i(x)) = \sum A(x_i) h_i(0) = \sum a_i \frac{\partial f}{\partial x_i}(0) = \frac{d}{dt}|_{t=0} f((a_1, \ldots, a_n)t)$ □

The standard basis of T_pU is $\frac{\partial}{\partial x_i}|_p$.

Definition 2.14. *If M, N are smooth and $F : U \to V$ smooth where $U \subset M$, $V \subset N$ open, then it's derivative, $DF : T_pM \to T_{F(p)}N$ is defined by sending $v \in T_pM$, $DF(v)(f) = v(f \circ F)$ where $f \in C^\infty(V)$.*

Proposition 2.14 (Chain Rule)**.** *$DF \circ DG = D(F \circ G)$ when $M \xrightarrow{F} N \xrightarrow{G} L$ are smooth maps and $D(\mathrm{id}) = \mathrm{id}$*

Proof. $h \in C^\infty_{F(G(p))}(L)$ and $v \in T_pM$.

$D(F \circ G)(v)(h) = v(h(F \circ G)) = v((h \circ F) \circ G) = (DG(v))(h \circ F)) = DF(DG(v))(h) = (DF \circ DF)(v)(g)$. □

Corollary 2.15. *If $F : M \to N$ is smooth and a local diffeomorphism from a neighborhood of p to a neighborhood of $F(p)$, then $DF : T_pM \to T_{F(p)}N$ is a linear isomorphism.*

Proposition 2.16. *If M^n is an n-dimensional manifold, then $T_pM^n \simeq \mathbb{R}^n$ as an n-dimensional vector space.*

Proof. $\forall p \in M$, take a smooth chart (U, ϕ) at p. Then $D\phi : T_pM \to T_{\phi(p)}\phi(U) = T_{\phi(p)}\mathbb{R}^n$ is an isomorphism by the corollary. □

Proposition 2.17. *U is open in $\mathbb{R}^n$, $F = (F_1, \ldots, F_m) : U \to \mathbb{R}^m$ smooth.*

Then $DF(\frac{\partial}{\partial x_i}|_p) = \sum_{j=1}^m \frac{\partial F_j}{\partial x_i}(p)\frac{\partial}{\partial x_j}|_{F(p)}$, the coefficient matrix is the Jacobian matrix.

Proof. Take a smooth $u(x_1, \ldots, x_m)$ defined in a neighborhood of $F(p)$. $DF(\frac{\partial}{\partial x_i})(u) = \frac{\partial}{\partial x_i}(u(F_1, \ldots, F_m))$, and the chain rule gives $\sum_{j=1}^m \frac{\partial u}{\partial x_j}\frac{\partial F_j}{\partial x_i}(p) = \left(\sum_{j=1}^m \frac{\partial F_j}{\partial x_i}(p)\frac{\partial}{\partial x_j}(u)\right)$ □

Tangent Bundle:

Let $M = M^n$ a smooth manifold. $TM = \cup_{p \in M} T_pM$ be the set of all the tangent vectors in M. $\pi : TM \to M : T_pM \mapsto p$.

eg: U open in $\mathbb{R}^n$, $TU \simeq U \times \mathbb{R}^n$, $\sum_{i=1}^n a_i \frac{\partial}{\partial x_i}|_p \mapsto (p, a_1, \ldots, a_n)$.

Suppose $F : U \to V \subset \mathbb{R}^n$ is open. Then $DF : TU \to TV$ by the identification $U \times \mathbb{R}^n$.

Definition 2.15. *Suppose M^n is smooth. Then TM is a smooth $2n$-manifold such that $\pi : TM \to M^n$ is smooth and $i : T_pM \to TM$ is smooth $\forall p \in M$.*

Proof. For a smooth chart (U, ϕ) of M, produce a smooth chart for TM as $(TU, D\phi)$. $D\phi : TU \to TV$ is a one to one and onto map $V = \phi(U)$ in $\mathbb{R}^n$.

The transition $D\phi \circ (D\psi)^{-1} = D\phi \circ D(\psi^{-1}) = D(\phi \circ \psi^{-1})$ is a diffeomorphism, by Prop 4.

In terms of (U, ϕ) and $(TU, D\phi)$ the map π becomes $\phi \circ \pi \circ (D\phi)^{-1}(p, a) = p$

The rest is easy. □

Definition 2.16 (Smooth Vector Field)**.** *M^n is smooth, then a smooth vector field X on M^n is a smooth map $X : M \to TM$ such that $\pi \circ X = id$. ie, $\forall p \in M$, $X(p) = X_p \in T_pM$.*

Eg U open in $\mathbb{R}^n$ a vector field X on U, $X(p) = \sum_{i=1}^n a_i(p)\frac{\partial}{\partial x_i}|_p$ where $a_i \in C^\infty(U)$ for all i.

Eg. $Y(x, y) = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ defines a nowhere 0 vector field on S^1.

We let $\frac{d}{dt}$ be the standard vector field on $\mathbb{R}^1$. Define $f : \mathbb{R}^1 \to S^1$ smooth map by $f(t) = (\cos t, \sin t)$. Claim: $Df : T\mathbb{R}^1 \to TS^1$ produce a well defined vector field $Df(\frac{d}{dt})$ on S^1 which is equal to Y.

Indeed, $Df(p)(\frac{d}{dt}|_p) = D(f(p + 2\pi n))(\frac{d}{dt}|_{p+2\pi n})$ for all $n \in \mathbb{Z}$.

$D(f)(\frac{d}{dt}) = (-\sin t, \cos t) = -\sin t\frac{\partial}{\partial x}|_p + \cos t\frac{\partial}{\partial y}|_p = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$.

eg. If $F : M \to N$ diffeomorphism, each vector field X on M produces a vector field $DF(X)$ on N.

Consider $f(x) = x^2 : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$. Diffeomorphism. $X = \frac{d}{dx}$ is the standard vector field. $DF(\frac{d}{dx}|_x) = 2x\frac{d}{dx}|_{x^2}$. So $DF(X) = 2\sqrt{x}\frac{d}{dx}$.

eg. $X = a(t)\frac{d}{dt}$ on $\mathbb{R}^1$. $X_0 = a(0)\frac{\partial}{\partial t} \neq 0$ for $a(0) \neq 0$.

Claim: $\exists$ smooth chart (U, ϕ) near 0 such that $(D\phi)(\frac{d}{dt}) = X$ near 0, i.e. $(D\phi^{-1})(X|_v) = \frac{d}{dt}$.

So $(D\phi)(\frac{d}{dt}) = \frac{d\phi(t)}{dt}\frac{d}{dt}|_{\phi(0)} = \phi'(t)\frac{d}{dt}|_{\phi(t)} = X_{\phi(t)} = a(\phi(t))\frac{d}{dt}|_{\phi(t)}$.

So we have $\phi'(t) = a(\phi(t))$ and $\phi(0) = 0$. By Picard's Theorem, we can solve this in $(-\epsilon, \epsilon)$ for some $\epsilon > 0$. Also $\phi'(0) = a(\phi(0)) = a(0) \neq 0$, so $(\phi, (-\delta, \delta))$ is a diffeomorphism by the Inverse Function Theorem.

Computation: $U \subseteq \mathbb{C}$ is open, $z = (x, y) = x + iy$. $F(z) : U \to \mathbb{C}$ holomorphic. $F'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$.

Theorem 2.18 (Poincare-Hopf)**.** *There are no smooth vector fields on S^2 such that $X_p \neq 0$ for all $p \in S^2$.*

Proof. Suppose not. Say, X is a vector field such that $X_p \neq 0, \forall p \in S^2$. Specifically, $X_N \neq 0$.

Consider the vector field $Y = D\phi_N(X|_{S^2 \setminus \{N\}})$ is a vector field on $\mathbb{C}$ which is nowhere vanishing.

Take a large ball $B_\gamma(0)$ with $\gamma >> 1$. Then $Y_p|_{\partial B_\gamma(0)}$ looks like FIGURE ONE by Picard's Theorem. So $Y_{B_\gamma(0)}$ is a nowhere vanishing vector field on $B_\gamma(0)$. Near $\partial B_\gamma(0)$ it looks like the above picture.

So we can define a map $f : B_\gamma(0) \to S^1 \subset \mathbb{C}^* : p \mapsto Y_p/\|Y_p\|$.

The winding number of f on $\partial B_R(0)$ for $R \leq \gamma$ is $\frac{1}{2\pi i}\int_{\partial B_R(0)} \frac{dz}{z}$. It is equal to ± 2 by figure and is continuous in $\mathbb{R}$ by complex analysis.

But the winding number is equal to zero for $R \to 0$, since $Y|_0$ are parallel.

Thus we get a contradiction. □

Corollary 2.19. $TS^2 \neq S^2 \times \mathbb{R}^2$

Corollary 2.20. *There is a vector field X on S^2 such that $D\phi_N(X|_{S^2 \setminus \{N\}}) = \frac{\partial}{\partial x}$.*

The Lie Bracket

Suppose X is a vector field on M^n, then $X : C^\infty(M) \to C^\infty(M) : f \mapsto X(f)$ where $X(f)(p) = X_p(f)$. X satisfies the following:

1. $X(f + kg) = X(f) + kX(g)$
2. $X(fg) = X(f)g + fX(g)$

If X, Y are vector fields on M, then $\forall f \in C^\infty(M)$, $[X, Y](f) = X(Y(f)) - Y(X(f))$ is a vector field. We call this the Lie Bracket of X and Y.

Proposition 2.21. *For X, Y, Z vector fields on M^n, then $[X, Y] = -[Y, X]$, $[X, Y]$ is bilinear in X, Y and it satisfies the Jacobi Identity.*

Theorem 2.22. *If $F : U \to V$ is a diffeomorphism and X, Y are vector fields on U, then $[DF(X), DF(Y)] = DF([X, Y])$.*

Proof. By diagram chasing. Diagram omitted. □

Definition 2.17 (Left Invariant)**.** *G is a Lie Group. $\forall g \in G$ let $\ell_g : G \to G : x \mapsto gx$, then ℓ_g is a diffeomorphism with inverse $(\ell g)^{-1} = \ell g^{-1}$. A vector field X is left invariant if we have $D\ell_g(X) = X$*

eg. $G = GL(n,\mathbb{R})$. X left invariant and $X_{id} = v \in T_{id}GL = \mathbb{R}^{n\times n}$. What is X_A for $A \in GL(n,\mathbb{R})$?

The solution is $X_A = A \cdot v \in T_A(GL)$. As $GL(n,\mathbb{R}) \subset \mathbb{R}^{n\times n}$, we identify $TGL \simeq GL \times \mathbb{R}^{n\times n}$ in a natural way.

Proposition 2.23. *The space of all left-invariant vector fields on G is linearly isomorphic to $T_{id}G$, where the isomorphism π sends $X \mapsto X_{\mathrm{id}}$.*

Proof. π is onto: Let $v \in T_{id}G$. Define a vector field (not yet smooth) $X_p = (D\ell_p)(v)$. We must verify that this is left invariant and smooth. Smoothness is trivial. Suppose $g \in G$, then $(D\ell_g)(X_p) = X_{gp} = D(\ell_{gp})v$ by the chain rule and the definition of a Lie Group. □

Definition 2.18 (Lie Algebra)**.** $(T_{id}G, [])$ *is defined to be the Lie Algebra of G.*

Theorem 2.24. *$T_{id}GL(n,\mathbb{R}) = \mathbb{R}^{n\times n}$ then $[A,B] = AB - BA$ matrix multiplication.*

$T_{id}SL(n,\mathbb{R}) = \{A \in \mathbb{R}^{n\times n} : \operatorname{tr} A = 0\}$

$T_{id}O(n,\mathbb{R}) = \{A \in \mathbb{R}^{n\times n} : A + A^T = 0\}$.

Proof. $GL(n,\mathbb{R}) \subset \mathbb{R}^{n\times n}$ open, $T_{id}GL \subset T_{id}\mathbb{R}^{n\times n}$, basis $\frac{\partial}{\partial x_{ij}}$. $x = (x_{ij}) \in \mathbb{R}^{n\times n}$.

Given $A \in \mathbb{R}^{n\times n} = T_{id}GL$, let $\tilde{A}$ be the left invariant vector field on GL with $\tilde{A}_{id} = A$, $\tilde{A}_X = XA \in T_X GL = T_X\mathbb{R}^{n\times n}$.

Take $B \in T_{id}$, $\tilde{B}_X = XB$ the left invariant vector field. Choose $f = f(x) \in C^\infty(GL)$. $[A,B](f) = [\tilde{A},\tilde{B}](f)|_{id} = \tilde{A}|_{id}(\tilde{B}f) - \tilde{B}|_{id}(\tilde{A}f) = A(\tilde{B}(f)) - B(\tilde{A}(f))$

$\tilde{B}(f) = \tilde{B}_X(f)$, to find $A = [a_{ij}]$ and $B = [b_{ij}]$, let $\tilde{B}_X = XB = [\sum_{k=1}^n x_{ik}b_{kj}]_{n\times n}$

So $A(\tilde{B}(f)) = \sum_{r,s} a_{rs}\frac{\partial}{\partial x_{rs}}(\tilde{B}(f))|_{X=\mathrm{id}} = \sum_{r,s,i,j,k} a_{rs}\frac{\partial x_{ik}}{\partial x_{rs}} b_{rj}\frac{\partial f}{\partial x_{ij}}(\mathrm{id}) + \frac{\partial^2}{\partial x_{rs}\partial x_{rk}}(f\ldots)$

This is the same as $\sum_{i,j,k,r,s} a_{rs}b_{kj}\frac{\delta_{ir}}{i}\frac{\delta_{ks}}{k}\frac{\partial f}{\partial x_{ij}}(\mathrm{id}) + \frac{\partial^2}{\partial\partial}$. Thus,

$[\tilde{A},\tilde{B}]_{id}(f) = \sum_{i,j,k} a_{ik}b_{kj}\frac{\partial f}{\partial x_{ij}}(id) + \frac{\partial^2}{\partial\partial}$. So $= \sum_{i,j,k}(a_{ik}b_{kj} - b_{ik}a_{kj})\frac{\partial f}{\partial x_{ij}}(id) = [A,B](f)$ by definition. $[A,B]_{ij} = \sum_k(a_{ij}b_{kj} - b_{ik}a_{kj})$.

To see $T_{id}SL(n,\mathbb{R})$... □

Lemma 2.25. *If U open in $\mathbb{R}^n$ and $F : U \to \mathbb{R}$ with regular value p, then $\iota : F^{-1}(p) \to U$ inclusion $D\iota : T_xF^{-1}(p) \to T_xU$ is injective and has image $\ker(DF : T_xU \to T_{F(x)}\mathbb{R})$.*

Proof. $F \circ \iota$ =constant= p. So $DF \circ D\iota = 0$, so $\mathrm{Im}(D\iota) \subset \ker(DF)$. After a change of coordinates $i : (x_1,\ldots,x_{n-m}) \mapsto (x_1,\ldots,x_{n-m},0,\ldots,0)$ so $D\iota$ is injective.

But $\dim\mathrm{Im}(D\iota) = \dim F^{-1}(p) = n - m$, and also $\dim\ker(DF) = n - m$, so $\ker(DF) = \mathrm{Im}(D\iota)$. □

e.g. $S^n = f^{-1}(1) \subset \mathbb{R}^{n+1}$ for $f(x) = x \cdot x : \mathbb{R}^{n+1} \to \mathbb{R}$. $DF_x : y \mapsto 2x \cdot y$. So $T_x S^n = \{y \in \mathbb{R}^{n+1} : y \cdot x = 0\}$.

e.g. $f : \mathbb{R}^{n\times n} \to \mathbb{R} : A \mapsto \det A$. So $Df(id) : T_{id}\mathbb{R}^{n\times n} \to T_1\mathbb{R}$, $Df(id) : B \to \operatorname{tr} B$.

$Df(id)(B) = \frac{d}{dt}|_{t=0}(f(I+tB)) = \frac{d}{dt}|_{t=0}\det(I+tB) = \frac{d}{dt}|_{t=0}\left(\prod_{i=1}^n(1+tb_{ii}) + t^2 \ldots\right) = b_{11} + \ldots + b_{nn} = \operatorname{tr} B$.

This shows, b Lemma 3, that $T_{\mathrm{id}}SL(n,\mathbb{R}) = \{A \in \mathbb{R}^{n\times n} : \operatorname{tr} A = 0\}$

3 Riemannian Geometry

Tensors

V a vector space over $\mathbb{R}$ with basis $v_1, \ldots, v_n$. $V^* = \hom(V, \mathbb{R})$ the dual vector space of all linear functionals on V with dual basis $v_1^*, \ldots, v_n^*$ where $v_i^*(v_j) = \delta_{ij}$.

e.g. The dual space of T_pU is T_p^*U, the cotangent space. The dual basis of $\frac{\partial}{\partial x_i}$ is dx_i, and $dx_i\left(\frac{\partial}{\partial x_i}\right) = \delta_{ij}$.

Definition 3.1 (k-linear functions)**.** *A k-linear function $f : V \times \ldots \times V \to \mathbb{R}$ with k copies of V, $f(x_1, \ldots, x_i, \ldots, x_n)$ is linear in x_i when other variables are fixed.*

f is called symmetric if $f \circ \sigma = f$ for all $\sigma \in S_k$.

f is called alternating if $f = \operatorname{sign}(\sigma) f \circ \sigma$.

Alternating k-linear functions are sometimes called k-forms.

$\otimes^k V^*$ =vector space of all k-linear functions on V. $(f+g)(\alpha) = f(\alpha)+g(\alpha)$.

$\bigwedge^k V^*$ =vector space of all alternating k-forms on V.

The tensor product $\otimes : \otimes^k V^* \times \otimes^\ell V^* \to \otimes^{k+\ell} V^* : (\alpha, \beta) \mapsto \alpha\beta$.

$(\alpha \otimes \beta)(x_1, \ldots, x_k, y_1, \ldots, y_\ell) = \alpha(x_1, \ldots, x_k)\beta(y_1, \ldots, y_\ell)$.

e.g. $V = \mathbb{R}^n$, then $\det(v_1, \ldots, v_n) = \det[v_1, \ldots, v_n]$, and $\det \in \bigwedge^n(\mathbb{R}^n)^*$. $b : V \times V \to \mathbb{R}$ is called bilinear, and it is the same as $b \in \otimes^2 V^*$.

Lemma 3.1. $\dim \otimes^k(V^*) = n^k$.

Proof. $f \in \otimes^k V^* \Rightarrow f$ is determined by $f(v_{i_1}, \ldots, v_{i_k})$ where $i_j \in \{1, \ldots, n\}$ by linearity, so $\dim \otimes^k V^* \leq n^k$ the number of possible choices of $i_1, \ldots, i_k$. But $(v_{i_1}^* \otimes \ldots \otimes v_{i_k}^*)(v_{j_1}, \ldots, v_{j_k})$ is 1 if all the $i_k = j_k$ and 0 else. So $\dim \otimes^k V^* \geq n^k$. □

Suppose U, V are vector spaces and $F : U \to V$ is linear. Then it induces a linear map $F^* : \otimes^k V^* \to \otimes^k U^* : \alpha \mapsto \alpha(F, F, \ldots, F) = F^*(\alpha)$, F^* is linear and satisfies $(id)^* = id$ and $F^*(\alpha \otimes \beta) = F^*(\alpha) \otimes F^*(\beta)$ and $(F \circ G)^* = G^* \circ F^*$.

Lemma 3.2. *U has basis $u_1, \ldots, u_m$ and U^* has dual basis $u_1^*, \ldots, u_m^*$. $F(u_i) = \sum_{k=1}^n a_{ij} v_j$ then $F^*(v_j^*) = \sum_{i=1}^m a_{ij} u_i^*$*

Proof. $F^*(v_j^*)(u_k) = v_j^*(F(u_k)) = v_k^*(\sum_{i=1}^n a_{ki} v_i) = \sum_i a_{ki}\delta_{ij} = a_{kj}$ □

e.g. $F : \mathbb{R}^n \to \mathbb{R}^m$ smooth with $F = (F_1, \ldots, F_m)$. We have $DF_x(\frac{\partial}{\partial x_i}) = \sum_{j=1}^n \frac{\partial F_j}{\partial x_i} \frac{\partial}{\partial y_j}|_{F(x)}$.

Notation, F^* is defined yo be $(DF)^*$. So $F^*(dy_j) = \sum_{i=1}^m \frac{\partial F_j}{\partial x_i} dx_i = d(F_j)$.

So $df = \sum \frac{\partial f}{\partial x_i} dx_i$, just as in calculus.

Dual of DF by the Substitution Rule.

e.g. $F(z) = \frac{1}{z}$. $F(x, y) = \frac{1}{x^2+y^2}(x, -y) = (u, v)$. What is $F^*(du)$? It is $F^*(du) = d\left(\frac{x}{x^2+y^2}\right) = \frac{y^2-x^2}{(x^2+y^2)^2} dx - \frac{2xy}{(x^2+y^2)^2} dy$. It is discontinuous at 0.

Corollary 3.3. *There are no smooth* 1*-forms* ω *on the Riemann Sphere such that* $\omega|_{S^2 \setminus \{N\}} = \phi_N^*(dx)$.

e.g. If $\alpha, \beta \in V^*$ then $\alpha \wedge \beta$ is a 2-form and it is given by $\alpha \otimes \beta - \beta \otimes \alpha$.

e.g. The standard symplectic 2-forms ω on $\mathbb{R}^{2n} \ni (x_1, y_1, x_2, \ldots, y_n)$ $\omega = \sum_{i=1}^n dx_i \wedge dy_i \in \bigwedge^2 (\mathbb{R}^{2n})^*$.

ω is nonsingular, that is, $[\omega(e_i, e_j)]_{2n \times 2n}$ has nonzero determinant for a basis $\{e_1, \ldots, e_{2n}\}$ of $\mathbb{R}^{2n}$.

The wedge product $\wedge : \bigwedge^k V^* \times \bigwedge^\ell V^* \to \bigwedge^{k+\ell} V^*$. We must define a projection $A : \otimes^k V^* \to \bigwedge^k V^*$.

$A(f)(x_1, \ldots, x_k) = \sum_{\sigma \in S_k} \operatorname{sign}(\sigma) f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$

Definition 3.2 (Wedge Product)**.** *If* $\alpha \in \bigwedge^k V^*, \beta \in \bigwedge^\ell V^*$*, then* $\alpha \wedge \beta := \frac{1}{k!\ell!} A(\alpha \otimes \beta)$.

Definition 3.3 (Pullback)**.** *If* $f : U \to V$*, then* $f^* : \otimes^k V^* \to \otimes^k U^*$, $f^*(\alpha)(u_1, \ldots, u_k) = \alpha(f(u_1), \ldots, f(u_k))$. *If* $k = 1$*, then* $f : U \to V$*, then* $f^* : V^* \to U^*$*, so* $f(u_i) = \sum_{j=1}^n a_{ij} v_j \Rightarrow f^*(v_j^*) = \sum_{i=1}^m a_{ij} u_i^*$.

Proposition 3.4. *If* $\alpha \in \bigwedge^k V^*$ *and* $\beta \in \bigwedge^\ell V^*$*, then*

1. $\alpha \wedge \beta = (-1)^{k\ell} \beta \wedge \alpha$
2. $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$.
3. $f : U \to V$ *linear, then* $f^*(\alpha \wedge \beta) = f^*\alpha \wedge f^*\beta$.

Proof. $f^*(\alpha \wedge \beta) = f^*\left(\frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} \operatorname{sign}(\sigma)(\alpha \otimes \beta) \cdot \sigma\right) = \frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} f^*((\alpha \otimes \beta) \cdot \sigma) = f^*(\alpha) \wedge f^*(\beta)$

$\alpha \wedge \beta = \frac{1}{k!\ell!} \sum \operatorname{sign}(\sigma)(\alpha \otimes \beta) \cdot \sigma$, $\beta \wedge \alpha = \frac{1}{k!\ell!} \sum (\beta \otimes \alpha) \cdot \sigma$, so the first conclusion holds. □

Example: suppose $f : V \to V$ linear. $f^*(v_1^* \wedge \ldots \wedge v_n^*) = \det(f) v_1^* \wedge \ldots \wedge v_n^*$. We sometimes call $v_1^* \wedge \ldots \wedge v_n^*$ the volume form.

Definition 3.4 (Tensor Bundle)**.** M^n *a smooth manifold. Then we let* $T^{(0,r)}M = \cup_{p \in M} \otimes^r (T_pM)^*$ *be the* $(0, r)$ *tensor bundle.*

$T^{(0,1)}$ *is the cotangent bundle.*

$\bigwedge^r M = \cup_{p \in M} \bigwedge^r (T_p^* M) \subset T^{(0,r)} M$.

Definition 3.5 (Smooth $(0, r)$ Tensor)**.** *A smooth $(0, r)$ tensor on M^n is a map $t : M \to T^{(0,r)}M$ such that $\forall p \in M$, $t_p = t(p) \in \otimes^r T_p^*M$ and varying smoothly in p. If (U, ϕ) is a smooth chart of M at p, then $(D(\phi^{-1}))^*(t|_U) = \sum a_{i_1,\ldots,i_r}(x)dx_{i_1} \otimes \ldots \otimes dx_{i_r}$ in an open set $\phi(U) \in \mathbb{R}^n$ so that $a_{i_1,\ldots,i_k} \in C^\infty(\phi(U))$*

e.g. $\Omega \subset \mathbb{R}^n$ open, then $T^{(0,r)}\Omega \simeq \Omega \times \bigotimes^r (T_p\mathbb{R}^n)^*$.

Lemma 3.5. *If $f : \tilde{\Omega} \to \Omega$ is a smooth function, then $Df : T_p\tilde{\Omega} \to T_{f(p)}\Omega$ gives $(Df)^* : T^*_{f(p)}\Omega \to T_p^*\tilde{\Omega}$ for $t = \sum a_{i_1,\ldots,i_k}(x)dx_{i_1} \otimes \ldots \otimes dx_{i_k}$.*

$(Df)^*(t) = \sum_{i_1,\ldots,i_r} a_{i_1,\ldots,i_r}(f(u))df_{i_1}(u) \otimes \ldots \otimes df_{i_r}(u)$ where $f(u) = (f_1(u), \ldots, f_n(u))$, $u \in \tilde{\Omega} \subset \mathbb{R}^n$.

Substitution: $x_i = f_i(u)$ just replace it in the formula for t.

Proof. Use $(Df)^*$ is linear and $(Df)^*(\alpha \otimes \beta) = ((Df)^*(u)) \otimes ((Df)^*p)$. It suffices to show: $(Df)^*(dx_i) = df_i(u)$. We've done this previously. □

Corollary 3.6. *If f is smooth, then $(Df)^*(t)$ is again smooth. Thus, the smoothness of $(0, r)$ tensor in M^n is well-defined.*

Definition 3.6 (Riemannian Metric)**.** *M^n a smooth manifold. A smooth r-form ω on M^n is a $(0, r)$ smooth tensor such that $\forall p \in M^n$, $\omega_p \in \bigwedge^r T_p^*M$. A Riemannian metric g on M^n is a smooth $(0, 2)$ tensor such that $\forall p \in M^n$, $g_p \in \otimes^2 T_p^*M$ and satisfies the following:*

1. *$g_p(u, v) = g_p(v, u) \forall u, v \in T_pM$ (symmetric)*
2. *$g_p(u, u) > 0$ for all $u \in T_pM \setminus \{0\}$*

That is, g_p is a positive definite symmetric bilinear form on T_pM.

Notation: $\alpha, \beta \in V^*$ we use $\alpha \cdot \beta := \frac{1}{2}(\alpha \otimes \beta + \beta \otimes \alpha)$, and notice that $\alpha \cdot \beta = \beta \cdot \alpha$.

In $\mathbb{R}^n$, we use $dx_idx_j = \frac{1}{2}(dx_i \otimes dx_j + dx_j \otimes dx_i)$.

If $\Omega \subset \mathbb{R}^n$ is open, a Riemannian metric g can be written $g(x) = \sum a_{ij}(x)dx_idx_j$ where $a_{ij}(x) = a_{ji}(x) = g\left(\frac{\partial}{\partial x_i}|_x, \frac{\partial}{\partial x_j}|_x\right)$ such that the matrix $[a_{ij}(x)]_{n\times n}$ is positive definite $\forall x \in \Omega$.

e.g. if $\mathbb{E}^n$ is euclidean n-space, then $g = \sum_{i=1}^n (dx_i)^2$.

Basic: If U open in $\mathbb{R}^n$, $F : U \to \mathbb{R}^M$ is smooth $(y_1, \ldots, y_m) = (F_1(x), \ldots, F_m(x))$, and call $F^* = (DF)^*$ the pullback.

$F^*(a(y)dy_{i_1} \otimes \ldots \otimes y_{i_k}) = a(F(x))dF_{i_1}(x) \otimes \ldots \otimes F_{i_k}(x)$ and $F^*(a(y)dy_{i_1} \wedge \ldots \wedge y_{i_k}) = a(F(x))dF_{i_1}(x) \wedge \ldots \wedge F_{i_k}(x)$

Recall a Riemannian Metric (M^n, g) g is a symmetric positive definite $(0, 2)$ tensor on M^n.

e.g. 1: Classical Differential Geometry:

S a smooth submanifold in $\mathbb{E}^n$, and $i : S \to \mathbb{E}^n$, $S = \{(x, y, f(x, y)) : (x, y) \in U \text{ open in } \mathbb{R}^3\}$

$(i)^*(\sum dx_j^2)$ gives a Riemannian Metric on S. $F : U \to S \hookrightarrow \mathbb{E}^3$, so $F^*(\sum dx_j^2) = dx^2 + dy^2 + (df(x,y))^2 = (1+f_x)^2 dx^2 + (1+f_y)^2 dy^2 + 2f_x f_y dxdy$.

e.g. 2: S^1 with standard metric $g = (d\theta)^2$, the invariant vector field is $\frac{\partial}{\partial\theta}$.

$i : S^1 \to \mathbb{E}^2$ by $\theta \mapsto (\cos\theta, \sin\theta)$. $i^*(dx^2 + dy^2) = d\theta^2$.

However, if t is a $(0,2)$ symmetric tensor on M^n which is nondegenerate at each point $p \in M$, then (M^n, t) is a semi-Riemannian metric.

e.g. Killing form on $GL(n, \mathbb{R})$.

$T_{\mathrm{id}}GL \simeq \mathbb{R}^{n\times n}$, so $\alpha_0 : T_{\mathrm{id}}GL \times T_{\mathrm{id}}GL \to \mathbb{R}$ by $\alpha_0(A,B) = \mathrm{tr}(A,B)$ symmetric bilinear form. It is nondegenerate iff $\forall A \neq 0$ there is a B such that $\mathrm{tr}(AB) \neq 0$. We can achieve this by taking $B = A^t$.

This is not positive definite, since $\mathrm{tr}(A^2) < 0$ can occur.

Produce a left invariant $(0,2)$ tensor α such that $(\ell_g)^*\alpha = \alpha$ at $g \in GL$ by $\alpha_g = (\ell_{g^{-1}})^*\alpha_0$.

Proposition 3.7. *α is also right-invariant, ie, α is bi-invariant.*

Proof. $(R_g)^*\alpha = \alpha$ where $R_g : x \mapsto xg$

$F_g : GL \to GL : x \mapsto g^{-1}xg$, and $(DF_g)^*\alpha_0 = \alpha_0$. So this is what we need to prove.

Take $A, B \in T_{\mathrm{id}}GL$, $F_g^*(\alpha_0)(A,B) \stackrel{?}{=} \alpha_0(A,B) = \mathrm{tr}(AB)$. We know $F_g^*(\alpha_0)(A,B) = \alpha_0(DF_g(A), DF_g(B)) = \mathrm{tr}(DF_g(A), DF_g(B))$. But what is $DF_g(A)$? It is $g^{-1}Ag$. So $= \mathrm{tr}(g^{-1}Agg^{-1}Bg) = \mathrm{tr}(g^{-1}ABg) = \mathrm{tr}(AB)$. □

The same expression is bi-invariant $(0,2)$ forms on $SL(n,\mathbb{R})$ and $O(n)$, that is, $\mathrm{tr}(AB)$.

e.g. Killing form is negative definite on $O(n)$, because the Lie Algebra is given by $\{A \in \mathbb{R}^{n\times n} : A = -A^t\}$, and here $\mathrm{tr}(AA) = \mathrm{tr}(A(-A^t)) = -\mathrm{tr}(AA^t) < 0$.

Killing form is then the BEST Riemannian metric on $O(n)$. For $n = 2$, this is $d\theta^2$.

Homework: Show that there are no Riemannian Metrics g on $SL(2,\mathbb{R})$ which is bi-invariant. Q: What is the signature of $\mathrm{tr}(A,B)$ on $\mathbb{R}^{n\times n}$?

Differential Forms:

Suppose V is an n-dimensional vector space with basis $v_1, \ldots, v_n$. Then $\bigwedge^r V^*$ is $\binom{n}{r}$ dimensional with a basis $v_{i_1}^* \wedge \ldots \wedge v_{i_r}^*$ where $i_1 < i_2 < \ldots < i_r$.

Proof. If $f \in \bigwedge^r V^*$, then f is determined by $f(v_{i_1}, \ldots, v_{i_r})$, and so $\dim \bigwedge^r V^* \leq \binom{n}{r}$. And $(v_{i_1}^* \wedge \ldots \wedge v_{i_r}^*)(v_{j_1}, \ldots, v_{j_r}) = 0$ if $i_\mu \neq j_\mu$ for some μ and 1 otherwise. This implies that $\{v_{i_1}^* \wedge \ldots \wedge v_{i_r}^*\}$ linearly independent, so $\dim \bigwedge^r V^k \geq \binom{n}{r}$. □

Suppose U open in $\mathbb{R}^n$. Then $\bigwedge^r U = U \times \bigwedge^r (\mathbb{R}^n)^*$. $\sum_{i_1 < \ldots, i_r} a_{i_1,\ldots,i_r} dx_{i_1} \wedge \ldots \wedge dx_{i_r}|_p \mapsto (p, \sum a_{i_1,\ldots,i_r} dx_{i_1} \wedge \ldots \wedge dx_{i_r})$.

A smooth r-form Ω in U, $\Omega = \sum_{i_1,\ldots,i_r} a_{i_1,\ldots,i_r}(x) dx_{i_1} \wedge \ldots \wedge dx_{i_r}$ where $a_{i_1,\ldots,i_r}(x) \in C^\infty(U)$.

e.g. 4. If $f \in C^\infty(U)$, then what is df? A $(0,1)$-form= $(0,1)-$tensor.

$f(x_1, \ldots, x_n)$, so $df = \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i}\right) dx_i$ by either $f : U \to \mathbb{R}$ then $df = (Df)^*(dt) = f^*(dt)$ or If X is a vector field on U, then $(df)(X) = X(f)$.

Let $\Gamma(\bigwedge^r M)$ be the vector space of all smooth r-forms on M^n.

Theorem 3.8. *For smooth M^n, $\exists$ linear map $d : \Gamma(\bigwedge^r M) \to \Gamma(\bigwedge^{r+1} M)$ the exterior derivative, such that*

1. $d \circ d = 0$
2. $f \in \Gamma(\bigwedge^0 M) = C^\infty(M)$, $df = (Df)^*(dt)$
3. $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^r \alpha \wedge d\beta$, *with α and r-form*
4. *If $F : N \to M$ smooth, then $F^*(d\alpha) = d(F^*(\alpha))$*

This was motivated by integration and partly due to Cartan.

Proof. Part 2 has been checked.

Case 1: M^n open in $\mathbb{R}^n$. $\omega = a(x)d_{i_1} \wedge \ldots \wedge dx_{i_r}$, r-form on M^n. Define $d\omega = \sum_{j=1}^n \left(\frac{\partial a}{\partial x_j}\right) dx_j \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_r}$.

$d \circ d(\omega) = \sum_{k=1}^n \sum_{j=1}^n \left(\frac{\partial^2 a}{\partial x_j \partial x_k}\right) (dx_k \wedge dx_j) \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_r} = 0$

Now we move on to 3, and take $\eta = b(x)dx_{j_1} \wedge \ldots \wedge dx_{j_s} = b(x)dx_J$. Then $d(\omega \wedge \eta) = d(a(x)b(x)dx_I \wedge dx_J) = \sum_{k=1}^n \frac{\partial}{\partial x_k}(ab)dx_k \wedge dx_I \wedge dx_J = \sum_{k=1}^n \left(\frac{\partial a}{\partial x_k} dx_k \wedge dx_I\right) (b dx_J) + \sum_{k=1}^n \left(a \frac{\partial b}{\partial x_k} dx_k \wedge dx_I \wedge dx_J\right) = d\omega \wedge \eta + (-1)^r \omega \wedge d\eta$.

For part 4, $F(u_1, \ldots, u_m) = (F_1(u), \ldots, F_n(u)) : U \to \mathbb{R}^n$. $\alpha = a(x)dx_{i_1} \wedge \ldots \wedge dx_{i_r} \in \Gamma(\bigwedge^r \mathbb{R}^n)$

$F^*(d\alpha) = F^*\left(\sum_{j=1}^n \frac{\partial a}{\partial x_j} dx_j \wedge dx_I\right) = \sum_{j=1}^n \frac{\partial a}{\partial x_j}(F(u))dF_j(u) \wedge dF_{i_1} \wedge \ldots \wedge dF_{i_r}(u)$.

And this is $\sum_{j,k} \frac{\partial a}{\partial x_j}\frac{\partial F_j}{\partial u_k} du_k \wedge dF_{i_1} \wedge \ldots \wedge dF_{i_r} = \sum_j \frac{\partial}{\partial u_k}(a(F(u))du_k) \wedge dF_{i_1} \wedge \ldots \wedge dF_{i_r} = d(a(F(u)) \wedge dF_{i_1} \wedge \ldots \wedge dF_{i_r}) = d(F^*(\alpha))$.

Case 2: Any manifold M^n. Suppose that $\alpha \in \Gamma(\bigwedge^r M)$. We define $d\alpha$ locally as follows. Take a chart (U, ϕ), then $d\alpha|_U = \phi^*(\alpha(\phi^{-1})^*(\alpha))$.

Claim: $d\alpha$ is independent of the choice of charts.

Suppose (V, ψ) is another chart with $U \cap V \neq \emptyset$.

We want to have that $\phi^* d(\phi^{-1})^*\alpha = \psi^* d(\psi^{-1})^*\alpha$ in $U \cap V$. We apply $(\phi^{-1})^*$, that is, $d(\phi^{-1})^*\alpha = (\phi^*)^{-1}\psi^* d(\psi^{-1})^*\alpha = (\phi^{-1})^*\psi^* d(\psi^{-1})^*\alpha = (\psi\phi^{-1})^* d(\psi^{-1})^*\alpha$
$= \alpha(\psi\phi^{-1})^*(\psi^{-1})^*(\alpha) = d((\psi^{-1})\psi\phi^{-1})^*(\alpha) = d(\phi^{-1})^*(\alpha)$ ◻

Q: Does every smooth manifold M^n have a Riemannian Metric? Vector Field? Smooth Forms?

Definition 3.7. *$f \in C^\infty(M)$ has suppose* $\text{supp}(f) = \overline{\{x : f(x) \neq 0\}}$.

Theorem 3.9 (Partition of Unity). *If $\{U_\alpha : \alpha \in I\}$ is an open cover of M^n, then there exists a partition of unity subordinate to $\{U_\alpha : \alpha \in I\}$, $\lambda_\alpha \in C^\infty(M) : \alpha \in I\}$ such that*

1. $\mathrm{supp}(\lambda_\alpha)$ *is compact and* $\mathrm{supp}(\lambda_\alpha) \subseteq U_\alpha$
2. $\forall x \in M$, $\exists$ *a neighborhood V of x such that $\{\alpha : \mathrm{supp}(\lambda_\alpha) \cap V \neq \emptyset\}$ is finite*
3. $\sum_{\alpha \in I} \lambda_\alpha(x) \equiv 1$ *on M^n.*

Corollary 3.10. *Every smooth manifold M^n has a Riemannian Metric, and nonzero vector field, and a non-zero k-form for $k \leq n$.*

Proof. Cover M^n by charts $\{(U_\alpha, \phi_\alpha) : \alpha \in I\}$. Let $g_{st} = \sum_{i=1}^n dx_i^2$, the standard metric on $\mathbb{R}^n$. $(\phi_\alpha)^* g_{st}$ is the pullback and gives us a metric on U_α.

Let $\{\lambda_\alpha : \alpha \in I\}$ be a partition of unity subordinate to $\{U_\alpha\}$. Then $g = \sum_{\alpha \in I} \lambda_\alpha (\phi_\alpha)^* g_{st}$ is a Riemannian Metric. It is well defined, clearly, as it is a locally finite sum, and it is a symmetric $(0,2)$ tensor.

It is positive definite because $\lambda_\alpha \geq 0$. □

Remark: If M^2 is a Riemann surface, then there is a nonconstant analytic map $\phi : M^2 \to S^2$.

We will not prove the theorem on partitions of unity:

We will only prove the result for compact manifolds.

We will need the following lemma

Lemma 3.11. *Let $C(r)$ be the cube of half-side length r and center at the origin.*
$\exists \lambda \in C^\infty(\mathbb{R}^n)$ *such that $\lambda \geq 0$ and $\lambda|_{C(1)} \equiv 1$ and $\lambda|_{C(2)^c} \equiv 0$*

Proof. Define $f(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \leq 0 \end{cases}$ is C^∞ and $g(x) = \frac{f(x)}{f(x)+f(1-x)}$ is also C^∞. So $h(x) = g(x+2)g(x-2)$ is λ on $\mathbb{R}^1$.

So $\lambda(x_1, \ldots, x_n) = h(x_1) \ldots h(x_n)$. □

Proof. M^n compact implies that there exist finite subcover $\{U_1, \ldots, U_N\}$ of $\{U_\alpha\}$.

$\forall x \in M$, $\exists$ a charge (V_x, ϕ_x) with $x \in V_x$ and $\overline{V_x} \subset U_i$ for some i and $\phi_x(V_x) \supset C(2)$ with $\phi_x(x) = 0$. Now $\{\phi_x^{-1}(C(1)) : x \in M\}$ is an open cover of M^n. M^n compact implies that $\exists$ finite subcover $\{(V_1, \phi_1), \ldots, (V_m, \phi_m)\}$ such that $M = \cup \phi_i^{-1}(C(1))$.

Let $\gamma : \{1, \ldots, m\} \to \{1, \ldots, N\}$ such that $V_i \subset U_{\gamma(i)}$ for all i.

Now define $h_i = \begin{cases} \lambda \circ \phi_i & x \in V_i \\ 0 & x \notin V_i \end{cases}$.

$h_i \in C^\infty(M)$ and $\mathrm{supp}(h_i) \subset \overline{V}_i \subset U_{\gamma(i)}$.

Claim: $\sum h_i(x) > 0$ for all $x \in M$. Indeed, $\forall x \in M$, $x \in \phi_i^{-1}(C(1))$ for some i, so $h_i(x) = 1$. Let $g(x) = \sum g_i(x) \in C^\infty(M)$. $g(x) > 0$ so $1 = \sum_{i=1}^m \frac{h_i(x)}{g(x)}$. $\mathrm{supp}(h_i/g) \subset \mathrm{supp}\, h_i$.

For U_u, $\lambda_i(x) = \sum_{\gamma(j)=i} \frac{h_i(x)}{g(x)}$, so $\overline{V}_j \subset U_i$.

For the noncompact case, we note that M^n has a countable basis and is locally compact, so the proof goes through, just more technically difficult.

M^n is locally compact, so $M = \cup_{i=1}^{\infty} N_i$ for N_i compact and $N_i \cap N_j \neq \emptyset$ if $|i-j| \geq 2$.

We apply the compact case inductively. □

Orientation on Manifolds

Orientation on a finite dimensional vector space V with basis $v_1, \ldots, v_n$:

A nonzero vector in $\bigwedge^n V^*$ is called a volume form. Dimension if $\binom{n}{n} = 1$.

$v_1^* \wedge \ldots \wedge v_n^*$ is a volume form.

Two volume forms $\alpha, \beta \in \bigwedge^n V^* \setminus \{0\}$ are equivalent if $\alpha = k\beta$ for $k \in \mathbb{R}_{>0}$. The equivalence class of volume forms is an orientation on V.

Definition 3.8 (Orientable Manifold)**.** *M^n is orientable if $\exists$ smooth charts $\{(U_\alpha, \phi_\alpha) : \alpha \in I\}$ covering M^n such that* $\det(D(\phi_\alpha \circ \phi_\beta^{-1})) > 0$ *for all α, β. (transition functions are orientation preserving).*

A volume form ω on M^n is a smooth n-form on M^n such that $\omega_p \neq 0$ for all $p \in M$.

Theorem 3.12. *M^n is orientable iff $\exists$ a volume form.*

e.g. $\mathbb{R}^n$ is orientable. $U \subset \mathbb{R}^n$ is orientable. S^n is orientable with $\{(U_N, \overline{\phi}_N), (U_S, \phi_S)\}$.

In fact, if M^n is covered by two charts, U, V such that $U \cap V$ is connected, then M^n is orientable.

Every Riemann Surface is orientable. This is due to the fact that $f : U \to \mathbb{C}$ with U open in $\mathbb{C}$ is an analytic homeomorphism, and so $\det(Df) > 0$, due to the Cauchy Riemann equations.

Homework: If $A : \mathbb{C}^n \to \mathbb{C}^n$ is $\mathbb{C}$-linear isomorphism, then $\det_{\mathbb{R}}(A) > 0$ for a real $(2n) \times (2n)$ matrix.

Thus, all complex analytic manifolds are orientable.

All Lie Groups are orientable.

We will now prove the theorem:

Proof. Orientable $\Rightarrow$ volume form: Let $\{(U_\alpha, \phi_\alpha) : \alpha \in I\}$ be the orientable charts covering M. Let $\{\lambda_\alpha : \alpha \in I\}$ be a partition of unity associated to $\{U_\alpha\}$.

Let $\omega = \sum_{\alpha \in I} \lambda_\alpha \phi_\alpha^*(dx_1 \wedge \ldots \wedge dx_n)$.

ω is well defined n-form.

Claim: $\omega_p \neq 0$ for all $p \in M$. At p, choose an orientable chart $(U_{\alpha_0}, \phi_{\alpha_0})$ and consider $(\phi_{\alpha_0}^{-1})^*(\omega) = \sum_\alpha \lambda_\alpha(\phi_{\alpha_0}^{-1})(\phi_{\alpha_0}^{-1})^*(\phi_\alpha^*)(\eta)$ where $\eta = dx_1 \wedge \ldots \wedge dx_n$. Then this equals $\sum_\alpha \lambda_\alpha(\phi_{\alpha_0}^{-1})(\phi_\alpha \circ \phi_{\alpha_0}^{-1})^*(\eta)$.

$F : \mathbb{R}^n \to \mathbb{R}^n$ is smooth, then $F^*(dx_1 \wedge \ldots \wedge dx_n) = \det(D(F)) dx_1 \wedge \ldots \wedge dx_n$.

So the above is $\left[\sum_\alpha \lambda_\alpha(\phi_{\alpha_0}^{-1}) \det(\phi_\alpha \circ \phi_{\alpha_0}^{-1})\right](\eta) = \mu\eta$ for $\mu \neq 0$ at p. So $\omega_p \neq 0$.

Volume Form $\Rightarrow$ Orientable: If M^n has a volume form ω, then we construct charts for M^n as follows:

$\forall p \in M$, choose a connected chart (V, ϕ) at p. Now $(\phi^{-1})^*(\omega|_V)$ volume form $\phi(V) \subset \mathbb{R}^n$ is open, then $(\phi^{-1})^*(\omega|_V) = h(x)dx_1 \wedge \ldots \wedge dx_n$.

$h \in C^\infty$, $h(x) \neq 0$, and V connected so $h(x) > 0$ for all $x \in \phi(V)$ or $h(x) < 0$ for all $x \in \phi(V)$.

If $h > 0$ we choose (V, ϕ) as an orientable chart. If $h < 0$ we choose $(V, A \circ \phi)$ as an orientable chart, where A is a reflection.

Claim, these charts satisfy $\det(D(\phi_\alpha \circ \phi_\beta^{-1})) > 0$ for all α, β.

$(\phi_\alpha^{-1})^*(\omega)h_\alpha(x)dx_1 \wedge \ldots \wedge dx_n$ with $h_\alpha > 0$ and $(\phi_\beta^{-1})^*(\omega) = h_\beta(x)dx_1 \wedge \ldots \wedge dx_n$ and $h_\beta > 0$.

But $(\phi_\alpha \circ \phi_\beta^{-1})((\phi_\alpha^{-1})^*\omega) = (\phi\beta^{-1})^*\omega$, so $(\phi_\alpha \circ \phi_\beta^{-1})^*(h_\alpha \eta) = h_\beta \eta = h_\alpha(\phi_\alpha \circ \phi_\beta^{-1}) \det(D(\phi_\alpha \circ \phi_\beta^{-1}))\eta$, so $\det(D(\phi_\alpha \circ \phi_\beta^{-1})) = \frac{h_\beta}{h_\alpha(\phi_\alpha} \circ \phi_\beta^{-1}) > 0$. □

Symplectic Manifolds:

Let V be a vector space with basis $v_1, \ldots, v_m$ and A a non-degenerate 2-form $\phi : V \times V \to \mathbb{R}$.

$A = [\phi(v_i, v_k)]_{m \times m}$ satisfies $A^T = -A$ and $\det A \neq 0$.

$\det(A) = \det(A^T) = \det(-A) = (-1)^m \det A$, so, as $\det A$ is nonzero, $m = 2n$ is even.

Linear Algebra: There exists a basis $w_1, \ldots, w_{2n}$ of V such that $[\phi(w_i, w_j)]$ is block diagonal with each block 2×2 and equal to $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Now $V' = \{v : \phi(v, w_1) = \phi(v, w_2) = 0\}$. $\det A \neq 0$ so $\dim V' = 2n - 2$, and $\phi|_{V' \times V'}$ nondegenerate.

i.e. $\phi = w_i^* \wedge w_2^* + \ldots + w_{2n-1}^* \wedge w_{2n}^*$. There is only one such 2-form.

$\phi \wedge \phi \wedge \ldots \wedge \phi$ n-times gives a $2n$-form. This is nonzero, and so we have a volume form.

Definition 3.9 (Symplectic Manifold)**.** *M^{2n} is a symplectic manifold if there is a 2-form ω on M^{2n} such that $d\omega = 0$ and ω_p is nondegenerate at all $p \in M$.*

Theorem 3.13. *M^{2n} symplectic$\Rightarrow$ M^{2n} orientable.*

Proof. $\omega \wedge \ldots \wedge \omega$ n-times is a volume form. □

e.g. 1: All orientable surfaces are symplectic.

e.g. 2: M^n a manifold, then T^*M is a symplectic manifold.

Homework TM is diffeomorphic to T^*M.

We will now prove the example

U open in $\mathbb{R}^n$. Then $T^*U \cong U \times \mathbb{R}^n$.

$\sum a_i dx_i|_x \mapsto (x_1, \ldots, x_n, a_1, \ldots, a_n) = (x, a)$.

We write down the symplectic 2-form $\omega_U = \sum_{i=1}^n da_i \wedge dx_i$.

Lemma 3.14. *If $F : U \to V$, V open in $\mathbb{R}^n$ is a diffeomorphism, then $F^* : T^*V \to T^*U$ is a diffeomorphism.*

$(F^*)^*(\omega_U) = \omega_V$.

Proof. Write $y = F(x)$, and $y \in V$. $T^*V \simeq V \times \mathbb{R}^n : \sum b_i dy_i \mapsto (y, b)$.

$F : U \to V : x \mapsto F(x)$ and $F^* : T^*V \to T^*U : (y_1, \ldots, y_n, b_1, \ldots, b_n) \mapsto (x_1, \ldots, x_n, a_1, \ldots, a_n)$. And $\sum a_i dx_i$ is a 1-form on T^*U.

Definition of F^*: $\sum b_i dy_= (F^*)^*(\sum a_i dx_i)$, then apply d to it, $d\phi^* = \phi^* d$ implies $d(\sum b_i dy_i) = d((F^*)^*(\sum a_i dx_i)) = (F^*)^*(d \sum a_i dx_i) = (F^*)^*(\omega_U)$. □

For any smooth manifold M with smooth charts $\{(U_\alpha, \phi_\alpha) : \alpha \in I\}$ covering it, let $\phi_\alpha(U_\alpha) = V_\alpha$ open in $\mathbb{R}^n$.

$\phi_\alpha^* : T^*V_\alpha \to T^*U_\alpha$ and $(\phi_\alpha^*)^{-1} = (\phi_\alpha^{-1})^*$.

Define a symplectic 2-form ω_α on $T^*U_\alpha \subset T^*M$ by $\omega_\alpha = ((\phi_\alpha^{-1})^*)^*(\omega_{V_\alpha})$.

Claim $\omega_\alpha|_{T^*U_\alpha \cap T^*U_\beta} = \omega_\beta|_{T^*U_\alpha \cap T^*U_\beta}$ implies that there is a globally well defined symplectic 2-form.

Proof. Apply $\omega_{V_\alpha} = (\phi_\alpha^*)^*(\omega_\alpha) \stackrel{?}{=} (\phi_\alpha^*)^*(\omega_\beta) = (\phi_\alpha^*)^*((\phi_\beta^{-1})^*)^*(\omega_{V_\beta}) = ((\phi_\beta^{-1})^*(\phi_\alpha^*))^*(\omega_{V_\beta}) = ((\phi_\alpha \circ \phi_\beta^{-1})^*)^*(\omega_{V_\beta})$, which is the first thing, by the lemma. □

Integration

Definition 3.10 (Haar Measure)**.** *If G is a Lie Group, a Haar Measure on G is a left invariant volume form.*

e.g. $\mathbb{R}^n$ has $dx_1 \wedge \ldots \wedge dx_n$.

S^1 has $d\theta$. $GL(n, \mathbb{R})$ say ω is the left invariant volume form $\omega_{\text{id}} = dx_{11} \wedge \ldots \wedge dx_{nn}$. $\omega_A = \frac{1}{\det A}\omega_{\text{id}}$.

Indeed, $\ell_{A^{-1}} : x \mapsto A^{-1}x$ sends A to id, so $\omega_A = (\ell_{A^{-1}})^*\omega_{\text{id}}$ is left invariant.

The volume form of an oriented Riemannian manifold M^n.

Fix a volume form ω. Now, $\forall p \in M$, choose orthogonal basis of T_pM $e_1, \ldots, e_n$ such that $e_1^* \wedge \ldots \wedge e_n^* = K\omega_p$ where $K > 0$.

Chaim $\tilde{\omega} = e_1^* \wedge \ldots \wedge e_n^*$ is the volume form, independent of the choices of orthogonal basis.

Proof. Suppose $\epsilon_1, \ldots, \epsilon_n$ is a different orthogonal basis. Then $\epsilon_i = \sum_j a_{ij} e_j$. Then $A = [a_{ij}]$ is orthogonal, so $AA^T = \text{id}$.

Thus, $\det(A)\epsilon_1^* \wedge \ldots \wedge \epsilon_n^* = e_1^* \wedge \ldots \wedge e_n^*$. As A is orthogonal, $\det A = \pm 1$. But $\det A > 0$, due to choice of bases, so $\epsilon_1^* \wedge \ldots \wedge \epsilon_n^* = e_1^* \wedge \ldots \wedge e_n^*$. □

The volume forms ω_1, ω_2 on M^n are equivalent iff $\omega_1 = h(x)\omega_2$ for $h \in C^\infty(M)$ and $h(x) > 0$.

An orientation on M^n is an equivalence class of a volume form.

Fix an orientation. e.g, on $\mathbb{R}^n$, the orientation is $[dx_1 \wedge \ldots \wedge dx_n]$.

To integrate on an open set $U \subset \mathbb{R}^n$, we let $C_0(M) =$ the vector space of all compactly supported continuous functions on M. and $\Gamma_0(\bigwedge^n M) =$ the vector space of all compactly supported continuous n-forms on M.

If $\omega \in \Gamma_0(\bigwedge^n U)$ iff $\omega = f(x)dx_1 \wedge \ldots \wedge dx_n$ for $f \in C_0(U)$.

We define $\int_U \omega$ to be $\int_U f(x)dx_1 dx_2 \ldots dx_n$, as in calculus.

Theorem 3.15 (Change of Variables)**.** *If $F : V \to U$ is an orientation preserving diffeomorphism, where V is open in $\mathbb{R}^n$, then $\int_U \omega = \int_V F^*(\omega)$.*

$F^*(\omega) = f(F(x))F^*(dx_1 \wedge \ldots \wedge dx_n) = f(F(x)) \det(DF(x))dx_1 \wedge \ldots \wedge dx_n$.

Now we suppose that M^n is an oriented manifold and and $\omega \in \Gamma_0(\bigwedge^n M)$. Compactly supported n-form on M^n.

We choose oriented charts $\{(U_\alpha, \phi_\alpha) : \alpha \in I\}$ covering M such that $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha) \subset \mathbb{R}^n$ is orientation preserving.

Let $\{\lambda_\alpha : \alpha \in I\}$ be a partition of 1 associated to $\{U_\alpha\}$. Let $V_\alpha = \phi_\alpha(U_\alpha)$.

We define $\int_M \omega = \sum_\alpha \int_{V_\alpha} (\phi_\alpha^{-1})^*(\lambda_\alpha \omega)$.

Proposition 3.16. *This is well-defined and independent of the choices.*

Proof. Suppose $\{(W_\beta, \psi_\beta) : \beta \in B\}$ is another set of charts and $\{\mu_\beta : \beta \in B\}$ is an associated partition of unity.

$$\sum_\beta \int_{\psi_\beta(W_\beta)} (\psi_\beta^{-1})^*(\mu_\beta \omega) = \sum_\alpha \int_{\psi_\alpha(U_\alpha)} (\phi_\alpha^{-1})^*(\lambda_\alpha \omega)$$

Note that $\det(D(\phi_\alpha \circ \psi_\beta^{-1})) > 0$ so $\sum_\alpha \int_{\phi_\alpha(U_\alpha)} (\psi_\alpha^{-1})^*(\lambda_\alpha \omega) = \sum_{\alpha,\beta} \int_{\phi_\alpha(U_\alpha)} (\phi_\alpha^{-1})^*(\lambda_\alpha \mu_\beta \omega)$

Applying similar identities, the change rule, and the change of variables theorem, we obtain the desired result. □

Proposition 3.17.

1. *$F : N \to M$ orientation preserving diffeomorphim implies that $\int_M \omega = \int_N f^*(\omega)$*
2. *$k_1, k_2 \in \mathbb{R}$ implies $\int_M (k_1\omega_1 + k_2\omega_2) = k_1 \int_M \omega_1 + k_2 \int_M \omega_2$*
3. *$-M^n$ the negatively oriented manifold $(-\eta)$ gives $\int_{-M} \omega = -\int_M \omega$.*

e.g., the area of (S^2, g_{st}) under stereographic projection ϕ_N, $(\phi_N^{-1})^*(g_{st}) = \frac{4(dx^2+dy^2)}{(1+x^2+y^2)^2}$, and the volume form then is $\frac{4dx \wedge dy}{(1+x^2+y^2)^2}$, so $Area(S^2) = 4 \iint_{\mathbb{R}^2} \frac{dx \wedge dy}{(1+x^2+y^2)^2} = 4 \int_0^{2\pi} \left(\int_0^\infty \frac{rdr}{(1+r^2)^2} \right) d\theta = 4\pi$

Stokes' Theorem

Definition 3.11 (Smooth Function)**.** *$X \subset \mathbb{R}^n$ closed, $F : X \to \mathbb{R}^m$ is called smooth if $\exists$ open set $U \supset X$ and smooth $f : U \to \mathbb{R}^m$ such that $f|_X = F$.*

Definition 3.12 (Smooth Manifold with Boundary)**.** *M^n is a smooth manifold with boundary if it is Hausdorff with a countable basis and covered by charts $\{(U_\alpha, \phi_\alpha) : \alpha \in A\}$ such that $\phi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}$ an open subset is a homeomorphism with $\phi_\alpha \circ \phi_\beta^{-1}$ is smooth.*

We define $\partial M^n = \{x \in M : \phi_\alpha(x) \in \mathbb{R}^{n-1} \times 0$ for some $\alpha\}$.

Lemma 3.18. *∂M is a smooth $(n-1)$-manifold with $\partial^2 M = 0$.*

Proof. Define smooth charts for ∂M: $(U_\alpha \cap \partial M, pr \circ \phi_\alpha)$ where $pr : \mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0} \to \mathbb{R}^{n-1}$.

Verifying $(pr \circ \phi_\alpha) \circ (pr \circ \phi_\beta)^{-1} = pr \circ (\phi_\alpha \circ \phi_\beta^{-1})$, which is smooth. □

e.g. $\mathbb{D}^2 = \{x | \|x\| \leq 1\}$ is a smooth manifold with boundary. One chart $(int(\mathbb{D}^n) = \{x||x| < 1\}, \mathrm{id})$. Take $a \in S^{n-1} = \partial\mathbb{D}^n$, $a \neq 0$. So say $a_n \neq 0$.

$F(x) = (x_1, \ldots, x_{n-1}, 1 - \sum_{i=1}^n x_i^2)$ defined near 0 is in $\mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}$. So $\det DF(a) \neq 0$, and so it is a local diffeomorphism at a.

e.g. If $f : \mathbb{R}^n \to \mathbb{R}^1$ with regular value $b \in \mathbb{R}^1$. Then $\{x | f(x) \leq b\}$ is a smooth manifold with boundary $f^{-1}(b)$.

e.g. Take any closed orientable surface $\Sigma_g = \partial H_g$ is he boundary of an orientable compact 3-manifold.

Lemma 3.19. *If M^n is a manifold with boundary and M^n is orientable (i.e. there exist charts (U_α, ϕ_α) covering M^n such that $\det(D(\phi_\alpha \circ \phi_\beta^{-1})) > 0$ for all α, β), then ∂M^n is orientable.*

Proof. Take $p \in \partial M^n$. $\phi_\alpha \circ \phi_\beta^{-1}(x_1, \ldots, x_n) = (f(x_1, \ldots, x_n), h(x_1, \ldots, x_n)) \in \mathbb{R}^{n-1} \times \mathbb{R}^1$ where $h \geq 0$. $(pr\phi_\alpha)(pr\phi_\beta)^{-1}(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, 0)$.

Also, $h(x_1, \ldots, x_{n-1}, 0) = 0$, so $D(\phi_\alpha \circ \phi_\beta^{-1}) = \begin{bmatrix} Df & * \\ \frac{\partial h}{\partial x_i}{}_{1 \leq i \leq n-1} & \frac{\partial h}{\partial x_n} \end{bmatrix}$. At the boundary, $h(x_1, \ldots, x_{n-1}, 0) = 0 \Rightarrow \frac{\partial h}{\partial x_i}(x_1, \ldots, x_{n-1}, 0) = 0$ for $i < n$.

Also, $h(x) \geq 0$, and $h(x_1, \ldots, x_{n-1}, 0) = 0$, so $\frac{\partial h}{\partial x_n} \geq 0$. Thus, $D(\phi_\alpha \circ \phi_\beta^{-1}) = \begin{bmatrix} Df & * \\ 0 & \frac{\partial h}{\partial x_n} \end{bmatrix}$, which means that their determinant is $\det(Df)\frac{\partial h}{\partial x_n} > 0$, and so $\det(Df) > 0$. □

Convention: The standard orientation on $\mathbb{R}^n$, $dx_1 \wedge \ldots \wedge dx_n$. Ω open in $\mathbb{R}^n$ has standard orientation $dx_1 \wedge \ldots \wedge dx_n$.

The induced orientation of $dx_1 \wedge \ldots \wedge dx_n$ in $\mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}$ on $\mathbb{R}^{n-1} \times 0$ is $(-1)^n dx_1 \wedge \ldots \wedge dx_{n-1}$.

Suppose that M^n is an oriented manifold with ∂M and orientation given by n-form ω. Let η be the $(n-1)$-form on ∂M corresponding to the induced orientation. Then $\omega = -dh \wedge \eta$ near ∂M. Where $h : M \to \mathbb{R}_{\geq 0}$ is a smooth function with $h^{-1}(0) = \partial M$.

Lemma 3.20. *If $F : N \to M$ is an orientation preserving diffeomorphism, then $\int_M d\omega = \int_{\partial M} i^*(\omega)$ iff $\int_N dF^*(\omega) = \int_{\partial N} i^*(f^*\omega)$.*

Theorem 3.21 (Stokes' Theorem)**.** *Let M^n be an oriented n-manifold such that ∂M^n has the induced orientation. Then for any $(n-1)$-form ω on M with compact support,*

$$\int_M d\omega = \int_{\partial M} i^*(\omega) = \int_{\partial M} \omega|_{\partial M} = \int_{\partial M} \omega$$

where $i : \partial M \to M$ is the inclusion.

Proof. Case 1: $M = \mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}$ with the standard orientation, and $\partial M = \mathbb{R}^{n-1} \times 0$ has the induced orientation.

ω is a finite sum of I: $f(x)dx_1 \wedge \ldots \wedge dx_{n-1}$ and II: $f(x)dx_1 \wedge \ldots \wedge \hat{dx_i} \wedge \ldots \wedge dx_n$.

If ω is type I, the $d\omega = \frac{\partial f}{\partial x_n} dx_n \wedge dx_1 \wedge \ldots \wedge dx_{n-1} = (-1)^{n-1} \frac{\partial f}{\partial x_n} dx_1 \wedge \ldots \wedge dx_n$, and $\int_M d\omega = (-1)^{n-1} \int_{\mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}} \frac{\partial f}{\partial x_n} dx_1 \wedge \ldots \wedge dx_n = (-1)^{n-1} \int_{\mathbb{R}^{n-1}} \left(\int_0^\infty \frac{\partial f}{\partial x_n} dx_n \right) dx_1 \wedge \ldots \wedge dx_{n-1}$. f is on compact support, so this gives

$$(-1)^{n-1} \int_{\mathbb{R}^{n-1}} (-f(x_1, \ldots, x_{n-1}, 0)) dx_1 \ldots dx_{n-1} = \int_{\partial M} (-1)^n f(x_1, \ldots, x_{n-1}, 0) dx_1 \wedge \ldots \wedge dx_{n-1} = \int_{\partial M} i^*(\omega$$

If ω is type II, then $\omega|_\partial M = 0$, so $\int_{\partial M} i^*(\omega) \equiv 0$. Now $d\omega = \frac{\partial f}{\partial x_i} d_i \wedge dx_1 \wedge \ldots \wedge \hat{dx_i} \wedge \ldots \wedge dx_n = (-1)^{i-1} \frac{\partial f}{\partial x_i} dx_1 \wedge \ldots \wedge dx_n$., so $\int_{\mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}} d\omega = (-1)^{i-1} \int_{\mathbb{R}^{n-1}} \left(\int_0^\infty \frac{\partial f}{\partial x_i} dx_i \right) dx_n = 0$.

$\int_{-\infty}^\infty \frac{\partial f}{\partial x_i} (x_1, \ldots, x_i \ldots, x_n) dx_i = 0$ due to compact support.

Case 2: General. Take orientation preserving charts $\{(U_\alpha, \phi_\alpha) | \alpha \in A\}$ covering M and a partition of unity $\{\lambda_\alpha\}$ associated to $\{U_\alpha\}$

$$\int_M d\omega = \sum_\alpha \int_M d(\lambda_\alpha \omega) = \sum_\alpha \int_{U_\alpha} d(\lambda_\alpha \omega) = \sum_\alpha \int_{u_\alpha \cap \partial M} (\lambda_\alpha \omega) = \sum_\alpha \int_{\partial M} (\lambda_\alpha \omega) = \int_{\partial M} \omega$$

□

In particular, if $\partial M = \emptyset$, then $\int_M d\omega = 0$.

e.g. Gauss-Bonnet for $\mathbb{H}^2$.

A hyperbolic triangle in $\mathbb{H}^2 = \{z | \Im z \geq 0\}$. We have metric $\frac{dx^2 + dy^2}{y^2}$, and area form is $\frac{dx \wedge dy}{y^2} = d\left(\frac{dx}{y}\right)$. And so we have

Theorem 3.22. *$Area(\Omega) = \pi - \theta_1 - \theta_2 - \theta_3$, where Ω is a hyperbolic triangle.*

Proof.

$$Area(\Omega) = \int_\Omega \frac{dx \wedge dy}{y^2} = \oint_{\partial \Omega} \frac{dx}{y} = \int_a + \int_b + \int_c$$

And so we have $\int_a \frac{dx}{y} = \int_\alpha^\beta \frac{d(\gamma \cos t + A}{\gamma \sin t} = - \int_\alpha^\beta dt = \alpha - \beta$, and so the theorem holds. □

e.g. If $\omega = \sum_{i=1}^3 a_i(x) dx_i$ is a closed 1-form in $\mathbb{R}^3$.

$d\omega = 0 = \sum_i da_i(x) \wedge dx_i = \sum_{i,j} \frac{\partial a_i}{\partial x_j} dx_j \wedge dx_i \equiv 0$ iff $\frac{\partial a_i}{\partial x_j} = \frac{\partial a_j}{\partial x_i}$ for all i, j.

Fix $o \in \mathbb{R}^3$. If $\alpha : [0,1] \to \mathbb{R}^3$ is a smooth path from o to x, then define $F_\alpha(x) = \int_\alpha \omega$.

Lemma 3.23. *$F_\alpha(x) = F_\beta(x)$ if $\beta : ([0,1], 0, 1) \to (\mathbb{R}^3, 0, x)$.*

Proof. Let $H(s,t) = (1-t)\alpha(s) + t\beta(s)$ smooth. $H(s,0) = \alpha(s)$ and $H(s,1) = \beta(s)$, $H(0,t) = 0$ and $H(1,t) = x$.

By Stokes' Theorem, $\int_{I^2} d(H^*\omega) = \int_{I^2} H^*(d\omega) = 0 = \int_{\partial I^2} H^*\omega = \int_\alpha \omega - \int_\beta \omega$. □

Definition 3.13 (Exact Form)**.** *M^n a smooth manifold. An i-form ω is called exact if $\omega = d\eta$ where η is an $(i-1)$-form.*

4 Algebraic Topology

Definition 4.1 (de Rham Cohomology)**.** *The n^{th} de Rham cohomology of M is $H^n_{dR}(M)$ =the quotient of the smooth closed n-forms with the smooth exact n-forms.*

See Topic 1: Chain Complexes, in Homological Algebra notes.

The de Rham Cohomology assigns each smooth M a vector space for each i, $H^i_{dR}(M)$ and assigns each smooth map $F : M \to N$, a linear transformation $F^* : H^i_{dR}(N) \to H^i_{dR}(M)$ such that $(\mathrm{id})^* = \mathrm{id}$ and $(F \circ G)^* = G^* \circ F^*$.

Theorem 4.1. *If $F : M \to N$ is a diffeomorphism, then $F^* : H^i_{dR}(N) \to H^i_{dR}(M)$ is an isomorphism.*

Easy fact: $H^i_{dR}(M) = 0$ for $i > n$, as there are no forms at all.

Lemma 4.2. *If M^n is connected, then $H^0_{dR}(M) = \mathbb{R}$*

Lemma 4.3 (Poincare's Lemma)**.** *$H^i_{dR}(\mathbb{R}^n)$ is $\mathbb{R}$ if $i = 0$ and 0 for $i > 0$.*

Proof. We will perform induction on n. $n = 1$ is ok from the above.

If it holds for n, then we let $\pi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be the canonical projection map.

Then $\pi^* : H^i_{dR}(\mathbb{R}^n) \to H^i_{dR}(\mathbb{R}^n \times \mathbb{R})$ is an isomorphism. □

There is a linear map, integration, $K : \Omega^i(\mathbb{R}^n \times \mathbb{R}) \to \Omega^{i-1}(\mathbb{R}^n \times \mathbb{R})$ such that $\mathrm{id} - (s \circ \pi)^* = dK + Kd$ on $\Omega^i(\mathbb{R}^n \times \mathbb{R})$.

If we assume this, and take cohomology classes $[\omega] \in H^i(\mathbb{R}^n \times \mathbb{R})$, $d\omega = 0$ adn aply it, we get $\omega - (s \circ \pi)^*\omega = dK\omega + Kd\omega = d(K\omega)$, and so $[\omega] = [(s \circ \pi)^*\omega] = (s \circ \pi)^*[\omega]$, done.

Take $\omega \in \Omega^(\mathbb{R}^n \times \mathbb{R})$.

Notation: $J = (i_1, \ldots, i_k)$ with $1 \leq j \leq n$, then $dx_J = dx_{i_1} \wedge \ldots \wedge dx_{i_k}$. ω is a finite linear combination of $f(x,t)dx_J$ with $f(x,t) \in C^\infty(\mathbb{R}^n \times \mathbb{R})$ and $f(x,t)dt \wedge dx_J$.

We define $K(f(x,t)dx_J) = 0$ and $K(f(x,t)dt \wedge dx_J) = \left(\int_0^t f(x,s)ds\right) dx_J$, and we can extend K linearly to $\Omega^i(\mathbb{R}^n \times \mathbb{R})$.

Remark: If M^n is closed and orientable, then $H^n_{dR}(M^n) \neq 0$.

Theorem 4.4. *If $\pi : M \times \mathbb{R} \to M$ is the projection map, then $\pi^* : H^i_{dR}(M) \to H^i_{dR}(M \times \mathbb{R})$ is an isomorphism.*

Definition 4.2 (Homotopic Maps)**.** *Two smooth maps $F, G : M \to N$ are homotopic if $\exists$ smooth map $H : M \times I \to N$ such that $H(x, 0) = F(x)$ and $H(x, 1) = G(x)$.*

Theorem 4.5 (Homotopy Invariance)**.** *If $F, G : M \to N$ are homotopic, then $F^* = G^* : H^*_{dR}(N) \to H^*_{dR}(M)$.*

Proof. Let $H : M \times I \to N$ be a homotopy from F to G, let s_1 and s_2 be the inclusions, $F = H \circ s_1$ and $G = H \circ s_2$, then $F^* = s_1^* \circ H^*$ and $G^* = s_2^2 \circ H^*$ but $s_1^* = s_2^* = (\pi^*)^{-1}$, where π is the projection. $\square$

Theorem 4.6. *M^n any smooth manifold $I \subset \mathbb{R}$ an interval, then the projection $\pi : M \times I \to M$ induces $\pi^* : H^i_{dR}(M) \to H^i_{dR}(M \times I)$, an isomorphism.*

Proof. Let $s : M \to M \times I$, $s(x) = (x, c)$ for $c \in I$ fixed. Then $\pi \circ s = \text{id}$ implies that $s^* \circ \pi^* = \text{id}$, and so π^* is 1-1.

We claim that π^* is onto. There exists $\tilde{K} : \Omega^i(M \times I) \to \Omega^{i-1}(M \times I)$ integrading along a fiber which is linear such that $\text{id} - (s \circ \pi)^* = d\tilde{K} + \tilde{K}d$ which implies that $(s \circ \pi)^* = \text{id}^*$ in H^i_{dR}.

Poincare's Lemma states that this condition holds for $M = \mathbb{R}^n$.

Let $\{(U_\alpha, \phi_\alpha) : \alpha \in A\}$ smooth charts convering M such that $\phi_\alpha(U_\alpha) = \mathbb{R}^n$. Let $\{\lambda_\alpha : \alpha \in A\}$ partition of unity associated to $\{U_\alpha : \alpha \in A\}$. For each α, we we can define $K_\alpha : \Omega^i(U_\alpha \times I) \to \Omega^{i-1}(U_\alpha \times I)$.

Now, on $\mathbb{R}^n \times I$, we have $\text{id} - (\tilde{\pi} \circ s)^* = dK - Kd$ by Poincare's Lemma and so $(\phi_\alpha \times 1)^*((\phi_\alpha \times 1)^*)^{-1} - (\phi_\alpha \times 1)^*(\tilde{\pi} \circ s)^*((\phi_\alpha \times 1)^*)^{-1} = (\phi_\alpha \times 1)^* dK((\phi_\alpha \times 1)^*)^{-1} + (\phi_\alpha \times 1)^* Kd((\phi_\alpha \times 1)^*)^{-1}$.

Thus, $\text{id} - (\pi \circ s)^* = dK_\alpha + K_\alpha d$.

For $\omega \in \Omega^i(M \times I)$ define $K(\omega) = \sum_\alpha K_\alpha(\lambda_\alpha \omega)$ Now $dK(\omega) + Kd(\omega) = \sum_\alpha (dK_\alpha + K_\alpha d)(\lambda_\alpha \omega) = \sum_\alpha (\lambda_\alpha \omega - (\pi \circ s)^*(\lambda_\alpha \omega)) = \omega - (\pi \circ s)^* \omega$. $\square$

Corollary 4.7. *If $f \simeq g : M \to N$ then $f^* = g^* : H^i_{dR}(N) \to H^i_{dR}(M)$.*

Definition 4.3 (Homotopic Equivalent)**.** *Two smooth manifolds M, N are smooth homotopic equivalent ($M \simeq N$) if $\exists F : M \to N$, $G : N \to M$ smooth such that $F \circ G \simeq \text{id}_N$ and $G \circ F \simeq \text{id}_M$.*

e.g. $M = \mathbb{R}^n$ and N is a point, then $\mathbb{R}^n \simeq$point as $F : \mathbb{R}^n \to \{0\}$ by $F(x) = 0$ and $G : \{0\} \to \mathbb{R}^n$ by $G(0) = 0$.

Theorem 4.8. *If M, N smooth manifolds and $M \simeq N$, then $H^i_{dR}(M) \simeq H^i_{dR}(N)$.*

Application:

Theorem 4.9 (Brouwer Fixed Point Theorem)**.** *If $f : \mathbb{D}^n \to \mathbb{D}^n$ is a smooth map, then $\exists p \in \mathbb{D}^n$ such that $f(p) = p$.*

Proof. If not, then $f(x) \neq x$ for all $x \in \mathbb{D}^n$. Let $g(x)$ be the intersection of the ray from $f(x)$ to x with $\partial\mathbb{D}^n = S^{n-1}$.

Claim: $g(x)$ is smooth in x.

Assuming this, then we look at $i : \partial\mathbb{D}^n \to \mathbb{D}^n$ inclusion $i(x) = x$, then $g \circ i(x) = x$.

$H^{n-1}(\partial\mathbb{D}^n) \xrightarrow{g^*} H^{n-1}_{dR}(\mathbb{D}^n) \xrightarrow{i^*} H^{n-1}(\partial\mathbb{D}^n)$, but the middle term is zero and the composition is the identity, and as $\partial\mathbb{D}^n = S^{n-1}$ is orientable, $H^{n-1}_{dR}(S^{n-1}) \neq 0$, contradiction. $\square$

Poincare Duality:

Definition 4.4 (Cup Product)**.** *Let M^n be smooth. Then the cup product is a map $H^i_{dR}(M) \times H^j_{dR}(M) \to H^{i+j}_{dR}(M)$ by $[\omega] \times [\eta] \mapsto [\omega \wedge \eta] = [\omega] \cup [\eta]$.*

It is well defined, and also bilinear.

Furthermore, if $F : M \to N$ is smooth, then $F^*([\omega] \cup [\eta]) = F^*([\omega]) \cup (F^*([\eta]))$, due to $F^*(\omega \wedge \eta) = F^*(\omega) \wedge F^*(\eta)$ on forms.

Theorem 4.10 (Poincare Duality)**.** *Suppose that M^n is a closed orientable and connected n-manifold (compact and no boundary). Then:*

1. $\dim H^i_{dR}(M) < \infty$
2. *The integration $\int H^n_{dR}(M) \to \mathbb{R} : [\omega] \mapsto \int_M \omega$ is an isomorphism*
3. *The cup product $H^i_{dR}(M) \times H^{n-i}_{dR}(M) \to H^n_{dR}(M) \simeq \mathbb{R}$ is nondegenerate. In particular, $\dim H^1_{dR}(M) = \dim H^{n-1}_{dR}(M)$.*

e.g. define $b_i = \dim H^i_{dR}(M^n)$ and the Euler characteristic of M is $\chi(M) = b_0 - b_1 + b_2 - \ldots + (-1)^n b_n$ is a topological invariant.

Consequence: If M^{2n+1} is a closed orientable connected manifold, then $\chi(M^{2n+1}) = 0$.

And now we will lead geometry behind and do more general algebraic topology:

Definition 4.5 (Homotopic)**.** *If $f, g : X \to Y$ are tow continuous maps, then they are homotopic if there exists a continuous map $H : X \times I \to Y$ such that $H(x, 0) = f(x)$, $H(x, 1) = g(x)$.*

We can define the continuous homotopy equivalences of two topological spaces by $X \simeq Y$ iff $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$.

Classifying Spaces Up to Homotopy Equivalence:

$S^{n-1} \simeq \mathbb{R}^n \setminus \{0\}$.

Proposition 4.11. *$O(n) \simeq GL(n, \mathbb{R})$.*

Proof. Let $f(x) = x$ is a map from $O(n) \to GL(n, \mathbb{R})$ be the inclusion. Let $g : GL(n, \mathbb{R}) \to O(n)$ by the Gram-Schmidt Process. Take $A \in GL(n, \mathbb{R})$. Then $A = [v_1, \ldots, v_n]$ such that $\{v_1, \ldots, v_n\}$ is a basis of $\mathbb{R}^n$.

Use Gram-Schmidt to take $\{v_1, \ldots, v_n\}$ to $\{w_1, \ldots, w_n\}$ an orthonormal basis and then set $A \mapsto B = [w_1, \ldots, w_n] \in O(n)$.

So $g(A)$ is smooth in A.

$g \circ f = \mathrm{id}_{O(n)}$.

Now we must show that $f \circ g(A) = B$ is homotopic to $\mathrm{id}_{GL(n,\mathbb{R})}$. We define $H(A, t) = (1-t)A + tB$. This is smooth such that $H(A, 0) = A$ and $H(A, 1) = fg(A)$. Claim $\det(H(A, t)) \neq 0$.

Gram-Schmidt says that $u_i = \sum_{j \leq i} a_{ij} v_j$, and so $a_{11} = 1$, so $w_i = \sum_{j=1}^{i} b_{ij} v_j$ where $b_{ii} > 0$. So $H(x, t) = [w_1(t), \ldots, w_n(t)]$ where $w_i(t) = \sum_{j=1}^{u} b_{ij}(t) v_j$,

$b_{ii}(t) = (1-t) + tb_{ii} > 0$, and so $w_i(t) = (1-t)v_i + tw_i$, giving us $(1-t)v_i + t\sum_{j=1}^{i} b_{ij}v_j$.

And so $\det H(x,t) > 0$. □

Fundamental Group:

Let X, Y be topological spaces. $f : X \to Y$ be a map, we will always assume that maps are continous.

A path in X is $a : [0,1] \to X$ such that $a(0) = p$, $a(1) = q$.

$a : ([0,1], 0, 1) \to (X, p, q)$.

Definition 4.6 (Path Homotopic)**.** *Two paths $a, b : ([0,1], 0, 1) \to (X, p, q)$ are path homotopic, $a \simeq_p b$ if $\exists$ a cont map $H : [0,1] \times [0,1] \to X$ such that $H(s,0) = a(s)$, $H(s,1) = b(s)$, $H(0,t) = p$ and $H(1,t) = q$.*

$[a]$ is the path homotopy class of a.

Lemma 4.12. *$\simeq_p$ is an equivalence relation.*

Lemma 4.13 (Gluing Lemma)**.** *If a topological space $X = A \cup B$ where A, B are closed and $f : A \to Z$, $g : B \to Z$ are two continuous maps such that $f|_{A\cap B} = g|_{A\cap B}$, then $h(x) = f(x)$ on A and $g(x)$ on B is continuous from X to Z.*

Proof. h is well defined. Take any closed set $c \subseteq Z$. Then $h^{-1}(c) = f^{-1}(c) \cup g^{-1}(c)$ which are each closed in A. □

Definition 4.7 (Loops)**.** *If $p = q$, then $a : ([0,1], 0, 1) \to (X, p)$ is a loop based at p. $[a]$ is the homotopy class of a loop.*

Definition 4.8 (Composition of Paths)**.** *Suppose $a : [0,1] \to X$ and $b : [0,1] \to X$ such that $a(1) = b(0)$. Then $a * b : [0,1] \to X$ is the path $(a*b)(t) = \begin{cases} a(2s) & 0 \le s \le 1/2 \\ b(2s-1) & 1/2 \le s \le 1 \end{cases}$*

By the Gluing Lemma, $a * b$ is continuous.

Lemma 4.14. *If $a \simeq_p a'$, $b \simeq_p b'$ and $a(1) = b(0)$, then $a * b \simeq_p a' * b'$.*

Corollary 4.15. *$[a] * [b] = [a * b]$ where $a(1) = b(0)$ is a well defined operation.*

Theorem 4.16. *The operation $*$ on the set of path homotopy classes of loops based at p satisfies:*

1. *Associative: $[a] * ([b] * [c]) = ([a] * [b]) * [c]$*
2. *Identity: $e = [p]$, $p : [0,1] \to X$ constant. Then $[a] * e = [a] = e * [a]$.*
3. *Inverse: Define $a^{-1} : [0,1] \to X$ by $a^{-1}(s) = a(1-s)$. Then $[a] * [a^{-1}] = [a^{-1}] * [a] = e$.*

That is, these classes form a group.

Before we prove this, we will need a lemma:

Lemma 4.17. *If $a : I \to X$ and $\phi : I \to I$, $\phi(0) = 0, \phi(1) = 1$, then $a \simeq_p a \circ \phi$. Thus, if $a \simeq_p b$, then $a \circ \phi \simeq_p b \circ \phi$.*

Proof. $H(s,t) = a(ts + (1-t)\phi(s))$ clearly continuous. $H(s,0) = a(\phi(s))$, $H(s,1) = a(s)$, $H(0,t) = a(0)$ and $H(1,t) = a(1)$. □

We can now prove the theorem:

Proof. The proof goes easily, and includes messy formulas. Thus, it is left to the reader. □

Definition 4.9 (Fundamental Group)**.** *X is a topological space, $p \in X$. The fundamental group of X at p is denoted by $\pi_1(X,p) = \{[a] : a : ([0,1],0,1) \to (X,p)\}$ with multiplication given by $*$.*

e.g. X is convex in $\mathbb{R}^n$ then $\pi_1(X,p) = 1$.

Lemma 4.18. *If p, q are in the same path component of X, then there is an isomorphism $\phi : \pi_1(X,p) \to \pi_1(X,q)$.*

Functorial Property:

Suppose that $f : (X,p) \to (Y,q)$ is continuous. Then f induces a group homomorphism $f_* : \pi_1(X,p) \to \pi_1(Y,q)$ where $f_*([a]) = [f \circ a]$.

Lemma 4.19.

1. *If $a \simeq_p b$, then $f \circ a \simeq_p f \circ b$.*
2. *$f(a * b) = f(a) * f(b)$.*

Definition 4.10 (Covering Space)**.** *A covering space map $p : X \to Y$ satisfies the following conditions:*

1. *$p(X) = Y$.*
2. *For any $y \in Y$, there exists open set U containing y (small) such that $p^{-1}(U) = \coprod_{\alpha \in A} V_\alpha$ is a disjoint union of oepn sets V_α such that $p|_{V_\alpha}(U)$ is a homeomorphism.*

Definition 4.11 (Elementary Neighborhood)**.** *The U's defined above are the elementary neighborhoods for $p : X \to Y$.*

Lemma 4.20.

1. *If $p : E \to B$ is a covering map and $Y \subset B$, then $p| : p^{-1}(Y) \to Y$ is a covering map.*
2. *If $p_i : E_i \to B_i$, $i = 1,2$ are covering maps, then $p_1 \times p_2 : E_1 \times E_2 \to B_1 \times B_2$ is a covering map.*

Proof.

1. Clearly, $p|$ is onto. The elementary naighborhoods for $p| : p^{-1}(Y) \to Y$ are $U \cap Y$, where U is elementary for p.

2. Definition! Elementary neighborhoods are products of elementary neighborhoods.

□

Proposition 4.21. *If $p(x) : \mathbb{C} \to \mathbb{C}$ is a nonconstant polynomial and $B = \{z : p'(z) = 0\}$, the set of branch points, then $p : \mathbb{C} \setminus p^{-1}(A) \to \mathbb{C} \setminus A$ where $A = p(B)$ is a covering map.*

Proof. Take $w \in \mathbb{C} \setminus A$, let $x_1, \ldots, x_n \in \mathbb{C}$ such that $p(x_i)w$, $\deg p = n$. Since $w \notin A$, we have $x_i \notin B$, so $p'(x_i) \neq 0$ for all i.

By inverse function theorem, there exist small neighborhood W of w and U_i of x_i such that $p| : U_i \to W$ is a diffeomorphism for all i.

Claim: W is an elementary neighborhood for p. $p^{-1}(W)$ contains the V_i by definition.

Let $z \in p^{-1}(W)$ iff $p(z) = \alpha \in W$. $p^{-1}(\alpha) \cap V_i \neq \emptyset \Rightarrow p^{-1}(\alpha)$ contains at least n elements, but $\deg(p) = n$, so $p^{-1}(\alpha) \subseteq \coprod V_i$, so we are done. □

Definition 4.12. *Suppose Γ is a countable group acting on a topological space X is called properly discontinuous if Γ acts properly discontinuously, that is, $\forall x \in X$ $\exists$nbhd U of x such that $\gamma(U) \cap U = \emptyset$ for all $\gamma \in \Gamma \setminus \{\mathrm{id}\}$.*

Proposition 4.22. *If Γ acts properly discontinuously on X, then the quotient map, $p : X \to X/\Gamma$ is a covering map.*

Proof. p is cont by deiniton, and onto.

The elementary neighborhood of a point $[x]$ in X/Γ is $p(U)$ where U satisfies $p^{-1}(p(U)) = \coprod_{\gamma \in \Gamma} \gamma U$. □

Proposition 4.23. *Suppose G is a topological group and $\Gamma < G$ a discrete subgroup. Then Γ acts on G by left multiplication properly discontinuously.*

Proof. Γ discrete $\iff$ there exists a nbhd W of id in G such that $\Gamma \cap W = \{\mathrm{id}\}$.

The map $F : G \times G \to G : (x, y) \mapsto xy^{-1}$ is continuous, so $F^{-1}(W)$ is a neighborhood of $(\mathrm{id}, \mathrm{id})$ in G.

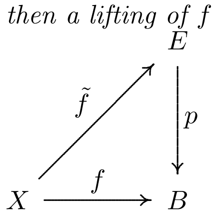
Then there is another nbhd U of id such that $U \times U \subset F^{-1}(W)$.

This is true iff $\forall u_1, u_2 \in U$, $u_1 u_2^{-1} \in W$.

Claim: if $\gamma(U) \cap U \neq \emptyset$, then $\gamma = \mathrm{id}$.

Indeed, $\gamma(u_1) = u_2$, then $\gamma u_1 = u_2$ iff $\gamma = u_2 u_1^{-1} \in W$, but $W \cap \Gamma = \{\mathrm{id}\}$, so $\gamma = \mathrm{id}$. □

Definition 4.13. *If $p : E \to B$ is a covering map and $f : X \to B$ continuous then a lifting of f is a continuous map $\tilde{f} : X \to E$ such that $p \circ \tilde{f} = f$.*



Theorem 4.24. *If X is connected and $\tilde{f}_1, \tilde{f}_2 : X \to E$ are two liftings of $f : X \to B$ wrt a cover $p : E \to B$ such that $\tilde{f}_1(x_0) = \tilde{f}_2(x_0)$*

Proof. Let $W = \{x \in X : \tilde{f}_1(x) = \tilde{f}_2(x)\} \neq \emptyset$.

Claim: W is open and closed (so $W = X$, since X is connected).

W open, take $x \in W$, we wil find an open nbhd V of $x \in X$ such that $V \subset W$. Consider $p(\tilde{f}_1(x)) = p(\tilde{f}_2(x)) = y$ in B. let U be an elementary neighborhood of y such that $p^{-1}(U) = \coprod_\alpha V_\alpha$, $p| : V_\alpha \to U$ homeo.

Say $\tilde{f}_1(x) = \tilde{(f)}_2(x) \in V_{\alpha_0}$. $V = \tilde{f}_1^{-1}(V_{\alpha_0}) \cap \tilde{f}_2^{-1}(V_{\alpha_0})$ is open and contains x. $V \subset W$. $z \in V \Rightarrow \tilde{f}_1(z), \tilde{f}_2(z) \in V_{\alpha_0}$.

$p(\tilde{f}_1(z)) = p(\tilde{f}_2(z)) \Rightarrow \tilde{f}_1(z) = \tilde{f}_2(z)$, $z \in W$.

Proof that W is closed is homework. □

Theorem 4.25 (Path-Lifting). *Suppose $p : E \to B$ is a covering map $p(e_0) = b_0$ and $a : ([0,1], 0) \to (B, b_0)$ is a path in B. Then there exists a unique lifting $\tilde{a} : ([0,1]0) \to (E, e_0)$ such that $p \circ \tilde{a} = a$.*

Proof. Uniqueness is clear by the previous theorem.

NEXT TIME □

Uniqueness of Lifting

Suppose $p : E \to B$ is a covering map, X is connected and $\tilde{f}_1, \tilde{f}_2 : X \to E$ so that $p \circ \tilde{F}_1 = p \circ \tilde{f}_2$ and $\tilde{f}_1(x_0) = \tilde{f}_2(x_0)$ for one $x_0 \in X$, then $\tilde{f}_1 = \tilde{f}_2$.

Theorem 4.26 (Path-Lifting Theorem). *Suppose $p : (E, e) \to (B, b)$ is a covering map $p(e) = b$ and $a : ([0,1], 0) \to (B, b)$ is a path in B. Then $\exists!$ lifting $\tilde{a} : ([0,1], 0) \to (E, e)$ such that $p \circ \tilde{a} = a$.*

Proof. Let $\mathcal{U} = \{$all elementary neighborhoods in $B\}$. This is an open cover of B.

Thus, $\mathcal{V} = \{a^{-1}(u) : u \in \mathcal{U}\}$ forms an open cover of I.

By Lebesgue lemma, there exists a partition $0 = t_0 < t_1 < \ldots < t_n = 1$ such that $a([t_i, t_{i+1}]) \subset U_i$ for $U_i \in \mathcal{U}$ for all i.

Construct $\tilde{a}$ inductively on $[t_0, t_n]$.

Step 1: $\tilde{a}|_{[t_0,t_1]}$. $a([t_0, t_1]) \subset U_1$, $p^{-1}(U_1) = \coprod_\alpha V_{1\alpha}$, the disjoint union of $V_{1\alpha}$, $p|_{V_{1\alpha}} : V_{1\alpha} \to U_1$ homeo.

Say $e \in V_{1\beta} \Rightarrow \tilde{a}|_{[t_0,t_1]} = (p|_{V_{1\beta}})^{-1} a|_{[0,t_1]}$.

Clearly, $\tilde{a}|_{[t_0,t_1]}$ is continuous, $p \circ \tilde{a} = a$ on $[t_0, t_1]$, $\tilde{a}(0) = e$.

Step 2: Suppose $\tilde{a}$ has been defined on $[t_0, t_{i-1}]$. Now $a([t_{i-1}, t_i]) \subset U_i$ - elementary neighborhood.

Let $p^{-1}(U_i) = \coprod_\alpha V_{i\alpha}$. $a(t_{i-1}) \in U_i$ and $p(\tilde{a}(t_{i-1})) = a(t_{i-1})$ by induction, and os we have $\tilde{a}(t_{i-1}) \in V_{i\beta}$ for some β. Define $\tilde{a}|_{[t_{i-1},t_i]} = (p|_{V_{i\beta}})^{-1} \circ a|_{[t_{i-1},t_i]}$.

By the gluing lemma, $\tilde{a}$ on $[t_0, t_i]$ is continuous and $p \circ \tilde{a} = a$. □

Theorem 4.27 (Homotopy Lifting). *Suppose $p : E \to B$ is a covering map, $p(e) = b$ and $H : [0,1] \times [0,1] \to B$ is contiuous so that $H(0,0) = b$, then $\exists!$ lifting $\tilde{H} : [0,1] \times [0,1] \to E$ such that $p \circ \tilde{H} = H$, $\tilde{H}(0,0) = e$*

Proof. $\mathcal{U}$ is the set of all elementary neighborhoods in B, and it is an open cover of B. Thus, $V = \{H^{-1}(U) : U \in \mathcal{U}\}$ forms an open cover $I \times I$. Lebesgue's Lemma gives a partition of I such that $H([t_{i-1}, t_i] \times [t_{j-1}, t_j]) \subset U_{ij}$ where $U_{ij} \in \mathcal{U}$.

Let order those small sequences as Q_k where $k \in [n^2]$ such that $Q_1 = [0, t_1] \times [0, t_1]$ and $Q_i \cap (Q_1 \cup \ldots \cup Q_{i-1})$ is connected. Say $H(Q_k) \subset U_k$.

Define $\tilde{H}$ on $Q_1 \cup \ldots \cup Q_k$ inductively as follows:

Step 1: $\tilde{H}|_{Q_1}$. $H(Q_1) \subset U_1$, so $p^{-1}(U_1) = \coprod_\alpha V_{1\alpha}$. $e \in V_{1\beta}$ for some β, since $b \in U$ is $H(0,0)$.

Define $\tilde{H}|_{Q_1} = (p|_{V_\beta})^{-1} \circ H|_{Q_1}$.

Step 2: Suppose $\tilde{H}$ has been defined on $Q_1 \cup \ldots \cup Q_{i-1}$. Take $v_i \in Q_i \cap (Q_1 \cup \ldots \cup Q_{i-1})$. Then, as $H(Q_i) \subset U_1$ and $p^{-1}(U_i) = \coprod_\alpha V_{i\alpha}$, $p(\tilde{H}(v_i)) = H(v_i) \in U_i$, and so, $\tilde{H}(v_i) \in V_{i\beta}$ for some β.

Now we define $g|_{Q_i} = (p|_{V_{i\beta}})^{-1} \circ H|_{Q_i}$.

$g|_{Q_i}$ cont, $p \circ g = H|_{Q_i}$.

Claim: $g|_{Q_i \cap (Q_1 \cup \ldots \cup Q_{i-1})} = \tilde{H}|_{Q_i \cap (Q_1 \cup \ldots \cup Q_{i-1})}$. indeed, both are liftings of $H|_{Q_i \cap (Q_1 \cup \ldots \cup Q_{i-1})}$ by definition, and they take the same value at v_i. Furthermore, $Q_i \cap (Q_1 \cup \ldots \cup Q_{i-1})$ is connected, and so by the uniqueness of lifting, we are done proving the claim.

Now define $\tilde{H}$ on $Q_1 \cup \ldots \cup Q_i$ by gluing, and so done. □

Corollary 4.28. *If $H : I \times I \to B$ is a path homotopy, $H(0,t) = b_0$, $H(1,t) = b$ for all t, and $\tilde{H} : I \times I \to E$ is a lifting of H with respect to a covering map $p : E \to B$, thenm $\tilde{H}(0,t) = y_0$ and $\tilde{H}(1,t) = y_1$ for all t.*

Proof. $p \circ \tilde{H}(0,t) = b_0$, $p \circ y_0 = b_0$, so $\tilde{H}(0,t) = y_0$ by uniqueness of lifting. □

Theorem 4.29. $\pi_1(S^1, b) \simeq \mathbb{Z}$.

Proof. Calculate $b = 1$. Let $p(x) = e^{2\pi i x} : \mathbb{R}^1 \to S^1$ be the covering, $p(0) = 1$. For any loop $a : [0,1] \to S^1$, $a(0) = a(1) = 1$. Let $\tilde{a} : [0,1] \to \mathbb{R}^1$ be the lifting of a with $\tilde{a}(0) = 0$. By the corollary, if $a \simeq_p b$, then $\tilde{a} \simeq_p \tilde{b}$. In particular, $\tilde{a}(1) = \tilde{b}(1) \in \mathbb{Z}$.

Define $\Phi : \pi_1(S^1, 1) \to \mathbb{Z}$. $\Phi([a]) = \tilde{a}(1)$.

Claim: Φ is a 1-1, onto, group homomorphism.

Φ is 1-1: Suppose $\Phi([a]) = \Phi([b])$ Then $\tilde{a}(1) = \tilde{b}(1)$. $\tilde{a}, \tilde{b} : ([0,1], 0, 1) \to (\mathbb{R}, 0, \tilde{a}(1))$, which implies that $\tilde{a} \simeq_p \tilde{b}$, and so $p \circ \tilde{a} \simeq_p p \circ \tilde{b}$ and thus, $a \simeq_p b$.

Φ is onto: suppose that $\tilde{a}(1) = n, \tilde{b}(1) = m$. $\Phi([a][b]) = \Phi([ab])$. Then $\tilde{ab}(1) = \tilde{a}(1) + \tilde{b}(1) = n + m$. □

Theorem 4.30. *Suppose Γ is a countable group active properly discontinuously on a simply connected manifold Ω. Then $\pi_1(\Omega/\Gamma) \simeq \Gamma$.*

Proof. $p : \Omega \to \Omega/\Gamma$, the quotient map is taken to be the covering map. The rest is homework. □

Corollary 4.31. *If E is path connected and $p : E \to B$ is a covering map, then the cardinality of $p^{-1}(b_1), p^{-1}(b_2)$ are the same.*

If the cardinality is finite, we say that p is an n-fold covering.

Proof. E path connected and p onto implies that B is path connected.

In particular, $\exists$ a path a in B from b_1 to b_2.

If $x \in p^{-1}(b_1)$, let $\tilde{a}_x$ be the lifting of a with $\tilde{a}_x(0) = x$. (path lifting) Since $a(1) = b_2 \Rightarrow p(\tilde{a}_x(1)) = b_2 \Rightarrow \tilde{a}_x(1) \in p^{-1}(b_2)$.

Let $\Phi : p^{-1}(b1) \to p^{-1}(b_2) : x \mapsto \tilde{a}_x(1)$.

Claim: Φ is 1-1, onto.

Φ is one to one: we take $\phi(x_1) = \phi(x_2)$. Take $\tilde{a}_{x_1}$ and $\tilde{a}_{x_2}$ are the two liftings of a such that $\tilde{a}_{x_1}(1) = \tilde{a}_{x_2}(1)$. By the uniqueness of lifting, $\tilde{a}_{x_1} = \tilde{a}_{x_2}$, so $x_1 = x_2$.

Φ is onto: $\forall y \in p^{-1}(b_2)$, let $\tilde{b}$ be the lifting of a^{-1} with initial point $\tilde{b}(0) = y$. Then $\tilde{b}(1) = x \in p^{-1}(b_1)$, then $\tilde{a}_x = \tilde{b}^{-1}$ satisfies $\phi(x) = y$. ☐

Definition 4.14 (Locally Path Connected)**.** *X is locally path connected if $\forall x \in X$ and any nbhd U of x, $\exists$ a path connected nbhd V of x such that $x \in V \subset U \subseteq X$.*

Theorem 4.32 (General Lifting)**.** *Suppose $p : E \to B$ is a covering map, $p(e_0) = b_0$ and $f : X \to B$ continuous with $f(x_0) = b_0$ and X is path connected and locally path connected. Then f has a lifting $\tilde{f} : (X, x_0) \to (E, e_0)$ iff $f_*(\pi_1(X, x_0)) \subset p_*(\pi_1(E, e_0))$.*

Proof. If $\tilde{f}$ exists, then $p \circ \tilde{f} = f$, so $p_*(\tilde{f}(\pi_1(X, x_0))) = f_*(\pi_1(X, x_0))$, which must be contained in $p_*(\pi_1(E, e_0))$.

That the condition is sufficient: take $x \in X$, let a be a path in X from x_0 to x. Then the path $f \circ a$ in B has a lifting $f \tilde{\circ} a$ starting at e_0. Define $\tilde{f}(x) = f \tilde{\circ} a(1)$. Clearly $p \circ \tilde{f} = f$. $\tilde{f}(x_0) = b_0$.

Claim 1: $\tilde{f}$ is well defined.

Proof of Claim 1: Suppose a, b are paths from x_0 to x so that $f \tilde{\circ} a(1) \neq f \tilde{\circ} b(1)$. Then, $\tilde{(f \circ ba^{-1})}$ lifts to a path with distinct endpoints.

But $[f \circ (ab^{-1})] \in p_*(\pi_1(E, e_0))$ says that the lifting of $f \circ (ab^{-1})$ is a loop at e_0. Contradiction.

Claim 2: $\tilde{f}$ is continuous.

Take open $W \subset E$. Say $\tilde{f}(x) \in W$. There exists a nbhd W' of x in X such that $\tilde{f}(W') \subset W$.

p is a local homeomorphism as it is a covering map, and so $p(W)$ is an open set containing $f(x)$. Now f is continuous implies that $f^{-2}(p(W))$ is an open set containing x.

X is locally path connected, and so $\exists$ path connected nbhd W' of x such that $x \in W' \subset f^{-1}(p(W))$.

Claim: $\tilde{f}(W') \subset W$.

p is a covering map, and so $\exists$ an elementary neighborhood U of $f(x)$ such that $p^{-1}(U) = \coprod_\alpha V_\alpha$ and one $V_\beta \subset W$.

Now, $z \in W'$. Let b be a path in W' form x to z, and let a be a path from x_0 to x.

$c = ab$ is a path from x_0 to z.

$b \subset W' \Rightarrow f \circ b \subset U$, so there is a lifting $\tilde{g}$ of $f \circ b$ starting at $\tilde{f}(x)$ lying in V_β. $\tilde{g} = \left(p|_{V_\beta}\right)^{-1}(f \circ b)$.

Then $f \circ \tilde{(ab)} = (f \tilde{\circ} a) \circ \tilde{g}$ by the gluing lemma. And so $\tilde{f}(z) = f \circ \tilde{(ab)}(1) = \tilde{g}(1) \in V_\beta \subset W$. □

Applications: Assume all spaces are path connected and locally path connected:

Definition 4.15 (Equivalent Covering Maps)**.** *Two covering maps $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ are equivalent iff there $\exists$ homeomorphism $h : E_1 \to E_2$ such that $p_1 \circ h = p_2$.*

Corollary 4.33. *$p_1 : (E_1, e_1) \to (B, b_0)$ and $p_2 : (E_2, e_2) \to (B, b_0)$ are equivalent iff $(p_1)_* \pi_1(E_1, e_1) = (p_2)_* \pi_1(E_2, e_2)$*

Proof. By the general lifting theorem, $\exists h : (E_2, e_2) \to (E_1, e_1)$ with $p_1 \circ h = p_2$ and $\exists h : (E_1, e_1) \to (E_2, e_2)$ such that $p_w \circ g = p_1$.

Claim: $h \circ g = \text{id}$, $g \circ h = \text{id}$. That is, h is a homeomorphism.

$h \circ g(e_1) = h(e_2) = e_1$. $p_1(h \circ g) = p_1 \circ h \circ g = p_2 \circ g = p_1 = p_1 \circ \text{id}$.

So both $h \circ g$ and id are liftings of $p_!$ and $h \circ g(e_1) = \text{id}(e_1)$. Uniqueness implies that $h \circ g = \text{id}$. □

Classification of Covering Spaces:

All spaces are path connected and locally path connected.

Theorem 4.34. *Suppose M is a connected manifold with $\Gamma = \pi_1(M, x_0)$. Then M has a universal cover $\tilde{M}$ which is simply connected and $p : \tilde{M} \to M$ such that Γ acts properly discontinuously on $\tilde{M}$ with $p(x_1) = p(x_2)$ iff $x_1 = \gamma x_2$ for $\gamma \in \Gamma$.*

This theorem actually holds for any space that is path-connected, locally path-connected and semi-locally simply connected. This last condition is that $\forall x \in M$, $\exists$ a path connected nieghborhood U of x such that $i_* : \pi_1(U, x) \to \pi_1(M, x)$ is trivial where $i : U \to M$ is the inclusion.

Proof. $\tilde{M} = \{[a] : a \text{ is a path in } M \text{ with } a(0) = x_0\}$.

$p : \tilde{M} \to M$ by $p([a]) = a(1)$.

$\Gamma = \{[\rho] : \rho \text{ a loop in } M\ \rho(0) = x_0\}$.

Let Γ act on $\tilde{M}$ by $[\rho][a] = [\rho * a]$. By definition of multiplication of paths, this is a group actions.

Claim: $p([a_1]) = p([a_2]) \iff [a_1] = [\rho][a_2]$ for $[\rho] \in \pi_1(M, x_0)$.

$\Leftarrow$ is clear, and so we will focus on $\Rightarrow$. $a_1(1) = a_2(1) \Rightarrow a_1 = (a_1 * a_2)^{-1} * a_2 = p * a_2$.

We must now define the topology. $[a] \in \tilde{M}$ and a simply connectd open set U containing $a(1)$. $[a] \in U_{[a]} = \{[ab] : b \text{ a path in } U \text{ with } b(1) = a(1)\}$.

Claim: $\{U_{[a]}\}$ form a basis.

$\tilde{M} = \cup U_{[a]}$. If $[c] \in U_{[a]} \cap V_{[b]}$ we wnat an open simnply connected W containing $c(1)$ and $W_{[c]} \subset U_{[a]} \cap V_{[b]}$.

$[c] \in U_{[a]}$ implies that $c \simeq_p aa' \simeq bb'$ and $c(1) \in U \cap V$. W open coordinate chart with $c(1) \in W$. $\pi_1(W) = 1$ such that $W \subseteq U \cap V$.

Then $\forall [f] \in W_{[f]}$ we have $f = cc' \simeq aa'c' \simeq a(a'c')$, and so $[f] \in U_{[a]}$. Similarly, $[f] \in V_{[b]}$.

Claim: $p : \tilde{M} \to M$ is continuous and $p|_{U_{[a]}} : U_{[a]} \to U$ is a homeomorphism.

$p| : U_{[a]} \to U$ is 1-1, $p([aa']) = a'(1)$. $p([aa']) = p([ab'])$ so $a'(1) = b'(1)$. U simply connected implies that $a' \simeq_p b'$, and so $aa' \simeq ab'$.

It is onto as U is path connected.

p continuous because we cna take an oepn simply connected set U, and then $p^{-1}(U) = \cup_{\gamma \in \Gamma} U_{\gamma[a]}$.

$\supseteq$ is clear as $[b] \in U_{\gamma[a]}$ imples that $b(1) \in U$ and so $p([b]) = b(1) \in U$. $\subseteq$ is because we take $[b] \in p^{-1}(U)$ and so $p([b]) \in U$. Thus, $b(1) \in U$. Since U is path connected, let c be a path in U from $a(1)$ to $b(1)$. $(a * c)(1) = b(1)$ and so $[b] = \gamma[ac]$, $\gamma \in \pi_1(M, x_0)$. Thus $[b] \in U_{\gamma[a]}$. Thus, p is continuous.

Since $p(U_{[a]}) = U$, p sends open sets to open sets, thus, $(p|_{U_{[a]}})^{-1}$ is continuous.

Claim: $p^{-1}(U) = \coprod_{\gamma \in \Gamma} U_{\gamma[a]}$ for open simply connected sets $U \subset M$.

Indeed, $U_{\gamma[a]} \cap U_{\gamma'[a]} \neq \emptyset$ implies $[b] \in U_{\gamma[a]} \cap U_{\gamma'[a]}$. $b \simeq \rho ac$ where $\rho \in \gamma$ and c is a path in U, and also $\simeq_p \rho' ac'$ where $c(1) = c'(1)$. U is simply-connected, and so $c \simeq_p c'$. $\rho ac \simeq \rho ac'$ and so $\rho acc^{-1} \simeq \rho' ac'c^{-1}$, thus, $\rho a \simeq \rho' a$ and so $\rho \simeq \rho'$, thus, $\gamma = \gamma'$. And so $p : \tilde{M} \to M$ is a covering map with elementary open sets U are open and simply connected.

Claim: $\tilde{M}$ path connected.

Take $[a] \in \tilde{M}$. The base point $y_0 = [x_0]$ a constant path. Define $b_t(s) = a(ts)$ for $t, s \in [0, 1]$.

$p([b_t]) = a(t \cdot 1) = a(t)$, and so $t \mapsto [b_t]$ is the required path in $\tilde{M}$.

$[b_1] = [a]$ and $[b_0] = [x_0]$. So $p(\alpha(t)) = a(t)$, and p a local homeomorphism implies that $\alpha(t)$ is continuous.

Claim: $\tilde{M}$ is simply connected.

We will show any $[a] \in \pi_1(M, x_0) \setminus \{\text{id}\}$ is lifted to a path, not a loop.

By the previous claim, the lifting of a is $\alpha(t) = [a(ts)|_{s \in [0,1]}]$. Then $\alpha(1) = [a] \neq [x_0]$ as $[a] \neq [\text{id}]$. So $p_* : \pi_1(\tilde{M}, [x_0]) \to \pi_1(M, x_0)$ is 1-1. The above implies that $p_*(\pi_1(\tilde{M}, [x_0])) \simeq \{\text{id}\}$, and so $\pi_1(\tilde{M}) = 1$.

Claim: $\tilde{M}$ is Hausdorff.

Follows from lemma which follows.

We also note that there is a countable basis, as Γ is countable, but will not prove it. □

Lemma 4.35. *$p : E \to B$ is a covering map and B is Hausdorff. Then E is Hausdorff.*

Proof. Take $x_1 \neq x_2$ in E. If $p(x_1) \neq p(x_2)$ then we are done.

If $p(x_1) = p(x_2)$, then we let U be an elementary neighborhood of $p(x_j)$ with respect to p. Then $p^{-1}(U) = \coprod_\alpha V_\alpha \Rightarrow x_1 \in B_{\beta_1}$, $x_2 \in V_{\beta_2}$. □

Definition 4.16 (Galois Covering)**.** *A covering map $p : E \to B$ is called Galois (regular) if $\exists$ a group Γ acting properly discontinuously on E such that $p(x_1) = p(x_2)$ iff $x_1 = \gamma x_2$ for some $\gamma \in \Gamma$. (Γ is the deck-transforamtion of p)*

Theorem 4.36. *Let B be a path connected manifold, and $\Gamma = \pi_1(B, x_0)$. Then*

1. *{ All conected covering spaces E of B, $p : (E, y_0) \to (B, x_0)$ up to equivalence} is bijective to { all subgroups of Γ}.*
2. *Under the correspondence, Galois covering maps correspond to normal subgroups of $\pi_1(B, x_0)$.*

Proof. Define the map Φ sending $p : E \to B$ with $p(y_0) = x_0$ $\Phi(p) = p_*(\pi_1(E, y_0)) < \pi_1(B, x_0)$. The General Lifting Theorem implies that Φ is 1-1.

Φ is onto, because if we take a subgroup G of $\pi_1(B, x_0)$ then we let $\tilde{B}$ be the universal cover of B with deck transformation $\Gamma = \pi_1(B, x_0)$.

$G < \Gamma$ implies that G act properly discontinuously on $\tilde{B}$.

Let $p : E = \tilde{B}/G \to \tilde{B}/\Gamma : [x] = G \cdot x \mapsto \Gamma \cdot x$. Let $\tilde{p} : \tilde{B} \to \tilde{B}/\Gamma$ be the universal cover.

p is a covering map is clear using $\tilde{p}$. $p_* : \pi_1(\tilde{B}/G) \to \pi_1(\tilde{B}/\Gamma)$.

By definition the only elements in $\pi_1(\tilde{B}/\Gamma) \simeq \Gamma$ which lifted to loops in $\tilde{B}/G$ are elements of G. Thus, $p_*(\pi_1) = G$. □

Theorem 4.37. *B is a connected manifold and $G = \pi_1(B, x_0)$.*

There exists a bijection Φ from the set of all connected covering spaces $p : E \to B$ up to equivalence to the subgroups of G, by $\Phi(p) = p_*(\pi_1(E, y_0))$, $p(y_0) = x_0$.

Furthermore, $|p^{-1}(x)| = [G : \Gamma]$ and $p : E \to B$ regular iff Γ a normal subgroup of G.

Proof. We proved 1-1 and onto before. To see that $|p^{-1}(x)| = [G : \Gamma]$, Recall $\Gamma \leq G$. The associated covering space $p : E \to B$ is $E = \tilde{B}/\Gamma$, $B = \tilde{B}/G$ where $\tilde{B}$ is the universal cover of B.

$p(\Gamma x) = Gx$, $p^{-1}(Gx) = \coprod_{g \in G/\Gamma}\{\Gamma(gx)\}$ $Gx = \coprod_{g \in G/\Gamma} \Gamma(gx)$. And so, the result follows.

To see that $p : E \to B$ is regular iff $\Gamma \trianglelefteq G$.

$\Rightarrow$: $p : (E, y_0) \to (B, x_0)$ is regular, then there is a group H acting properly discontinuously on E such that $p(x_1) = p(x_2)$ iff $x_1 = hx_2$ for $h \in H$. Take $[a] \in \pi_1(B, x_0)$, $[b] \in \pi_1(E, y_0)$. $[a]^{-1}p_*([b])[a] \in p_*\pi_1(E, y_0)$. Let $\tilde{a}$ be a lifting of a with $\tilde{a}(0) = t_0$). $\tilde{a}^{-1} * b * \tilde{a}$ is a loop based at $\tilde{a}(1)$, so $p(\tilde{a}^{-1} * b * \tilde{a}) = a^{-1}p \circ ba \in [a]^{-1}p_*[b][a]$. $p : E \to B$ regular implies that there is $h \in H$ such that $h\tilde{a}(1) = \tilde{a}(0) = y_0$, and so $p \circ (h \cdot (\tilde{a}^{-1} * b * \tilde{a})) = p(\tilde{a}^{-1} * b * \tilde{a}) \in [a]^{-1}p_*[b][a]$, and os $[a]^{-1}p_*([b])[a] = p_*([c]) \in p_*(\pi_1(E, y_0))$, where $c = p \circ (h \cdot (\tilde{a}^{-1} * b * \tilde{a}))$ is a loop based at y_0.

$\Leftarrow$: If $\Gamma \leq G$ normal with quotient group $H = G/\Gamma$, then construct the action of $H = \{\Gamma g : g \in G\} = \{[g] : g \in G\}$ on $E = \tilde{B}/\Gamma$ by $[g](\Gamma x) = \Gamma(gx)$, $p([g](\Gamma x)) = p(\Gamma x)$. Also $p(\Gamma x_1) = p(\Gamma x_2)$ so $Gx_1 = Gx_2$. That is, $x_2 = gx_1$ for some $g \in G$. H acts properly discontinuously on $\tilde{B}/\Gamma$. Take $\Gamma x \in \tilde{B}/\Gamma$, $x \in \tilde{B}$.

G acts properly discontinuously on $\tilde{B}$, so there exists a neighborhood U of x such that $gU \cap U = \emptyset$, $g \in G \setminus \{\text{id}\}$.

Take $V = \Gamma U$ to be open neighborhood of Γx. Claim: $[g] \in H \setminus \{\text{id}\}$. $[g]V \cap V = \emptyset$. If not $\exists y \in U, \Gamma y \in V$ with $[g](\Gamma y) = \Gamma(gy) \in V \cap \Gamma U$. And so $gy = \gamma y'$ where $\gamma \in \Gamma$, $y' \in U$.

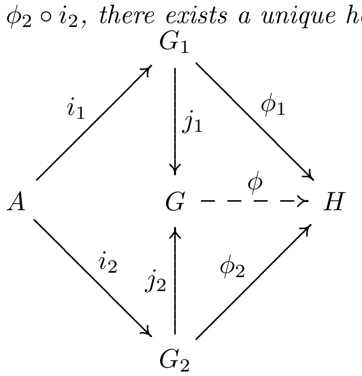
$(\gamma^{-1}g)y = y'$, $(\gamma^{-1}g)U \cap U \neq \emptyset$ so $\gamma^{-1}g = \text{id}$, so $g = \gamma \in \Gamma$, contradiction. $\square$

Computation of $\pi_1(X)$

Some algebra: The free group of k generators $x_1, \ldots, x_k$ will be denoted $F_k = \langle x_1, \ldots, x_k \rangle = \mathbb{Z} * \ldots * \mathbb{Z}$.

Given any finitely generated group G with generators $g_1, \ldots, g_k$, then $\Phi : F_k \to G$ by $x_i \mapsto g_i$ is an epimorphism. $\ker \Phi$ is a normal subgroup of F_k, which is normally generated by $\gamma_1, \ldots$. We say G is presented by $\langle x_1, \ldots, x_k : \gamma_1, \ldots \rangle$.

Definition 4.17 (Universal Extension)**.** *If A, G_1, G_2 are groups, and $i_1 : A \to G_1$ and $i_2 : A \to G_2$ are two homomrophisms, then there exists a unique, up to isomorphism, G and homomorphisms $j_1 : G_1 \to G$, $j_2 : G_2 \to G$ suc that $\forall$ groups H and any homomorphisms $\phi_1 : G_1 \to H$, $\phi_2 : G_2 \to H$ with $\phi_1 \circ i_1 = \phi_2 \circ i_2$, there exists a unique homomorphism $\phi : G \to H$ such that $\phi_k = \phi \circ i_k$.*



It turns out that this is $G_1 * G_2 / \langle i_1(a)(i_2(a))^{-1} \rangle_{a \in A}$.

Theorem 4.38 (Seifert, Van Kampen)**.** *Let X be a topological space, $X = U_1 \cup U_2$, where $U_1, U_2, U_1 \cap U_2$ are open, path connected. Then for any group H and group homomorphism $\phi_k : \pi_1(U_k, x_0) \to H$ such that $\phi_1(i_{1*}) = \phi_2(i_{2*})$, there exists a unique homomorphism $\phi : \pi_1(X, x_0) \to H$.*

Thus, $\pi_1(X) \simeq \pi_1(U_1) * \pi_2(U_2) / \langle (i_1)_*(a)(i_2)_*(a^{-1}) \rangle_{a \in \pi_1(U_1 \cap U_2)}$.

Corollary 4.39. *Under the same assumptions:*

1. *If $\pi_1(U_1 \cap U_2) = 1$, then $\pi_1(X) \simeq \pi_1(U_1) * \pi_1(U_2)$*

2. *If* $\pi_1(U_2) = 1$*, then* $\pi_1(X) = \pi_1(U_1)/\langle \pi_1(U_1 \cap U_2) \rangle$*.*

3. *If* $\pi_1(U_1) = \pi_1(U_2) = 1$*, then* $\pi_1(X) = 1$*.*

We will prove this corollary without using Seifert-Van Kampen.

Proof. Part 3: Take a loop $a : [0,1] \to X$, $a(0) = a(1) = x_0$. By Lebesgue Lemma, there is a partition $0 = t_0 < t_1 < \ldots < t_n = 1$ of $[0,1]$ such that $\forall i$, $a([t_1, t_{i+1}]) \subset U_1$ or U_2. $a_i(s) = a((1-s)t_i + st_{i+1})$ be the path of a from $a(t_i)$ to $a(t_{i+1})$.

$a \simeq_p a_0 * \ldots * a_{n-1}$. For each i, choose a path $b_)i$ from x_0 to $a(t_i)$ as follows: if $a(t_i) \in U_1 \cap U_2$, then $b_i \subset U_1 \cap U_2$ and if $a(t_i) \in U_k \setminus (U_1 \cap U_2)$, then $b_i \subset U_k$.

Let $c_i = b_i a_i b_{i+1}^{-1}$, this is a loop at x_0. Then $a_0 \ldots a_{n-1} \simeq_p (a_0 b_1)(b_1^{-1} a_1 b_2) \ldots (b_{n-1}^{-1} a_{n-1}) = c_0 c_1 \ldots c_{n-1}$.

Claim: c_i lies in U_1 or U_2. The claim implies that $c_i \simeq 1$, and so we are done.

Say $a[t_i, t_{i+1}] \subset U_k$, $a(t_i) \in U_k$, $a(t_{i+1}) \in U)k$, and so $b_i \subset U_k$ and $b_{i+1} \subset U_k$. Thus, $c_i = b_i a_i b_{i-1}^{-1} \subset U_k$, and so we are done. □

This proof shows that there is an epimorphism $\Phi : \pi_1(U_1) * \pi_1(U_2)/\langle (i_1)_*(a)(i_2)_*^{-1}(a) \rangle \to \pi_1(X)$. The difficulty is to show that the kernel is trivial.

Theorem 4.40. *If* $n \geq 3$*,* M_1, M_2 *are connected manifolds, then* $\pi_1(M_1 \# M_2) \simeq \pi_1(M_1) * \pi_1(M_2)$ *where* $M_1 \# M_2 = (M_1 \setminus \overset{\circ}{D_1^n}) \cup_h (M_2 - \overset{\circ}{D_2^n})$ *where* D_i^n *is a smooth* n*-ball in* M_i *and* $h : \partial D_2^n \to \partial B_1^n$ *is a diffeomorphism.*

Remark: The diffeomorphism type of $M_1 \# M_2$ is independent of the choices of D_1, D_2 and h.

Theorem 4.41 (Gurosh)**.** *Subgroups of a free group are free.*

Proof. Every connected graph is homotopy equivalent to a wedge of $S^1 \wedge \ldots \wedge S^1$ with countable many circles.

By S-vK, $\pi_1(S^1 \wedge \ldots \wedge S^1)$ is a free group. Suppose that G is a subgroup of $F_n = \mathbb{Z} * \ldots * \mathbb{Z} = \pi_1(S^1 \wedge \ldots \wedge S^1, p)$. Let $p : E \to S^1 \wedge \ldots S^1$ be the covering space with $p_*(\pi_1(E,q)) = G$. But E is a graph, and so $\pi_1(E,q)$ is a free group. □

Proposition 4.42. *If* X *is a compact connected surface with* $\partial X \neq \emptyset$*, then* $\pi_1(X)$ *is a free group.*

Theorem 4.43. $\pi_1(\Sigma_g, X) \simeq \langle a_1, \ldots, a_{2g} | a_1 a_2 \ldots a_{2g} a_1^{-1} \ldots a_{2g}^{-1} \rangle$.

Corollary 4.44. *If* $g \geq 2$*, then* $\pi_1(\Sigma_g)$ *is not abelian.*

e.g. The 3-Dimensional Lens Space $L(p,q)$ where $p, q \in \mathbb{Z}$ are relatively prime integers. Let $\mathbb{Z}_p = \langle \eta \rangle$ act on $S^3 = \{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$ $\eta(z,w) = (e^{2\pi i/q} z, e^{2\pi i q/p} w)$. $\mathbb{Z}_p$ acts freely.

So, $L(p,q) = S^3/(\mathbb{Z}_p)$, and so S^3 is the universal cover. Thus $\pi_1(L(p,q)) \simeq \mathbb{Z}_p$.

And so, we can construct 3-manifolds with fundamental group the free product of any finite number of cyclic groups.

The braid group, B_n, $n \in \mathbb{N}$

$X_n = \{(x_1, \ldots, x_n) \in \mathbb{C}^n : x \neq x_j, i \neq k\}$.

The symmetric group on n letters acts on X_n by permuting coordinates and is properly discontinuous.

We look at $Y_n = X_n/S_n = \{\{x_1, \ldots, x_n\} \subset \mathbb{C} : x_i \neq x_j, i \neq j\}$.

$(x_1, \ldots, x_n) \mapsto \{x_1, \ldots, x_n\}$. $p : X_n \to Y_n$ is a covering map, quotient by S_n, so S_n is the deck transformation group.

Take a base point $q = (1, \ldots, n) \in X_n$ and $\overline{q} = \{1, \ldots, n\} \in Y_n$.

Definition 4.18 (Braid Group)**.** *$B_n = \pi_1(Y_n, \overline{q})$, and the pure braid group $\pi_1(X_n, q)$.*

P_n is a subset of B_n by p_*.

Recall from Homological Algebra:

Suppose C^*, D^*, E^* are cochain complexes and $0 \to C^* \xrightarrow{i} D^* \xrightarrow{j} E^* \to 0$ a short exct sequence of chain maps.

Theorem 4.45. *There is an associated natural long exact sequence $\ldots \to H^n(C^*) \xrightarrow{i^*} H^n(D^*) \xrightarrow{j^*} H^n(E^*) \xrightarrow{\partial} H^{n+1}(C^*) \to \ldots$*

de Rham Cohomology:

Let M be a smooth manifold, U_1, U_2 open in M such that $M = U_1 \cup U_2$.

$i_k : U_1 \cap U_2 \to U_j$ and $j_k : U_k \to M$ be inclusions. $\Omega^i(M)$ is the space of all i-forms on M.

Proposition 4.46. *The following is an exact sequence:*

$$0 \to \Omega^i(M) \overset{j_1^* - j_2^*}{\to} \Omega^i(U_1) \oplus \Omega^i(U_2) \overset{i_1^* + i_2^*}{\to} \Omega^i(U_1 \cap U_2) \to 0$$

Mayer-Vietoris for de Rham Cohomology:

If $M = U_1 \cup U_2$ where U_1, U_2 are open in a smooth manifold M, then $\exists$ a natural long exact sequence as in theorem 23 and prop 6.

Corollary 4.47. *$H^i_{dR}(S^n) = \mathbb{R}$ for $i = 0, n$, 0 else.*