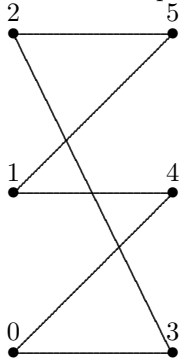


1 Chain Complexes

Poincaré 1900

Simplicial Homology. C_n = free abelian group on the n -simplex with ordered vertices, and $d : C_n \rightarrow C_{n-1}$ sends an n -simplex to $\sum_{\tau \in \partial\sigma} \pm \tau$.

The incidence matrix has every entry 0, +1, -1 depending on if the vertex is on the simplex and what the orientation is.



This graph has incidence matrix

$$\begin{bmatrix} 0 & 0 & 0 & -1 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \text{ where}$$

the rows are indexed by the edges and the columns are indexed by the vertices, with a 1 if an edge ends at that point, -1 if it begins there.

For a connected graph with V vertices and E edges, d has rank $V - 1$ and has a splitting, so $\mathbb{Z}^E = C_1 \rightarrow C_0 = \mathbb{Z}^V$ with $H_0(\text{graph}) \simeq \mathbb{Z}$ and $H_1(\text{graph}) \simeq \mathbb{Z}^{E-V+1}$.

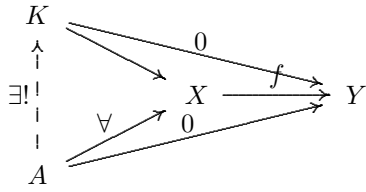
Definition 1.1 (Chain Complex). *A chain complex is a set of objects $\{C_n\}$ in a category like vector spaces, abelian groups, R -mod, and graded R -mod, with $d_n : C_n \rightarrow C_{n-1}$ maps such that the kernel of d_n is Z_n , the n -cycles of C_* , the image of d_{n+1} is B_n , the n -boundaries of C_* and $H_n(C) = Z_n/B_n$ is the kernel of the d_n (coker d_{n+1})*

An **Ab**-category is a category such that the Hom-sets are abelian groups and composition is bilinear.

An additive category is an **Ab**-category where finite products and coproducts exist and are the same.

Monics vs. Kernels - ι monic=cancellation, $X \xrightarrow{f} Y \xrightarrow{\iota} Z$, that is, $\iota f = 0 \Rightarrow f = 0$

$\ker(f)$ is a universal defined by the following diagram:



Epis vs. Cokernels - π epi=cancellation, $X \xrightarrow{f} Y \xrightarrow{\pi} Z$. Epis are the dual of monics and cokernels are the duals of kernels.

$$\text{Im}(f) = \ker(\text{coker } f)$$

Definition 1.2 (Abelian Category). An abelian category \mathcal{A} is an additive category in which monics=kernels, epis=cokernels, every monic $\iota : A \rightarrow B$ is the kernel of the map from B to $\text{coker}(\iota)$ and every epi $\pi : B \rightarrow C$ is the cokernel of $\ker(\pi) \rightarrow B$.

Theorem 1.1. **Ch**=the category of chain complexes in \mathcal{A} is an abelian category.

Proof. A morphism $C_* \xrightarrow{f} D_*$ of chain complexes is a family of maps $f_n : C_n \rightarrow D_n$ which commute with d , that is all squares below commute:

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{d} & C_n & \xrightarrow{d} & C_{n-1} & \xrightarrow{d} & \cdots \\
& & \downarrow f & & \downarrow f & & \\
\cdots & \xrightarrow{d} & D_n & \xrightarrow{d} & D_{n-1} & \xrightarrow{d} & \cdots
\end{array}$$

It is an **Ab**-category because it inherits the addition of morphisms, products and coproducts are done termwise, and we set the n^{th} term of the kernel to the kernel of the n^{th} term morphism, similarly with cokernels and images. \square

Operations

Shift - $C[p]$ is the complex which in degree n is C_{n+p} , $d_{C[p]} = (-1)^p d_C$

Good Truncation - $(\tau_{\geq p} C)$ is the complex which in degree n is $\begin{cases} 0 & n < p \\ Z_p & n = p \\ C_n & n > p \end{cases}$

$$\begin{array}{ccccccc}
\cdots & \longleftarrow & 0 & \longleftarrow & Z_p & \longleftarrow & C_{p+1} \longleftarrow \cdots \\
& & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longleftarrow & C_{p-1} & \longleftarrow & C_p & \longleftarrow & C_{p+1} \longleftarrow \cdots
\end{array}$$

Then $H_n(\tau C) = \begin{cases} 0 & n < p \\ H_n(C) & n \geq p \end{cases}$

Brutal Truncation - $(\beta_{\geq p} C)_n = \begin{cases} 0 & n < p \\ C_n & n \geq p \end{cases}$ and $H_p(\beta C) = C_p/B_p$ is different.

And the dual notions, $\tau_{<p}C = C/\tau_{\geq p}C$ has $H_n(\tau_{<p}C) = \begin{cases} H_n C & n < p \\ 0 & n \geq p \end{cases}$
and $\beta_{<p}C = \ker(C \rightarrow \beta_{\geq p}C)$ has $H_{p-1}(\beta_{<p}C = Z_{p-1})$ is $\dots \leftarrow C_{p-2} \leftarrow C_{p-1} \leftarrow 0$ is like a p -skeleton.

Definition 1.3 (Double Complex C_{**}). A double complex C_{**} is a $\mathbb{Z} \times \mathbb{Z}$ indexed family of objects C_{pq} and maps d^h, d^v such that $d^h \circ d^h = 0$, $d^v \circ d^v = 0$ and $d^h \circ d^v + d^v \circ d^h = 0$.

$$\begin{array}{ccccc}
& & \vdots & & \\
& & \downarrow d^v & & \\
C_{p+1,q} & \xleftarrow{d^h} & C_{p,q} & \xleftarrow{d^h} & C_{p-1,q} \\
\downarrow d^v & & \downarrow d^h & & \\
C_{p+1,q-1} & \xleftarrow{d^h} & C_{p,q-1} & &
\end{array}$$

Operations

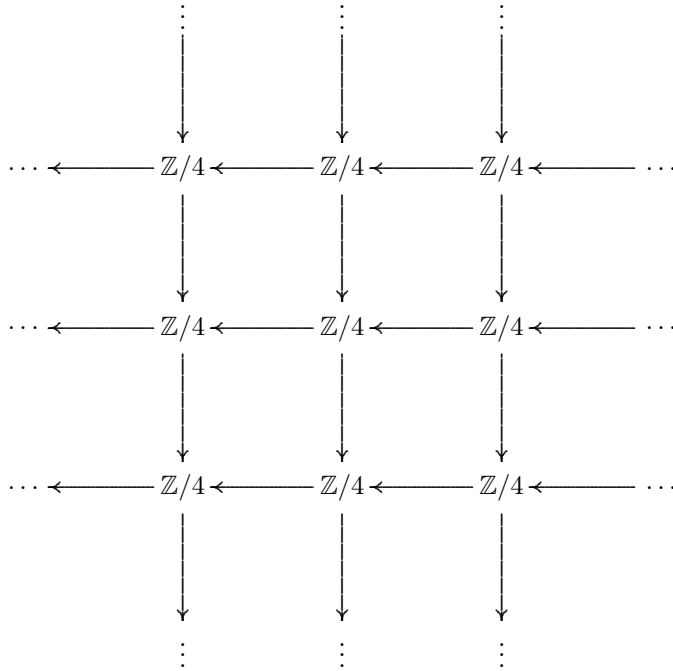
Total Complex - $(\text{Tot}^\oplus)_n = \bigoplus_{p+q=n} C_{p,q}$, $d = d^v + d^h$.

Definition 1.4 (Bounded Double Sequence). A double sequence is said to be bounded if $\forall n$ only finitely many C_{pq} with $p+q = n$ are nonzero.

Variant: $(\text{Tot}^\Pi)_n = \prod_{p+q=n} C_{p,q}$ is the product total complex. The product total complex and the total complex are the same iff C is bounded.

Example:

$C_{pq} = \mathbb{Z}/4$ for every p, q and $d^h = d^v = 2$.



Definition 1.5 (Acyclic). A complex C_* is acyclic if $H_n(C) = 0$ for all n .

Definition 1.6 (Exact). A sequence $C_{n-1} \leftarrow C_n \leftarrow C_{n+1}$ is exact if C_* is acyclic.

Compare to a chain complex of chain complexes.

We have \mathcal{A} an abelian category, and so we know that $\mathbf{Ch}(\mathcal{A})$ is an abelian category, and so we can take $\mathbf{Ch}(\mathbf{Ch}(\mathcal{A}))$

p^{th} object $D_p = C_{p,*}$ and $f_p : D_p \rightarrow D_{p-1}$ is a morphism of complexes, so it is really $f_{pq} : C_{pq} \rightarrow C_{p-1,q}$.

$$\begin{array}{ccccc}
\vdots & & \vdots & & \\
\downarrow & & \downarrow & & \\
C_{p-1,q-1} & & C_{p,q+1} & & \\
\downarrow & & \downarrow & & \\
C_{p-1,q} & \xleftarrow{f} & C_{p,q} & \xleftarrow{f} & C_{p+1,q} \\
\downarrow & & \downarrow & & \\
C_{p-1,q-1} & \xleftarrow{f} & C_{p,q-1} & &
\end{array}$$

BUT: $fd = df$ so it is not quite a double complex. We use “the sign trick” $d^v = (-1)^p d_C$ to make it into a double complex.

Proposition 1.2. *The category of double complexes is equivalent to the category $\mathbf{Ch}(\mathbf{Ch}(\mathcal{A}))$ via the sign trick.*

Aside: There is a “geometric realization” functor from $\mathbf{Ch}_{\geq 0}(\mathbf{Ab})$ to spaces, $C \mapsto |C|$

Short Exact Sequences of complexes \leftrightarrow fibrating.

Acyclic complex \leftrightarrow contractible space

$\Omega|C[-1]| \leftrightarrow |C|$ and $\Omega^n|C[-n]| \leftrightarrow |C|$

$\pi_n|C| = H_n(C)$ and $|\tau_{>k}C| \rightarrow |C|$ isomorphism on π_n $n > k$, 0 on $n \leq k$, and we have the k -connective cover of $|C|$. $k = 0$ is the basepoint component and $k = 1$ is the universal cover.

Given a \mathbb{Z} -indexed chain complex C , $\tau_{\geq 0}C$ $0 \leftarrow Z_0 \leftarrow C_1 \leftarrow \dots$

$(\tau_{\geq 0}C)[-1]$ $0 \leftarrow Z_{-1} \leftarrow C_0 \leftarrow \dots$

$E_0 = |\tau_{\geq 0}C|$ is an (infinite loop) space.

$\Omega(\tau_{\geq -1}C)[-1] \simeq |\tau_{\geq 0}C|$

$E_n = |(\tau_{\geq -n}C)[-n]|$ is the infinite loop spectrum. This gives a fully faithful embedding of $\mathbf{Ch}(\mathbf{Ab}) \subset$ spaces/homotopy.

Definition 1.7 (Quasi-Isomorphism). *A morphism $C \xrightarrow{f} D$ of complexes is a quasi-isomorphism if $H_n(C) \xrightarrow{f_*} H_n(D)$ for all n .*

Aside: $\mathbf{Ch}^{(d)} \xrightarrow{H_n} \mathcal{A}$ is an additive functor $f_* + g_* = (f + g)_*$

$$\begin{array}{ccccccc}
C_{n-1} & \longleftarrow & C_n & \longleftarrow & \cdots & & \\
\downarrow & & \downarrow & & \downarrow & & \\
D_{n-1} & \longleftarrow & D_n & \longleftarrow & \cdots & & \\
& & & & & & \downarrow \\
& & & & & & d
\end{array}$$

So $f_* : H_n(C) \rightarrow H_n(D)$ is well-defined.

Definition 1.8 (Mapping Cone). *The mapping cone of $C \xrightarrow{d} D$ is $\text{Tot}(D \xleftarrow{f} C)$
 $\text{cone}_n = C_{n-1} \oplus D_n$.
 $\text{cone}(C)$ means $\text{cone}(C \xrightarrow{\text{id}} C)$*

Lemma 1.3. *$\text{cone}(C)$ is acyclic.*

Proof. $d(C_{n-1}, C_n) = (-dC_{n-1}, dC_n - C_{n-1}) = (0, 0)$ so $dC_{n-1} = 0$, so $(C_{n-1}$ is acyclic), $C_{n-1} = dC_n$
 $d(-C_n, 0) = (dC_n, C_n) = (C_{n-1}, C_n)$ a boundary, so $Z_n = B_n, H_n(\text{cone}) = 0$. \square

Remark: D is a subcomplex of $\text{cone}(f)$, and the quotient complex is $C[-1]$.
 $0 \rightarrow D \rightarrow \text{cone}(f) \rightarrow C[-1] \rightarrow 0$.

Lemma 1.4 (Snake Lemma). *Given the following commutative diagram:*

$$\begin{array}{ccccccc}
\cdot & \longrightarrow & \cdot & \longrightarrow & \cdot & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
& & f & & g & & h \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot
\end{array}$$

Then \exists rational map $\partial : \ker(h) \rightarrow \text{coker}(f)$ so that the sequence

$$\ker(f) \rightarrow \ker(g) \rightarrow \ker(h) \xrightarrow{\partial} \text{coker}(f) \rightarrow \text{coker}(g) \rightarrow \text{coker}(h)$$

Proof. Watch Jill Clayburgh in "It's My Turn" from 1980, by diagram chasing. \square

Theorem 1.5. *If $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is an exact sequence of complexes, then \exists a natural map $\partial : H_n(C) \rightarrow H_{n-1}(A)$ so that the following sequence is exact:*

$$\dots \xrightarrow{g_*} H_{n+1}(C) \xrightarrow{\partial} H_n(A) \xrightarrow{f_*} H_n(B) \xrightarrow{g_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{f_*} \dots$$

Application 1: $0 \rightarrow C \rightarrow \text{cone}(C) \rightarrow C[-1] \rightarrow 0$ is exact, $\text{cone}(C)$ acyclic gives us a long exact sequence by the theorem

$$H_{n+1}(\text{cone}) \rightarrow H_{n+1}(C[-1]) \simeq H_n(C) \rightarrow H_n(\text{cone})$$

which gives $H_{n+p}(C[-p]) \xrightarrow{\partial^p} H_n(C)$

Application 2: Any map $f : C \rightarrow D$ has f_* fitting into a long exact sequence

$$\dots \rightarrow H_{n+1}(\text{cone}(f)) \rightarrow H_n(C) \xrightarrow{f_*} H_n(D) \rightarrow H_n(\text{cone}(f)) \rightarrow \dots$$

by application of $0 \rightarrow D \rightarrow \text{cone}(f) \rightarrow C[-1] \rightarrow 0$ is exact.

And now we will prove the theorem.

Proof. Rows exact

$$\begin{array}{ccccccc}
 & Z_n(A) & \longrightarrow & Z_n(B) & \longrightarrow & Z_n(C) & \longrightarrow & 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & A_n & \longrightarrow & B_n & \longrightarrow & C_n & \longrightarrow & 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & A_{n-1} & \longrightarrow & B_{n-1} & \longrightarrow & C_{n-1} & \longrightarrow & 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & \frac{A_{n-1}}{dA_n} & \longrightarrow & \frac{B_{n-1}}{dB_n} & \longrightarrow & \frac{C_{n-1}}{dC_n} & \longrightarrow & 0 &
 \end{array}$$

$$\begin{array}{ccccccc}
 & H_n(A) & \longrightarrow & H_n(B) & \longrightarrow & H_n(C) & \longrightarrow & 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & \frac{A_{n-1}}{dA_n} & \longrightarrow & \frac{B_{n-1}}{dB_n} & \longrightarrow & \frac{C_{n-1}}{dC_n} & \longrightarrow & 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & Z_{n-1}(A) & \longrightarrow & Z_{n-1}(B) & \longrightarrow & Z_{n-1}(C) & \longrightarrow & 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & H_{n-1}(A) & \longrightarrow & H_{n-1}(B) & \longrightarrow & H_{n-1}(C) & \longrightarrow & 0 &
 \end{array}$$

And so, by the Snake Lemma sequences we get from the above, we are done. \square

Definition 1.9 (Split Complex). A complex C_* is split if \exists decomposition $C_n = B_N \oplus H_n \oplus B'_{n-1}$ so that $C_n \xrightarrow{d} C_{n-1}$ is projection to B'_{n-1} , isomorphism $B'_{n-1} \xrightarrow{d} B_{n-1}$ inclusion $B_{n-1} \subset C_{n-1}$.

$$\begin{array}{ccc}
 B'_{n+1} & & B'_n \\
 & \searrow \cong & \oplus \\
 & & H_n \\
 & & \oplus \\
 & & B_n \\
 & \searrow \cong & B_{n-1}
 \end{array}$$

So H_n is $H_n(C_*)$.

A sequence is split exact if it is split and acyclic ($H_n = 0$). These are “nice” complexes or null homotopic. This is equivalence to the existence of maps $s_n : C_n \rightarrow C_{n+1}$ such that $d = dsd$.

Definition 1.10. Given arbitrary maps $s_n : C_n \rightarrow D_{n+1}$ define $\eta : sd + ds : C_n \rightarrow D_n$.

$$\begin{array}{ccccccc}
 \dots & \xrightarrow{d} & C_n & \xrightarrow{d} & C_{n-1} & \xrightarrow{d} & \dots \\
 & & \searrow s & & \searrow s & & \\
 \dots & \xrightarrow{d} & D_{n+1} & \xrightarrow{d} & D_n & \xrightarrow{d} & \dots
 \end{array}$$

Lemma 1.6. η is a chain map.

Proof. $d\eta = dsd + dds = dsd$, $\eta d = sdd + dsd = dsd$ so $d\eta = \eta d$. \square

Lemma 1.7. $H(C) \rightarrow H(D)$.

$$\eta_* = 0$$

Proof. If $c \in C_n$ is a cycle, $\eta(c) = d(sc) + s(dc) = d(sc)$ boundary, so $[\eta(c)] = 0$ in $H(D)$. \square

Definition 1.11 (null and chain homotopic). A chain map $\eta : C \rightarrow D$ is null homotopic if $\exists s$ such that $\eta = sd + ds$.

$f, g : C \rightarrow D$ are chain homotopic if $\exists s$ $f = g + \eta$, $\eta = sd + ds$. Note, $f_* = g_* + \eta_* = g_*$

Exercise: Chain homotopy is an equivalence relation.

Consequence: We define the category \mathbf{K} of chain homotopy equivalence classes of maps. Objects same as \mathbf{Ch} but $\text{hom}(C, D) = \text{hom}_{\mathbf{Ch}}(C, D)/\text{chain homotopy}$. This is an additive category.

Definition 1.12. C and H are chain homotopy equivalent (c.h.e.) $\exists C \xrightarrow{f} D, D \xrightarrow{g} C$ so that $fg = 1_D$ and $gf = 1_C$.

2 Triangulated Category

Definition 2.1 (Triangle). Let \mathbf{D} be an additive category with an auto-functor $\mathbf{D} \rightarrow \mathbf{D} : D \mapsto D[-1]$ and inverse $D \mapsto D[+1]$

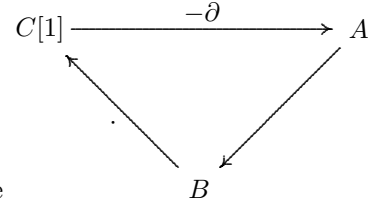
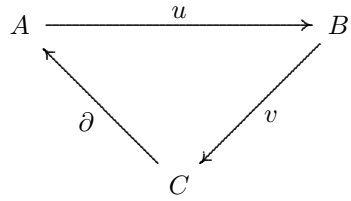
A triangle in \mathbf{D} is a sequence $A \rightarrow B \rightarrow C \xrightarrow{\partial} A[-1]$ in \mathbf{D} such that each composite is zero.

A morphism of triangles is a commutative diagram

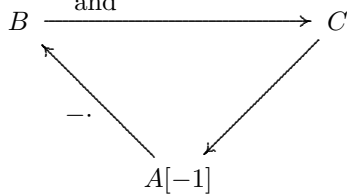
$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[-1] \\ \downarrow f & & \downarrow g & & \downarrow h & & \downarrow f[-1] \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & A'[-1] \end{array}$$

ξ = a distinguished collection of triangles called exact triangles should satisfy:
(note, when we say “triangle” we will often mean an exact triangle)

1. ξ is closed under \simeq, \oplus . $A \rightarrow A \rightarrow 0 \rightarrow A[-1]$ is always a triangle, and every morphism in \mathbf{D} fits into an exact triangle.



2. If $\begin{array}{ccc} A & \xrightarrow{u} & B \\ & \searrow \partial & \swarrow v \\ & & C \end{array}$ is a triangle, then so are



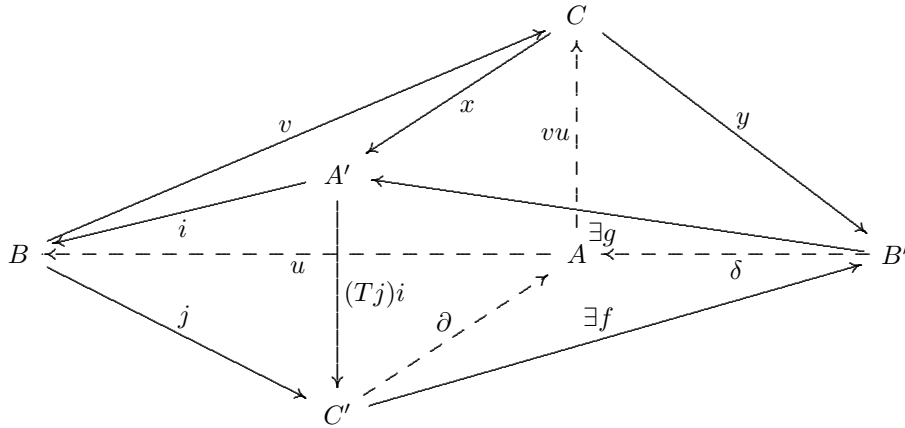
3. Fill-in axiom: given two triangles and maps f, g making the following commute

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[-1] \\ \downarrow f & & \downarrow g & & \vdots & & \downarrow f[-1] \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & A'[-1] \end{array}$$

then there exists a morphism (the dotted arrow) from C to C' .

Definition 2.2 (Triangulated Category). \mathbf{D} is a triangulated category if it has an ξ satisfying the above axioms and the octahedral axiom, which follows:

Given $A \xrightarrow{u} B \xrightarrow{v} C$, then



(see page 375 of Weibel)

Then, the maps exist so that the primed loop is an exact triangle and there are two ways of going from C' to C .

Lemma 2.1 (5-Lemma). *Given a morphism of triangles, if $A \rightarrow A'$ and $B \rightarrow B'$ are isomorphisms, then so is $C \rightarrow C'$.*

Corollary 2.2. *The completion C of $A \rightarrow B$ to a triangle is unique up to isomorphism (the isomorphism may not be unique).*

Example: Vector Spaces over a field k . $V = V[1]$. Exact means that the sequence is exact. Axiom 1 and 2 are satisfied by long exact sequences of vector spaces, and axiom 3 is satisfied because $A \rightarrow B \rightarrow \ker f \oplus \text{coker } f \rightarrow A$ gives natural isomorphisms.

Example: Graded Vector spaces by shifting the grading. Similar to Vector Spaces.

Theorem 2.3. $\mathbf{K} = \mathbf{K}(\mathbf{A})$ satisfied Triangle Axioms 1,2, and 3 when \mathbf{A} is an abelian category.

Note: \mathbf{K} is the category of chain complexes with morphisms being equivalence classes of morphisms modulo chain homotopy.

Proof. It has a shift functor.

It is exact: Isomorphic to $A \rightarrow B \rightarrow \text{cone}(f) \rightarrow A[-1]$.

For a morphism of complexes $A \xrightarrow{f} B$. Note: $\text{cone}(A \xrightarrow{1} A) \simeq 0$ so

$$\begin{array}{ccccccc}
 A & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & A[-1] \\
 \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
 A & \longrightarrow & A & \longrightarrow & \text{cone}(1) & \longrightarrow & A[-1]
 \end{array}$$

is an isomorphism of triangles. Thus, \mathbf{K} satisfies Axiom 1.

$$\begin{array}{ccccccc}
& & & \text{cone}(A)[-1] \simeq 0 & & & \\
& & & \uparrow & & & \\
\text{cone}(f)[1] & \xrightarrow{\delta[1]} & A & \longrightarrow & \text{cone}(\delta) & \longrightarrow & \text{cone}(f) \\
& & \uparrow \simeq & & \uparrow \simeq & & \uparrow \simeq \\
& & A & \xrightarrow{f} & B & \xrightarrow{v} & \text{cone}(f) \xrightarrow{\delta} A[-1] \longrightarrow B[-1] \\
& & \uparrow \simeq & & \uparrow \simeq & & \uparrow \simeq \\
& & B & \xrightarrow{v} & \text{cone}(f) & \longrightarrow & \text{cone}(v) \longrightarrow B[-1] \\
& & & & & & \uparrow \\
& & & & & & \text{cone}(B) \simeq 0
\end{array}$$

Which holds due to the following three diagrams:

$\text{cone}(f)$

$$\begin{array}{ccc}
B_n & \xrightarrow{+} & B_{n-1} \\
\oplus & \nearrow - & \oplus \\
A_{n-1} & \xrightarrow{-} & A_{n-2}
\end{array}$$

$\text{cone}(v)$

$$\begin{array}{ccc}
B_n & \xrightarrow{+} & B_{n-1} \\
\oplus & \nearrow - & \oplus \\
A_{n-1} & \xrightarrow{-} & A_{n-2} \\
\oplus & \nearrow - & \oplus \\
B_{n-1} & \xrightarrow{-} & B_{n-2} \\
& \text{cone}(\delta) &
\end{array}$$

$$\begin{array}{ccc}
A_{n-1} & \xrightarrow{+} & A_{n-2} \\
\oplus & \nearrow - & \oplus \\
B_{n-1} & \xrightarrow{-} & B_{n-2} \\
\oplus & \nearrow - & \oplus \\
A_{n-2} & \xrightarrow{-} & A_{n-3}
\end{array}$$

And so we have the second axiom. □

Lemma 2.4. *The mapping cone is a natural construction:*

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B & & cone(f) \\
 \downarrow i & & \downarrow j & & \downarrow \\
 A' & \xrightarrow{f'} & B' & & cone(f')
 \end{array}$$

Proof. We may assume the rows are mapping cones:

$$\begin{array}{ccccccc}
 A & \xrightarrow{f} & B & \longrightarrow & cone(f) & \longrightarrow & A[-1] \\
 \downarrow & & \downarrow & & \downarrow Nat & & \downarrow \\
 A' & \xrightarrow{f'} & B' & \longrightarrow & cone(f') & \longrightarrow & A'[-1]
 \end{array}$$

The left square only commutes up to chain homotopy. □

Lemma 2.5. *$f, g : A \rightarrow B$ are chain homotopic, $s_n : A_n \rightarrow B_{n+1}$ iff \exists map $(f, s, g) : cyl(A) \rightarrow B$ so that the inclusion maps fit into the following diagram*

$$\begin{array}{ccc}
 & & g \\
 & \searrow & \\
 A & & B \\
 & \nearrow & \\
 & & (f, s, g) \\
 & \nearrow & \\
 cyl(A) & \xrightarrow{\quad} & B \\
 & \searrow & \\
 & & f \\
 & \nearrow & \\
 A & & B
 \end{array}$$

Application: Given a commutative square

$$\begin{array}{ccccccc}
 A & \longrightarrow & B & \longrightarrow & \bullet & \longrightarrow & A[-1] \\
 \downarrow & & \downarrow & & \downarrow \text{---} & & \downarrow \\
 C & \longrightarrow & D & \longrightarrow & \bullet & \longrightarrow & C[-1] \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \bullet & \text{---} & \bullet & \longrightarrow & \exists & \longrightarrow & \bullet \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 A[-1] & \longrightarrow & B[-1] & \longrightarrow & \bullet & \longrightarrow & A[-2]
 \end{array}$$

Definition 2.3 (Homological Functor). *If \mathbf{D} is a triangulated category, a covariant functor $\mathbf{D} \xrightarrow{H} \mathbf{A}$ an abelian category is a homological functor if it converts exact triangles into long exact sequences.*

Convention, $H_0(A) \equiv H(A)$ and $H_n(A) \equiv H(A[n])$ so the long exact sequence is $H_n A \rightarrow H_n B \rightarrow H_n C \rightarrow H_{n-1} A \rightarrow \dots$

The prototype is the homology of chain complexes.

The octahedral axiom for $\mathbf{K}(\mathcal{A})$

Proposition 2.6. *Let \mathbf{D} be a triangulated category, and X an object. Then $\text{hom}_{\mathbf{D}}(X, -)$ is a homological functor to \mathbf{Ab} . That is, for all exact triangles, $A \rightarrow B \rightarrow C \xrightarrow{\partial} A[-1] \rightarrow B[-1] \rightarrow \dots$ goes to*

$$\dots \rightarrow \text{hom}(X, A) \rightarrow \text{hom}(X, B) \rightarrow \text{hom}(X, C) \rightarrow \dots$$

is an exact sequence of abelian groups.

Proof. Enough to show that it is exact at $\text{hom}(X, B)$, as rotations will give us the others.

Since $A \rightarrow C$ is zero, enough to show every $\beta : X \rightarrow B$ such that $X \rightarrow B \rightarrow C$ is zero, comes from $\alpha : X \rightarrow A$.

$$\begin{array}{ccccccccc} X[1] & \longrightarrow & 0 & \longrightarrow & X & \longrightarrow & X & \longrightarrow & 0 \\ \downarrow \beta[1] & & \downarrow & & \downarrow \alpha & & \downarrow \beta & & \downarrow \\ B[1] & \longrightarrow & C[1] & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \end{array}$$

□

3 Derived Functors

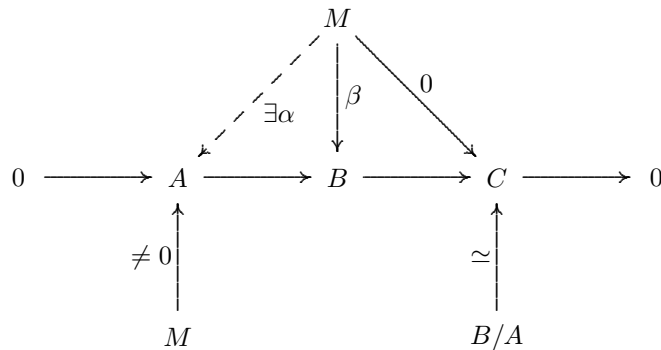
Projective Resolution as a functor $R\text{-mod} \rightarrow \mathbf{K}$. (Left R -modules)

Definition 3.1 (Projective Object). *In any abelian category, \mathcal{A} , an object P is projective if for every surjection $B \rightarrow C$, every map $P \rightarrow C$ lifts to a map $P \rightarrow B$.*

$$\begin{array}{ccc} & P & \\ & \swarrow \exists & \downarrow \forall \\ B & \longrightarrow & C \end{array}$$

$\text{hom}(P, -)$ is an exact functor, that is, it converts exact sequences to exact sequences.

Lemma 3.1. *$\text{hom}_{\mathcal{A}}(M, -)$ converts exact sequences in \mathcal{A} , $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$, to left exact sequences, $0 \rightarrow \text{hom}(M, A) \rightarrow \text{hom}(M, B) \rightarrow \text{hom}(M, C)$.*



Proof.

□

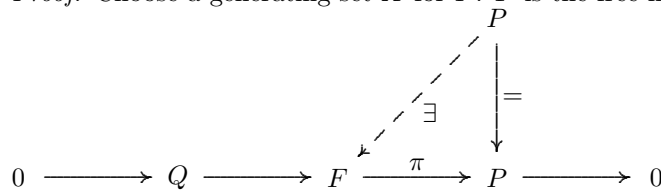
e.g. R^n has $\text{hom}(R^n, C) \simeq C \oplus C \oplus \dots \oplus C$. So the direct sum of n copies of a sequence, thus, R^n is projective.

P is the free R -module on a set X , then $\text{hom}_{R\text{-mod}}(P, C) = \text{hom}_{\text{sets}}(X, C)$ so that free modules are projective.

What about other projective modules?

Lemma 3.2. P is projective if and only if $\exists Q$ such that $P \oplus Q$ is a free module.

Proof. Choose a generating set X for P . F is the free module on X .



□

e.g. $R = M_3(F)$, $V = F^3$ as column vectors is a left R -module. $R \simeq V \oplus V \oplus V$, so V is projective.

e.g. $R \subset M_3(F)$ is a subring of upper triangular matrices. $R \simeq V_1 \oplus V_2 \oplus V_3$ where V_i is the i^{th} column. And there are homomorphisms between them, so $V_1 \subseteq V_2 \subseteq V_3$ and all are projective.

Definition 3.2 (Has Enough Projectives). \mathcal{A} has enough projectives if \forall objects M in \mathcal{A} , there is a projective P and a surjection $P \rightarrow M$.

Remark: $R\text{-mod}$ has enough projectives.

Definition 3.3 (Resolution). Given an exact sequence $\dots P_n \xrightarrow{d} P_{n-1} \rightarrow \dots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$, the chain complex P_* is called a resolution of M .

If all the P have the property \mathcal{P} , then we call it a \mathcal{P} -resolution.

Lemma 3.3. If \mathcal{A} has enough projectives, then every object M has a projective resolution.

Proof. We can construct a chain complex $P_n \rightarrow P_{n-1} \rightarrow \dots \rightarrow P_0 \rightarrow M \rightarrow 0$ by induction on M , the case $n = 0$ being the “enuf” hypothesis.

Inductively, let $Z_n = \ker(d_n)$, find $P_{n+1} \rightarrow Z_n$ onto, and then we get $P_{n+1} \rightarrow P_n \rightarrow P_{n-1} \rightarrow \dots$ \square

Proposition 3.4. *Any two projective resolutions of M are chain homotopy equivalent. (isomorphic in \mathbf{K}).*

This actually follows from:

Theorem 3.5 (Comparison Theorem). *Let P_* be a chain complex of projectives, $P_0 \xrightarrow{\epsilon} M$ a map sending $d(P_1)$ to 0, and $Q_* \rightarrow N$ a resolution.*

Then there exists a morphism of chain complexes $P_ \xrightarrow{f} Q_*$ over each $M \rightarrow N$*

$$\begin{array}{ccccccc} \dots & \xrightarrow{\partial} & P_0 & \xrightarrow{\epsilon} & M & \longrightarrow & 0 \\ & & \downarrow \exists & & \downarrow & & \\ \dots & \longrightarrow & Q_0 & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

and any two such maps are chain homotopy equivalent. $\text{hom}_{\mathbf{K}}(P, Q) = \text{hom}_{\mathcal{A}}(M, N)$.

Definition 3.4 (Homological δ -functor). *A homological δ -functor from an abelian category \mathcal{A} to an abelian category \mathcal{B} is a family of functors $H_n : \mathcal{A} \rightarrow \mathcal{B}$, $H_n = 0$ if $n < 0$, together with a natural $\partial : H_n(C) \rightarrow H_{n-1}(A)$ for every s.e.s. $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in \mathcal{A} converting s.e.s. to long exact sequences.*

$$\dots \rightarrow H_1(C) \xrightarrow{\partial} H_0(A) \rightarrow H_0(B) \rightarrow H_0(C) \rightarrow 0$$

e.g. $\mathbf{Ch}_{\geq 0} \rightarrow \mathcal{A}$ by homology.

Corollary 3.6. *Let F be any right exact functor. Then*

$$R\text{-mod} \xrightarrow{\text{resolution}} \mathbf{K}_{\geq 0}(R\text{-mod}) \xrightarrow{F} \mathbf{K}(\mathbf{B})_{\geq 0} \xrightarrow{H_*} \mathbf{B}$$

This will be the left derived functors of F .

Proof. Suppose by induction that (1) we have constructed

$$\begin{array}{ccccccc} P_{n+1} & \longrightarrow & \dots & \xrightarrow{\partial} & P_0 & \xrightarrow{\epsilon} & M \longrightarrow 0 \\ \downarrow a & \nearrow b & \uparrow & & \downarrow \exists & & \downarrow \\ Q_{n+1} & \longrightarrow & \dots & \longrightarrow & Q_0 & \longrightarrow & N \longrightarrow 0 \end{array}$$

and (2) any other such f'_n s has chain maps $P_i \xrightarrow{s_i} Q_{i+1}$, $i < n$ so that $f'_i - f_i = sd + ds$ for $i < n$.

Let $Z_n = \ker(Q_n \rightarrow Q_{n-1}) = \text{Im}(Q_{n+1} \rightarrow Q_n)$.

Proof. Take the cone sequence of $A \xrightarrow{f} B$, P_* a projective resolution of A and Q_* a projective resolution of B with $\varphi : P_* \rightarrow Q_*$.

$$\begin{array}{ccccccc}
0 & \longrightarrow & B & \longrightarrow & \text{cone}(f) & \longrightarrow & A[-1] \longrightarrow 0 \\
& & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & Q_* & \longrightarrow & \text{cone } \varphi & \longrightarrow & P_*[-1] \longrightarrow 0
\end{array}$$

So $\text{cone } \varphi$ is a resolution of $\text{cone}(f)$. By exactness, we get morphisms of resolutions, and so the homology of $\text{cone } \varphi \rightarrow \text{cone}(f)$ is always zero.

So we have a long exact sequence on H_* , and because $C \xrightarrow{\cong} \text{cone}(f)$ and $H_{n+1}(P_*) \xrightarrow{\cong} H_n(Q_*)$.

$$\text{So } L_n F(A) = H_n(P_*) \xrightarrow{f} L_n F(B) \quad \square$$

Definition 3.8 (Tor, Ext). *Let R be a commutative ring. Let $/I : R\text{-mod} \rightarrow R\text{-mod} : M \mapsto M/IM$. Then $\text{Tor}_n^R(M, R/I)$ is $L_n(/I)(M)$.*

If N is any R -module, $\text{Tor}_n^R(M, N)$ is $L_n(\otimes_R N)(M)$.

$\text{Ext}_R^n(M, N)$ is $L_n(\text{hom}(-, N))(M)$

Given an R -module M , choose projective $\rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$.

$\text{Tor}_n^R(M, R/I)$ is H_n of the complex $\dots \rightarrow P_1/IP_1 \rightarrow P_0/IP_0 \rightarrow 0$

$\text{Tor}_n^R(M, N)$ is H_n of the complex $P_1 \otimes N \rightarrow P_0 \otimes N \rightarrow 0$

Note: $P_1 \otimes N \rightarrow P_0 \otimes N \rightarrow M \otimes N \rightarrow 0$ is exact, so $\text{Tor}_0(M, N) = H_0(P_* \otimes N) \simeq M \otimes N$.

Given $\rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$, get chain complex $\leftarrow \text{hom}(P_1, N) \leftarrow \text{hom}(P_0, N) \leftarrow 0$, arrows reversed in $(R\text{-mod})^{op}$.

$\text{Ext}^n(M, N) = H^n$ of the cochain complex of modules $\text{hom}(P_*, N)$.

Example: $\mathbf{Ab}_{fg} \xrightarrow{F} \mathbf{Ab}_{fg} : M \mapsto M/pM$ (\mathbf{Ab}_{fg} is the category of finitely generated abelian groups).

$0 \rightarrow P_1 \rightarrow \mathbb{Z}^k \rightarrow M \rightarrow 0$. P_1 is finitely generated and must be torsion free as it is a subgroup of \mathbb{Z}^k , so it is isomorphic to \mathbb{Z}^ℓ .

Thus, every M has a projective resolution of length 1.

$0 \rightarrow F(P_1) = F(\mathbb{Z}^\ell) \rightarrow F(\mathbb{Z}^k) = F(\mathbb{Z})^k \rightarrow 0$, so $(L_n F)(M) = 0$ for $n \geq 2$ for all right exact F . $0 \rightarrow (M : p) \rightarrow \mathbb{Z}^\ell/p \rightarrow \mathbb{Z}^k/p \rightarrow M/p \rightarrow 0$ is exact, by

DIAGRAM THREE

So $\text{Tor}_1^{\mathbb{Z}}(M, \mathbb{Z}/p) \simeq (M : p) = \{m \in M : pm = 0\}$ for all M .

Homological functors: $\mathbf{K}(R\text{-mod}) \xrightarrow{H_*} R\text{-mod}$, additive $\Delta \rightarrow l.e.s.$

Spectra $\xrightarrow{\pi_*} \mathbf{Ab}$ homotopy

$R\text{-mod} \xrightarrow{\text{projres/replacement}} \mathbf{K}(R\text{-mod}) \xrightarrow{\otimes_R N} \mathbf{K}(R\text{-mod}) \xrightarrow{H_*} R\text{-mod}$.

The functor $M \mapsto P_* \otimes N$ is the total tensor product $M \otimes_R^L N$.

Theorem 3.8. *If $F : \mathcal{A} \rightarrow \mathcal{B}$ is right exact, the δ -functors $L_* F$ are universal.*

Proof. This means that given another δ -functor S_* with $S_0 \xrightarrow{\eta_0} F$ there is a unique morphism of δ -functors over η_0 .

Construct $S_n \xrightarrow{\eta_n} L_n F$ by induction on n . Given M in \mathcal{A} , choose $0 \rightarrow Z \rightarrow P \rightarrow M \rightarrow 0$, P projective,

DIAGRAM ONE

Compute $(L_* F)P$ using trivial projective resolution $(0 \rightarrow P) \rightarrow P \rightarrow 0$,

We will use the lemma: If P is projective, $(L_n F)(P) = 0$ for $n \neq 0$.

Thus $L_1 F(P) = 0$, and so the above diagram can be filled in.

We need to check that η_1 is a natural transformation.

DIAGRAM TWO

Also have to see, given DIAGRAM THREE that DIAGRAM FOUR commutes. See page 48 of Weibel. \square

Dimension Shifting

Given a short exact sequence $0 \rightarrow Z \rightarrow P \rightarrow M \rightarrow 0$ with P projective, $L_{n+1} F(M) \xrightarrow{\delta} L_n F(Z)$ for all $n \geq 1$ is an isomorphism.

The point: Now we have DIGRAM FIVE

Aside: Presheaves on a category \mathcal{T} like $\text{Top}(X) = \text{open subsets of } X$ and inclusions.

These are contravariant functors from $\mathcal{T} \rightarrow \mathcal{A}$. The category $\mathcal{A}^{\mathcal{T}^{op}} = \mathbf{PSh}$ is an abelian category. $0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0$ is exact iff $(\forall T \in \mathcal{T}) 0 \leftarrow F'(T) \leftarrow F(T) \leftarrow F''(T) \leftarrow 0$ is exact in \mathbf{Ab} .

There is a Yoneda Functor $h : \mathcal{T} \rightarrow \mathbf{PSh} : X \mapsto \text{hom}_{\mathcal{T}}(-, X)$.

$\text{hom}_{\mathbf{PSh}}(h_X, F) \simeq F(X)$. Given $f : h_X \rightarrow F$, $f_X : h_X(X) \rightarrow F(X)$, the identity is in $h_X(X)$.

Given $\gamma \in F(X)$, construct $h_X \xrightarrow{\eta} F$ by $\eta_T : h_X(T) \rightarrow F(T)$, $f \in h_X(T) = \text{hom}(T, X)$, REST OF ASIDE FROM JOE!

Given $g : T' \rightarrow T$, check that DIRAGRAM SIX commutes

Corollary 3.9. $\text{hom}_{\mathbf{PSh}}(h_X, h_Y) = h_Y(X) = \text{hom}_{\mathcal{T}}(X, Y)$.

That is, the Yoneda Embedding is full and faithful.

Corollary 3.10. *Each h_X is a projective object in \mathbf{PSh}*

DIAGRAM SEVEN

Corollary 3.11. *If \mathcal{T} is small, then \mathbf{PSh} has enough projectives.*

Proof.

$$\bigoplus_{T \in \mathcal{T}} \bigoplus_{x \in F(T)} h_T \rightarrow F$$

Thus we have $\text{hom}(\bigoplus h_T, F) = \prod_{T,x} \text{hom}(h_T, F) = F(T)$, so $\pi \leftrightarrow \{\gamma\}$.

Now, given $X \in \mathcal{T}$, $\zeta \in F(X)$ the component corresponding to the pair (X, ζ) is the Yoneda map $h_X(X) \rightarrow F(X) : 1 \mapsto \zeta$, so it is onto. \square

Lemma 3.12. *If P_i are projective and $\bigoplus P_i$ exists in \mathcal{A} then $\bigoplus P_i$ is projective.*

Proof. DIAGRAM EIGHT \square

Remark: Total Tensor Product of Sheaves (of abelian groups) is the sheafification of $M \otimes \coprod N$.

End Aside

Definition 3.9 (Adjoint Functors). *Functors $F : \mathcal{A} \rightarrow \mathcal{B}$, $G : \mathcal{A} \leftarrow \mathcal{B}$ are adjoint (F the left adjoint, G the right adjoint) if there is a natural isomorphism $\tau_{A,B} : \text{hom}_{\mathcal{B}}(F(A), B) \xrightarrow{\cong} \text{hom}_{\mathcal{A}}(A, G(B))$*

Example: $R\text{-mod} \xrightarrow{\text{forget}} \text{Set}$ and $R\text{-mod} \xleftarrow{\text{free}} \text{Set}$, so $\text{hom}_R(\text{free on } X, M) \simeq \text{hom}_{\text{sets}}(X, M)$

Theorem 3.13. *Let $L : \mathcal{A} \rightarrow \mathcal{B}$ be left adjoint to $R : \mathcal{B} \rightarrow \mathcal{A}$, both additive functors.*

1. *If R is exact, then $L(\text{projective})$ is projective in \mathcal{B} .*
2. *$L(\oplus A_i) \simeq \oplus L(A_i)$ and $L(\lim_{\rightarrow} A_i) \simeq \lim_{\rightarrow} L(A_i)$*
3. *L is always a right exact functor, R is always left exact*

Remark: $\text{Sheaf} \xrightarrow{\text{forget}} \mathbf{PSh}$ has adjoint $\text{Sheaves} \xleftarrow{\text{sheafify}} \mathbf{PSh}$, and sheafify is exact, so forgetful is left exact.

If $0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0$ is exact sequence of sheaves, then $(\forall X), 0 \rightarrow F'(X) \rightarrow F(X) \rightarrow F''(X)$ is exact. $F(X) = H^0(X, F)$ the (right derived functors are $H^n(X, F)$).

Proposition 3.14. *Let M be a left R -module and right S -module* **DIAGRAMS NINE**

Proposition 3.15. $\otimes_R M$ is left adjoint to $\text{hom}_S(M, -)$, that is, $\text{hom}_S(L \otimes M, N) \simeq \text{hom}_R(L, \text{hom}(M, N))$

Proof. $f : L \otimes M \rightarrow N$ goes to $\ell \mapsto (m \mapsto f(\ell \otimes m))$
 $\ell \otimes \mapsto \varphi(\ell)m \leftarrow \varphi$, for $\varphi(\ell) : M \rightarrow N$. □

Corollary 3.16. $\otimes_R M$ is always right exact and $\text{hom}(M, -)$ is always left exact.

$$(\oplus L_i) \otimes M \simeq \oplus (L_i \otimes M) \text{ and } (\lim_{\rightarrow} L_i) \otimes M \simeq \lim_{\rightarrow} (L_i \otimes M).$$

We will now prove the theorem.

Proof. (1) **DIAGRAM TEN**

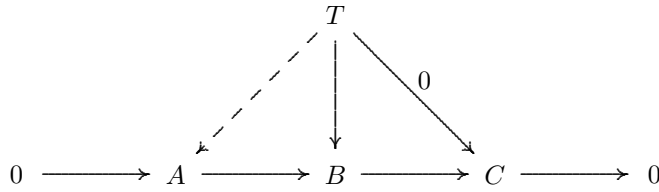
(2)

(3) **DIAGRAM ELEVEN** □

Yoneda Embedding: $h : \mathcal{T} \rightarrow \mathbf{PSh}(\mathcal{T})$ of abelian groups.

Lemma 3.17. *If \mathcal{T} is abelian, then h lands in additive presheaves.*

Lemma 3.18. *h is an exact functor. If \mathcal{T} is abelian, $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ exact in \mathcal{T} then $h_A \rightarrow h_B \rightarrow h_C$ exactly.*



Proof.

□

Lemma 3.19. *If \mathcal{T} is abelian and $A \rightarrow B \rightarrow C$ is in \mathcal{T} and $h_A \rightarrow h_B \rightarrow h_C$, then $A \rightarrow B \rightarrow C$ is exact.*

Corollary 3.20. *L is right exact (R is left exact). $\mathcal{A} \rightleftharpoons \mathcal{B}$ when L, R are additive and adjoint.*

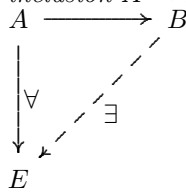
Proof. It is enough to show that $\forall B, \text{hom}(LA', B) \leftarrow \text{hom}(LA, B) \leftarrow \text{hom}(LA''B) \leftarrow 0$ and each is equal to $\text{hom}(A', RB)$ or the others, respectively. So $\text{hom}(-, RB)$ is left exact. □

Duality Remarks:

1. Dual of a category \mathcal{C} has the same objects and $\text{hom}_{\mathcal{C}^{op}}(A, B) = \text{hom}_{\mathcal{C}}(B, A)$
2. The axioms for an abelian category are self-dual, so \mathcal{A} is abelian iff \mathcal{A}^{op} is abelian.
3. $F : \mathcal{A} \rightarrow \mathcal{B}$ is right exact is the same as $F : \mathcal{A}^{op} \rightarrow \mathcal{B}^{op}$ is left exact.
4. So we can repeat the whole discussion for right exact F and L_*F to get left exact F and right derived functor R^*F .
5. If T_* is a homological δ -functor, then reversing arrows gives us T^* , a cohomological δ -functor.

So what is the dual of projective?

Definition 3.10 (Injective Object). *An object E of \mathcal{A} is injective if for every inclusion $A \hookrightarrow B$, every map $A \rightarrow E$ extends to a map $B \rightarrow E$.*

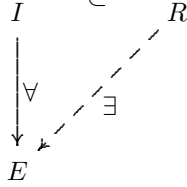


Problem: In \mathbf{Ab} and $R\text{-mod}$, we have free objects which are projectives. What are injective modules? (projective objects of $(R\text{-mod})^{op}, \mathbf{Ab}^{op}$)

We will see that $R\text{-mod}$ and \mathbf{Ab} and \mathbf{PSh} and Sheaves have enough injectives.

Remark: In adjoint situation, $\mathcal{A} \rightleftharpoons \mathcal{B}$, then if R is exact, L preserves projectives, so if L exact, then R preserves injectives.

Theorem 3.21 (Baer's Criterion 1942). *A left R -module E is injective if and only if $(\forall$ left ideals I of R) every map $I \rightarrow E$ extends to a map $R \rightarrow E$ of left R -modules.*



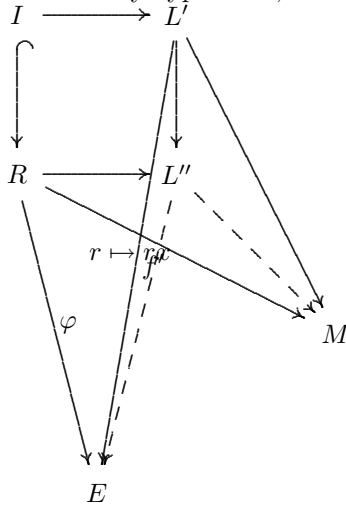
Proof. \Rightarrow is trivial.

Suppose given $L \subset M$ and $f_0 : L \rightarrow E$. Consider the set of all pairs $(L', f' : L' \rightarrow E)$ such that $L \subseteq L' \subseteq M$ and $f'|_L = f_0$. Then \mathcal{L} is partially ordered $(L', f') \leq (L'', f'')$ if $L' \subseteq L''$ and $f''|_{L'} = f'$ and $\mathcal{L} \neq \emptyset$ as $(L, f_0) \in \mathcal{L}$.

Given an ascending chain (L_α, f_α) , $(\cup L_\alpha, \cup_\alpha f_\alpha : \cup L_\alpha \rightarrow E)$ is an upper bound, so Zorn's Lemma says there is a maximal element (L', f') in \mathcal{L} . If $L' = M$ done. Assume $L' \neq M$. Then pick $x \in M, x \notin L'$.

Case 1: $Rx \cap L' = 0$. Then $L' \subsetneq L' \oplus Rx \subseteq M$ with $f' : L' \rightarrow E$ and $(f', 0) : L' \oplus Rx \rightarrow E$ commuting with the inclusion map, which gives a contradiction, as we are extending a maximal element.

Case 2: $Rx \cap L' = Ix$ for some $I \neq 0$ in R . Define $I \rightarrow E : r \mapsto f'(rx)$ as $rx \in L'$. By hypothesis, this extends to $R \xrightarrow{\varphi} E$.



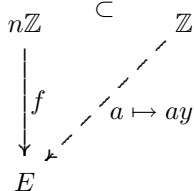
So $L'' = L' \oplus Rx$. We claim $L'' \subseteq M$. $L'' = Rx + L'$. But then $(L'', L'' \rightarrow E) \in \mathcal{L}$, contradiction. \square

Exercise: If R is a commutative domain, show that its field of fractions F is an injective R -module.

Definition 3.11 (Divisible). *M is a divisible R -module if $(\forall x \in M)(\forall r \in R)(\exists y \in M)x = ry$.*

Proposition 3.22. *An Abelian group E is injective (in \mathbf{Ab}) if and only if E is divisible.*

Proof. \leftarrow by Baer:



Set $x = f(n)$. Since E is divisible, choose y . If $a = bn$, $ay = b(ny) = bx = bf(n) = f(bn)$ \square

Poorism: Any $k[x]$ -module is divisible if and only if it is injective. This is true for any PID R .

Definition 3.12 (Pontrjagin Dual). *The Pontrjagin Dual A^* of an abelian group A is the abelian group $\text{hom}_{\mathbf{Ab}}(A, \mathbb{Q}/\mathbb{Z})$.*

Since \mathbb{Q}/\mathbb{Z} is injective, $\text{hom}(-, \mathbb{Q}/\mathbb{Z})$ is an exact contravariant functor from \mathbf{Ab} to \mathbf{Ab} .

Example: $(\mathbb{Z}/n)^* \simeq \mathbb{Z}/n$, and $\mathbb{Z}^* \simeq \mathbb{Q}/\mathbb{Z}$.

Also, $(A \oplus B)^* \simeq A^* \oplus B^*$. And if A is any finite abelian group, then $A \simeq A^*$.

Proposition 3.23. *The category \mathbf{Ab}_{fin} of finite abelian groups is equivalent to $(\mathbf{Ab}_{fin})^{op}$ by Pontrjagin Duality.*

$$(\lim_{\rightarrow} A_i)^* \simeq \lim_{\leftarrow} (A_i^*).$$

Facts:

1. In \mathbf{Ab} ever injective abelian group is a direct sum of \mathbb{Q} and the Sylow subgroups \mathbb{Z}_p^∞ of \mathbb{Q}/\mathbb{Z} , so \mathbf{Ab}_{fin} has no injectives.
2. Every abelian A embeds in an injective.
3. The product of injectives is injective.
4. If E is injective, so is every quotient of E . Thus $0 \rightarrow A \rightarrow E \rightarrow E/A \rightarrow 0$ is an injective resolution.

Given left exact $F : \mathbf{Ab} \rightarrow B$, then $0 \rightarrow A \rightarrow E^0 \rightarrow E^1 \rightarrow 0$ becomes $0 \rightarrow F(E^0) \rightarrow F(E^1) \rightarrow 0$. So the right derived functors $(R^0F)(A) = F(A)$, $(R^1F)(A) = \text{coker}(F(E^0) \rightarrow F(E^1))$ and $(R^nF)(A) = 0$ for $n \neq 0, 1$. Formula: $0 \rightarrow A \rightarrow E^*$ and injective resolution, $(R^nF)(A) = H^n[F(E^*)]$.

Lemma 3.24. *If F is a ring, $R^* = \text{hom}_{\mathbf{Ab}}(R, \mathbb{Q}/\mathbb{Z})$ is an injective left R -module.*

Proof. If $f \in R^*$, $r \in R$ define $rf \in \text{hom}(R, \mathbb{Q}/\mathbb{Z})$ $(rf)(x) = f(xr)$, $(r_1 + r_2)f = (r_1f) + (r_2f)$, $((rs)f)(x) = f(x(rs)) = (sf)(xr) = r(sf)(x)$. So R^* is a left R -module.

$\text{hom}_R(M, \text{hom}_{\mathbf{Ab}}(R, \mathbb{Q}/\mathbb{Z})) \simeq \text{hom}_{\mathbb{Z}}(M \otimes_R R, \mathbb{Q}/\mathbb{Z}) \simeq \text{hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) = M^*$.

So we end up with $\text{hom}(-, R^*)$ is exact, and so R^* is injective. \square

In fact, we have that if M is any right R -module then M^* is a left R -module.

Corollary 3.25. *Every R -module M embeds in an injective R -module, namely $E = \prod_{f \in M^*} R^*$ via the “ f ” coordinate of image of e is the element of $R^* = \text{hom}_{\mathbf{Ab}}(R, \mathbb{Q}/\mathbb{Z})$ sending r to $f(re)$.*

Corollary 3.26. *R -mod has enough injectives, and so right derived functors exists.*

Example: $\text{hom}_R(M, -) : R\text{-mod} \rightarrow \mathbf{Ab}$ is covariant and left exact.

Claim: $R^n(\text{hom}_R(M, -))(N) \simeq \text{Ext}_R^n(M, N) \simeq L_n(\text{hom}_R(-, N))(M)$.

Example, a Tower M_* of R -module is a diagram of modules $\rightarrow M_{k+1} \rightarrow M_k \rightarrow \dots \rightarrow M_0$. A contravariant functor from $I = (\dots \rightarrow k+1 \rightarrow k \rightarrow \dots \rightarrow 0)$ to $R\text{-mod}$ is contravariant from $0 \rightarrow 1 \rightarrow 2 \rightarrow \dots$

Definition 3.13 (k_*E). k_*E is $\dots = E = E \rightarrow 0 \rightarrow \dots \rightarrow 0 \rightarrow 0$ where the last E is in the k^{th} position.

Lemma 3.27. $\text{hom}_{\text{Tower}}(M_*, k_*E) \simeq \text{hom}_R(M_k, E)$.

Corollary 3.28. *If E is an injective tower R -module, k_*E is an injective tower. Hence, **Towers** has enough injectives.*

Definition 3.14. $\lim_{\leftarrow}^n(M_*) = (R^n \lim_{\leftarrow})(M_*)$ where $\lim_{\leftarrow} : \text{Tower} \rightarrow R\text{-mod}$ is inject limit.

Remark: $\lim_{\leftarrow} M_* = \{(\dots, m_k, \dots, m_1, m_0) \in \prod M_k : M_{k+1} \rightarrow M_k \text{ sends } m_{k+1} \rightarrow m_k\}$

Remark: \varprojlim^1 was discovered in 1961 by Milnor in studying $H^*(X)$, X infinite dimensional CW-complex (like $X = BG$).

Lemma 3.29 (Acyclic Assembly Lemma). *Finite Version:*

Take a first quadrant double complex with exact columns. Then the total complex is exact. DIAGRAM ONE

Infinite Version requires AB_4^ , but modules are ok.*

Take an upper half plane double complex with exact columns, then $\text{Tot}\Pi(C)$ is exact.

AB_3^* is just that \mathcal{A} is complete, in that For every set $\{A_i\}_{i \in I}$ of objects, then $\prod_{i \in I} A_i$ exists in \mathcal{A} . Note that $\text{hom}(B, \prod A) = \prod \text{hom}(B, A_i)$ so $\prod : \mathcal{A}^I \rightarrow \mathcal{A}$ is right adjoint to the diagonal $\mathcal{A} \rightarrow \mathcal{A}^I : A \mapsto \text{all } A_i = A$.

So \prod is a left exact functor.

AB_4^* holds if and only if AB_3^* holds and \prod is exactly for all index sets. (this is not true in Sheaves).

Proof. Pick and n -cycle $(C_{pq})_{\substack{p+q=n \\ q \geq 0}}$. Cycle: $d^v(C_{n0}) = 0$, $0 = d^v(C_{pq}) + d^h(C_{p+1,q-1})$ for $q > 0$.

We will prove by induction on q that there exists a $\beta_q = (\dots, 0, b_q, b_{q-1}, \dots, b_0)$ so that $d(\beta_q) = (\dots, 0, d^h b_q, C_{p,q-1}, \dots, C_{n,0})$.

We consider $C' = C - d\beta_q = (\dots, C_{q+1}, C'_q, 0, \dots, 0)$.

Now $dC' = dC = dd\beta = 0 - 0 = 0$, so $d^v(C'_q) = 0$ and it's column is exact, so $\exists b_{q+1}$ so $d^v(b_{q+1}) = C'$, so we set $\beta_{q+1} = (0, \dots, 0, b_{q+1}, \dots, b_1, 0)$.

Note that $\beta = (\dots, b_q, \dots, b_1, 0) \in \prod C_{pq}$ has $d\beta = C$. \square

$$\begin{array}{ccccc}
 & & \longleftarrow C_{n,0} & \longleftarrow C_{n+1,0} & \\
 & \downarrow & \downarrow & \downarrow & \\
 & & & &
 \end{array}$$

Corollary 3.30. *If we have the augmentation $A_{n-1} \longleftarrow A_n \longleftarrow A_{n+1}$ so that C is an upper half plane double complex and the augmented columns are exact, then $\text{Tot}(C) \xrightarrow{f} A$ is a quasi-isomorphism.*

Application: We take the upper half plane complex where all objects are $\mathbb{Z}/4$ and the maps are all multiplication by 2. Then we augment by adding in $A_n = \mathbb{Z}/2$ and so $H_n(\text{Tot } C) \simeq H_n(A_*)$

But A_* is $\mathbb{Z}/2 \xleftarrow{0} \mathbb{Z}/2 \xleftarrow{0} \mathbb{Z}/2$

Application: Choose projective resolution $M \leftarrow P_*, N \leftarrow Q_*$ form double complex $P \otimes Q$ in the right quadrant. We augment by adding a row of $P_n \otimes N$. Since $P_p \otimes R -$ is exact, so each column is $P_p \otimes (0 \leftarrow N \leftarrow Q_0 \leftarrow \dots)$ is exact, and so is each row $(0 \leftarrow M \leftarrow P_0 \leftarrow \dots) \otimes Q_q$. Hence $H_*(M \otimes Q_*) \simeq H_*(\text{Tot}(P \otimes Q)) \simeq H_*(N \otimes P_*)$

Recall, though, that $H_*(M \otimes Q_*)$ are the right derived functors of $M \otimes_R -$ applied to N and that $H_*(P_* \otimes N)$ are the left derived functors of $- \otimes_R N$ applied to M , that is, $\text{Tor}_*^R(M, N)$.

So Tor is balanced.

Pick an injective resolution $0 \rightarrow N \rightarrow E^0 \rightarrow E^1 \rightarrow \dots$. Form $\text{hom}_R(P_*, E^*)$.

DIAGRAM TWO

Hence $\text{Ext}_R^*(M, N)$ is $H^*(\text{hom}(M, E^*)) \simeq H^*(\text{hom}(P, E)) \simeq H^*(\text{hom}(P_*, N))$. The first are the right derived and the last the left derived functors of $\text{hom}(-, N)$. So Ext is Balanced.

Recall that Towers is the category of diagrams $M_* : \dots \rightarrow M_p \rightarrow \dots \rightarrow M_1 \rightarrow M_0$. $\text{hom}(M_*, N_*)$ are compatible families of $f_p : M_p \rightarrow N_p$.

$\Delta : \prod M_p \rightarrow \prod M_p$ with $\Delta(\dots, m_p, \dots, m_0) = (\dots, m_p - \bar{m}_{p+1}, \dots, m_0 - \bar{m}_1)$.

Definition 3.15. $\varprojlim^1(M) = \text{coker } \Delta$, $\varprojlim^n(M) = 0$ for $n \neq 0, 1$.

Lemma 3.31. *If all $M_{p+1} \rightarrow M_p$ are onto, then $\varprojlim^1(M) = 0$.*

Proof. Given $(\dots, x_p, \dots, x_1, x_0) \in \prod_{p=0}^{\infty} M_p$ we construct $(m_p) \in \prod M_p$ so $\Delta(m_p) = (x_p) \Rightarrow \Delta$ onto $\Rightarrow \varprojlim^1 = 0$.

Take $x_1 \in M_1$ mapping onto $-m_0$ and $(\dots, x_2, x_1, 0) \xrightarrow{\Delta} (\dots, m_0)$ with x_2, \dots not yet constructed.

Condition on x_2 is that $x_1 - \bar{x}_2 = m_1$, so we set $x_2 \in M_2$ a lift of $x_1 - m_1$. Continue inductively such that x_{n+1} is a lift of $x_n - m_n$. \square

Definition 3.16 (Trivial M-L Condition). M_* satisfies the trivial M-L condition if $\forall n, \exists N$ such that $M_N \rightarrow M_n$ is 0.

Note that then $\varprojlim(M_*) = 0$.

Lemma 3.32. If M_* has trivial M-L condition then $\varprojlim^1(M) = 0$.

Proof. Given $x = (x_n) \in \prod M_n$, want (m) so that $\Delta(m_n) = (x_n)$.

Set $m_n = \sum_{k=n}^{\infty} x_k = x_n + \bar{x}_{n+1} + \dots + \bar{x}_N + \dots$ in M_n . By hypothesis, this sum is actually finite.

$$m_n - \bar{m}_{n+1} = (\sum_{k=n}^{\infty} \bar{x}_k) - (\sum_{k=n+1}^{\infty} \bar{x}_k) = x_n. \quad \square$$

M-L stands for Mittag-Leffler (Stockhold 1880).

Corollary 3.33. $\varprojlim^1(\dots = E = E \rightarrow 0 \rightarrow 0 \rightarrow \dots \rightarrow 0) = 0$ where they're all injective.

Theorem 3.34 (Milnor). \varprojlim^n are the right derived functors of \varprojlim .

Proof. Step 1: Given a short exact sequence of towers

$$\begin{array}{ccccccc} 0 & \longrightarrow & M'_n & \longrightarrow & M_n & \longrightarrow & M''_n \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & M'_1 & \longrightarrow & M_1 & \longrightarrow & M''_1 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & M'_0 & \longrightarrow & M_0 & \longrightarrow & M''_0 \longrightarrow 0 \end{array}$$

DIAGRAM THREE NOTE: ADD SNAKE COMMAND!!!

Step 2: Hence, $\{\varprojlim^n\}$ are a cohomological δ -functor. But \varprojlim^n (enough injectives) = 0 so it is universal. \square

For a Tower $M_k \rightarrow \dots \rightarrow M_1 \rightarrow M_0 \rightarrow 0$. For each k , you get a chain of subobjects $M_k \supseteq \text{Im}(M_{k+1} \rightarrow M_k) \supseteq \text{Im}(M_{k+2} \rightarrow M_k) \supseteq \dots L_k = \bigcap_n \text{Im}(M_n)$.

We say that M_* has teh ML Consdition if $(\forall k)(\exists n)L_k = \text{Im}(M_n \rightarrow M_k)$.

Examples: If all $M_{k+1} \rightarrow M_k$ are onto, then $L1 : \varprojlim^1 M = 0$.

If $\forall k \exists n M_n \rightarrow M_k$ is zero, then $L2 : \varprojlim^1 M = 0$ the trivial ML condition.

In **Ab**, all M_k are finite abelian groups, DCC. In R -mod, all M_k have the DCC. In vector spaces, when $\dim M_k < \infty$.

Theorem 3.35. *If M has ML condition, then $\varprojlim^1 M = 0$.*

Proof. $0 \rightarrow \{L_k\} \rightarrow \{M_k\} \rightarrow \{M_k/L_k\} \rightarrow 0$ is a short exact sequence of towers and $L_{k+1} \rightarrow L_k$ onto. Then $L_{k+1} = \text{Im}(M_n \rightarrow M_{k+1})$ and $L_k = \text{Im}(M_n \rightarrow M_k)$ for all $N \gg 0$. Also, $\{M_k/L_k\}$ is trivial ML. $M_n/L_n \rightarrow M_k/L_k$ is zero if $\text{Im}(M_n) = L_k$.

Look at $0 \rightarrow \varprojlim L_k \rightarrow \varprojlim M_k \rightarrow \varprojlim M_k/L_k \rightarrow \varprojlim^1 L_k \rightarrow \varprojlim^1 M_k \rightarrow \varprojlim^1 M_k/L_k \rightarrow 0$. But $\varprojlim^1 L_k = \varprojlim^1 M_k/L_k = 0$, so $\varprojlim^1 M_k = 0$. \square

Remark: $\varprojlim^1 \{M_k\}$ is either zero or uncountable.

Theorem 3.36 (Eilenberg-Moore). *Suppose given a tower of chain complexes (=a chain complex of towers), set $C_k = \varprojlim_k C_n(k)$ so $C_* = \varprojlim C_*(k)$ is a natural exact sequence. Then there is a natural exact sequence $0 \rightarrow \varprojlim^1 H_{q+1}(C_*(k)) \rightarrow H_q(C_*) \rightarrow \varprojlim_k H_q(C_*(k)) \rightarrow 0$.*

Typical application: Given a second quadrant chain complex, $\beta_n C =$ brutal truncation. Then $\text{Tot}(\beta_n C)_N = \bigoplus_{p+q=n, p \geq n} C_{pq}$. Thus, $\text{Tot}(C)_N = \prod_{p+q=n} C_{pq} = \varprojlim \text{Tot}(\beta_n C)$.

$0 \rightarrow \varprojlim^1 H_{q+1} \text{Tot}(\beta_k C) \rightarrow H_q \text{Tot}(C) \rightarrow \varprojlim H_q(\beta_k C) \rightarrow 0$.

Recall for $\mathbb{Z} \xrightarrow{p} \mathbb{Z}$ we have $0 \rightarrow \varprojlim = 0 \rightarrow \prod_n \mathbb{Z} \xrightarrow{\Delta} \prod_n \mathbb{Z} \rightarrow \varprojlim^1 \{p^k \mathbb{Z}\} = \hat{\mathbb{Z}}/\mathbb{Z} \rightarrow 0$

LONG CONFUSING EXAMPLE.

4 Homology of Groups

Let G be a group.

Definition 4.1 (G -module). *A (left) G -module A is an abelian group with (left) action of G by homomorphisms $g(a+b) = g \cdot a + g \cdot b$.*

Remark: = left module over the group ring $\mathbb{Z}[G] : (\sum n_g g)a = \sum n_g (g \cdot a)$. So G -modules are the same thing as modules over $\mathbb{Z}[G]$.

Definition 4.2 (Universal Trivial G -module). *$A_G =$ coinvariants = quotient of A by $\{g \cdot a = a : \forall g \in G, a \in A\}$. This is the universal trivial G -module quotient of A . (Trivial means it acts trivially)*

$A^G = \{a \in A : ga = a \forall g \in G\}$ is the universal trivial G -submodule of A .

Lemma 4.1. *These are adjoint $\text{hom}_G(A, T) \simeq \text{hom}_{\mathbf{Ab}}(A_G, T)$, $\text{hom}_G(T, A) \simeq \text{hom}_{\mathbf{Ab}}(T, A^G)$, if T is a trivial G -module.*

Corollary 4.2. *A_G is right exact, A^G is left exact.*

Definition 4.3 ($H_n(G, A)$). *$H_n(G, A)$ is the n^{th} left derived functor applied to A . $H^n(G, A)$ is the n^{th} right derived functor applied to A .*

$$H_0(G, A) = A_G \text{ and } H^0(G, A) = A^G.$$

Topological Application: Let X be a topological space (in fact, a CW-Complex) with $\pi_1(X) = G$ and $\pi_n(X) = 0$ for $n \neq 1$. Usually write $X = BG$ or $X = K(G, 1)$. Referred to as Eilenberg-MacLane spaces.

Let \tilde{X} be the universal covering space. (exercise: \tilde{X} is contractible, but has a free G -action).

The singular complex $S_*(\tilde{X})$ is a complex of free $\mathbb{Z}G$ -modules, $(S_n \tilde{X})_G = S_n(X)$.

$H_*(\tilde{X}) = H_*(S_* \tilde{X}) = \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{else} \end{cases}$ so $\dots \rightarrow S_q(\tilde{X}) \rightarrow \dots \rightarrow S_0(\tilde{X}) \rightarrow \mathbb{Z} \rightarrow 0$ is exact. It is, in fact, a projective resolution in G -mod. Then $H_n(G, \mathbb{Z}) = H_n[(S_* \tilde{X})_G] = H_n(S_* X)$. Thus $H_n(G, \mathbb{Z}) \simeq H_n(X, \mathbb{Z})$, which is independent of X !

More Clever: If A is any G -module, take $H_n(S_*(\tilde{X}) \otimes_G A)$ and define this to be $H_n(X; A)$.

Lemma 4.3. $H_n(G, A) \simeq \text{Tor}_n^{\mathbb{Z}G}(\mathbb{Z}, A)$.

Proof. $H_0(G, A) = A_G$ and $\mathbb{Z} \otimes_{\mathbb{Z}G} A \simeq A_G$, and as $\mathbb{Z} \simeq \mathbb{Z}G/I$ for I the ideal of $\mathbb{Z}G$ generated by $\{g - 1\}$, the left derived functors are unique. \square

Lemma 4.4. $H^n(G, A) \simeq \text{Ext}_{\mathbb{Z}G}^n(\mathbb{Z}, A)$.

The proof is the same.

Corollary 4.5. $H_n(X, A) \simeq H_n(G, A)$ and $H^n(X, A) \simeq H^n(G, A)$ if $X = BG$.

Dumb Examples

1. If $G = 1$, then $\text{Tor}_1^{\mathbb{Z}}(\mathbb{Z}, A) = 0$ so $H_n(G, A) = 0$ for all $n \neq 0$.
2. G infinite cyclic group on t , then $\mathbb{Z}G \simeq \mathbb{Z}[t, t^{-1}]$. $0 \rightarrow \mathbb{Z}[t, t^{-1}] \xrightarrow{t-1} \mathbb{Z}[t, t^{-1}] \rightarrow \mathbb{Z} \rightarrow 0$. (G -modules are abelian groups with automorphism t).

We apply $\otimes_G A$ and take H_* to get $0 \rightarrow H_1(G, A) \rightarrow A \xrightarrow{t-1} A \rightarrow H_0(G, A) \rightarrow 0$. We get $H_1(G, A) \simeq \{a \in A : ta = a\} = A^G = H^0(G, A)$.

Apply $\text{hom}_G(-, A)$ and take H^* . $0 \leftarrow H^1(G, A) \leftarrow \text{hom}_G(\mathbb{Z}G, A) \leftarrow \text{hom}_G(\mathbb{Z}G, A) \leftarrow H^0(G, A) \leftarrow 0$, but $\text{hom}_G(\mathbb{Z}G, A) \simeq A$, so we get $H^1(G, A) \simeq A/(t-1)A \simeq A_G = H_0(G, A)$. $H_n(G, A) = 0$ for $n \neq 0, 1$.

3. $G = C_n$ is cyclic of index n on t . $\mathbb{Z}G \simeq \mathbb{Z}[t]/(t^n - 1)$, which has a \mathbb{Z} basis of $1, t, t^2, \dots, t^{n-1}$.

Definition 4.4 (Norm). If G is any finite group, then the "norm" is the element $N = \sum_{g \in G} g$ of $\mathbb{Z}[G]$.

Lemma 4.6. $(\mathbb{Z}G)^G \simeq \mathbb{Z} \cdot N$.

Apply $\otimes_G A$...See Joe's Notes

5 Spectral Sequences

Definition 5.1 (Homological Spectral Sequence). *A homological spectral sequence is a sequence of bigraded objects $\dots, E_{pq}^r, E_{pq}^{r+1}, \dots$, ($r \geq a$ for some a , typically $a = 2$) together with maps $d_{pq}^r : E_{pq}^r \rightarrow E_{p-r, q+r-1}^r$ with slope $-(r+1)/r$ with $d \circ d = 0$, the lines of slope $-1+1/r$ are chain complexes, and isomorphisms $E_{pq}^{r+1} = \text{homology of appropriate line at } p, q \text{ spot}$.*

The bidegree of E_{pq}^r is $n = p + q$, and it is bounded if $\forall n$, only finitely many (p, q) with $p + q = n$ have $E_{pq}^r \neq 0$ (for some r).

A first quadrant spectral sequence has $E_{pq}^r = 0$ unless $p \geq 0$ and $q \geq 0$.

Dumb Examples:

Consider a chain complex $C_n = E_{0,n}^0$ and $E_{pq}^0 = 0$ if $p \neq 0$ with differentials down (along C_*). Forced to have $E_{0,n}^1 = H_n(C_*)$ and $E_{p,q}^1 = 0$ if $p \neq 0$ with differentials to the left. Taking homology doesn't change anything, so $E_{p,q}^2 = E_{p,q}^1$. Repeating we get the same thing.

We say a spectral sequence collapses at level r is E_{**}^r has exactly one nonzero row or column. We write $E_{pq}^\infty = E_{pq}^r = E_{pq}^{r+1}$

Example 2: Chain map $B_* \rightarrow A_*$ form double complex in columns $p = 0, 1$.

DIAGRAM 1. We will call this E_{pq}^0 (the E^0 page), and $\text{Tot}(E^0) = \text{cone}(f)$.

$E_{0q}^1 = H_q(A_*)$. $E_{1q}^1 = H_q(B_*)$ and $E_{pq}^1 = 0$ else.

We choose differentials d^1 to be the maps induced by d^h on H_* . Get $E_{0q}^2 = \text{coker}$, $E_{1q}^2 = \text{ker}$, $E_{pq}^2 = 0$ else, and get short exact sequence $0 \leftarrow E_{0q}^2 \leftarrow H_q(A) \xleftarrow{f} H_q(B) \leftarrow E_{1q}^2 \leftarrow 0$, and there are no possible nonzero differentials at E^2 -level.

We say a spectral sequence is degenerate at E^r is all differentials d^r, \dots are zero. Then $E_{pq}^r = E_{pq}^{r+1} = E_{pq}^\infty$.

If $\{H_n\}_{n \in \mathbb{Z}}$ is a family of objects, we say that the spectral sequence converges to H_* we write $E_{pq}^r \Rightarrow H_n$ if $(\forall n)$ we have a finite filtration $0 \subseteq \dots \subseteq F_{p-1}H_n \subseteq F_pH_n \subseteq \dots \subseteq H_n$ and isomorphisms $E_{pq}^\infty \simeq F_pH_n / F_{p-1}H_n$, $n = p + q$.

In example 2, DIGRAM TWO.

Exact Sequence of Low Degree Terms of a 1st Quadrant Spectral Sequence $\Rightarrow H_*$, DIAGRAM THREE

The differential $E_{20}^2 \rightarrow E_{01}^2$ in a first quadrant spectral sequence of a double complexes

Theorem 5.1. *Given a bounded double complex E_{pq}^0 there is a spectral sequence $E_{pq}^0 \Rightarrow H_{p+q}(\text{Tot})$ with $d^0 = d^v$ and d^1 induced by d^h .*

DIAGRAMS FOUR

E^2 page, $E_{pq}^2 = H_p^h(H_q^v(E_{p*}^0))$ Represent $[x]$ by $x \in E_{20}^0$, we defined up to images of E_{30} and E_{21} . $d^h x = 0$ in $H_0^v(E_{1*}^0)$ so it is $d^v(y)$ for $y \in E_{11}^0$.

Application: If $R \rightarrow S$ is a ring homomorphism and M is an R -module, N an S -module. Then there is a first quadrant spectral sequence $\text{Tor}_p^S(\text{Tor}_q^R(M, S), N) \Rightarrow \text{Tor}_{p+q}^R(M, N)$ is a base change for Tor .

Suppose that S is a localization of R (e.g., S is a flat R -module), then $\text{Tor}_q^R(M, S) = 0$ for $q \neq 0$, so the spectral sequence collapses, and we get $\text{Tor}_p^R(M, N) \simeq \text{Tor}_p^S(M \otimes_R S, N)$.

Application 2: Suppose that $S = R/xR$, $0 \rightarrow R \xrightarrow{x} R \rightarrow S \rightarrow 0$, x not a zero divisor.

Then $\text{Tor}_0^R(M, S) = M/xM$ and $\text{Tor}_1^R(M, S) \simeq {}_xM = \{m \in M : xm = 0\}$.

Then the spectral sequence has two rows, DIAGRAM FIVE and get a long exact ‘‘Gysin’’ sequence

$$\dots \rightarrow \text{Tor}_n^R(M, N) \rightarrow \text{Tor}_n^S(M/xM, N) \xrightarrow{d^2} \text{Tor}_{n-2}^S(xM, N) \rightarrow \text{Tor}_{n-1}^R(M, N) \rightarrow \dots$$

In particular, $\text{Tor}^R(M, N) \neq \text{Tor}^S(M, N)$ even if $M = xM$.

$\mathbb{Z} \rightarrow \mathbb{Z}/25$ with M, N finite dimensional $\mathbb{Z}/25$ -modules. $\text{Tor}_1^{\mathbb{Z}}(M, N) = M^* \otimes_{\mathbb{Z}/25} N$ and $\text{Tor}_0^{\mathbb{Z}}(M, N) = M \otimes_{\mathbb{Z}/25} N$, they have the same dimension.

DIAGRAM SIX, every d^2 differential is an isomorphism.

Lyndon and Hochschild-Serre Spectral Sequence:

If $H \triangleleft G$ groups and A is a G -module, then

- 1) $H_*(H, A)$ are G/H -modules and
- 2) $E_{pq}^2 = H_p(G/H, H_q(H, A)) \Rightarrow H_{p+q}(G, A)$.

Theorem 5.2. A first quadrant double complex $\{C_{pq}\}$ yields a spectral sequence with $E_{pq}^0 = C_{pq}$ converging to $H_*(\text{Tot } C)$. $E_{pq}^1 = H_q^v(C_{p*})$ with d^h and $E_{pq}^2 = H_p^h(H_q^v(C))$.

Application:

Take M and R -module, N an S -module, and a map $R \rightarrow S$. Next take projective resolutions P_* of M , Q_* of N , and form the first complex double complex $P_* \otimes Q_*$.

Then $H_q^v(P_p \otimes Q_*) = P_p \otimes H_q^v(Q_*)$ which is $P_p \otimes N$ for $q = 0$ and 0 else. So it collapses at E^2 to $H_p(P_* \otimes N) = \text{Tor}_p^R(M, N)$.

Remark: Reflectinv about the $p = q$ axis gives us another spectral sequence, with $E_{pq}^0 = C_{qp}$, $E_{pq}^1 = H_q^h(C_{*q})$, and $E_{pq}^2 = H_p^v H_q^h(C)$.

In our case, $E_{pq}^1 = H_q^h(P_* \otimes_R Q_p) = H_q^h((P_* \otimes_R S) \otimes_S Q_p) = [H_q(P \otimes_R S)] \otimes_S Q_p$ as Q is S -projective.

So we have $\text{Tor}_q^R(M, S) \otimes_S Q_p$. And $E_{pq}^2 = \text{Tor}_p(\text{Tor}_q^R(M, S), N) \Rightarrow \text{Tor}^R(M, N)$.

Application 2:

Fix a projective chain complex P_* (Think $P_* = S_*(X)$, X a Top Space, the singular homology complex) and a module N with projective resolution Q_* . We want to compute $H_n(P \otimes Q)$ two ways.

$H_q^v H_p^h(P_* \otimes Q_*) = H_q[(H_p P) \otimes Q_*] = \text{Tor}_q^R(H_q P, N)$. We need to flip indices.

In the other direction $H_q^v(P_p \otimes Q_*) = P_p \otimes H_q^v(Q_*) = P_p \otimes N$ if $p = 0$ and 0 else.

Looking at $P_0 \otimes N \leftarrow P_1 \otimes N \leftarrow \dots$, we get $H_p(P \otimes Q) \cong H_p(P_* \otimes N)$.

Combinging thee, we get the Kunneth Spectral Sequence. $E_{pq}^2 = \text{Tor}_p^R(H_q P, N) \Rightarrow H_{p+q}(P_* \otimes N)$

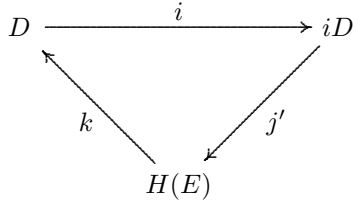
Example: $R = \mathbb{Z}$. $E_{pq}^2 = 0$ unless $p \in \{0, 1\}$.

Theorem 5.3. *If $0 \subseteq F_0 \subseteq \dots \subseteq F_p \subseteq \dots$ is a filtration of a chain complex C_* , each $F_*(C_n)$ is a finite filtration of C_n , then we get a first quadrant spectral sequence $E_{pq}^1 = H_q(F_p/F_{p-1}) \Rightarrow H_{p+q}(C)$.*

Definition 5.2 (Exact Couple). *An exact couple is a pair of objects D, E and maps $i : D \rightarrow D, j : D \rightarrow E, k : E \rightarrow D$ forming a long exact sequence.*

It has degree = r if $\text{bidegree}(i) = (+1, -1)$, $\text{bidegree}(j) = (-r, +r)$ and $\text{bidegree}(k) = (-1, 0)$.

Definition 5.3 (Derived Exact Couple). *The derived exact couple \mathcal{E}' has $D' = i(D)$ image and $E' = H(E, jk)$.*



where $j'(ix) = j(x), x \in D$. j' is well-defined since if $i(x) = i(y)$ then $i(x - y) = 0$ so $\exists e \in E$ with $k(e) = x - y$, so $j(x) = j(x - y) + j(y) = j(y)$. We should check if \mathcal{E} has degree r then \mathcal{E}' has degree r .

If this arises from a filtered complex $D'_{pq} = HF_p$, then $D'_{pq} = \text{Im}(H(F_{p-r}) \rightarrow H(F_{p,q}))$.

Fixing p , we see that for $r \geq p$, $D'_{pq} = 0$. Also, $E_{pq}^\infty \rightarrow \text{Im}(H(F_p C) \rightarrow H_C) \rightarrow \text{Im}(HF_{p-1} \rightarrow H_n) \rightarrow E_{pq}^\infty$

Conclusion: $E_{pq}^\infty = \text{Im}(H(F_p) \rightarrow H(C)) / \text{Im}(H(F_{p-1}) \rightarrow H(C))$.

The filtration on $H_n(C)$ is $F_p(H_n(C)) = \text{Im}(H_n(F_p) \rightarrow H_n(C))$

BIG EXAMPLE INVOLVING $SO(n)$.

Definition 5.4 (F -acyclic). *Any $B \in \mathcal{B}$ is called F -acyclic if the derived functors $(L_n F)(B) = 0, n \neq 0$ and $(L_0 F)(B) = F(B)$.*

Example: any projective is F -acyclic. if $F(B) = A \otimes B$, then any flat module is $(A \otimes)$ -acyclic.

Theorem 5.4. *Let $G : \mathcal{A} \rightarrow \mathcal{B}$ and $F : \mathcal{B} \rightarrow \mathcal{C}$ be additive right exact functors between abelian categories.*

Assume G takes projectives to projectives (or even to F -acyclics), then there is a first quadrant spectral sequence (for all $A \in \mathcal{A}$)

$$E_{pq}^2 = (L_p F)(L_q G)(A) \Rightarrow L_{p+q}(FG)(A).$$

example: $R \rightarrow S$ a ring homomorphism. N an S -module.

$R\text{-mod} \rightarrow S\text{-mod}$: $M \mapsto M \otimes_R S$, and $S\text{-mod} \rightarrow \mathbf{Ab}$ by $\otimes_S N$. The composition is the map tensoring with N .

So we obtain $E_{pq}^2 = \text{Tor}_p^S(\text{Tor}_q^R(-, S), N) \Rightarrow \text{Tor}_{p+q}^R(-, N)$, which is base change for Tor .

Theorem 5.5. *If F, G are left exact, and G takes injectives to injectives (or F -acyclics) then there is a first quadrant cohomological spectral sequence $E_2^{pq} = (R^p F)(R^q G)(A) \Rightarrow R^{p+q}(FG)A$.*

Application:

Claim: If E is an injective R -module, then $\text{hom}_R(S, E)$ is an injective S -module.

The proof lays in the fact that $M \mapsto \text{hom}_S(M, \text{hom}_R(S, E)) = \text{hom}_R(M, E)$.

So, we have $E_2^{pq} = \text{Ext}_p^S(M, \text{Ext}_q^R(S, N)) \Rightarrow \text{Ext}_*^R(M, N)$, base change for Ext.

$\pi : \tilde{X} \rightarrow X$, then we have $\pi^* : \text{sheaves over } \tilde{X} \text{ to sheaves over } X$, and functors Γ to **Ab**.

If \mathcal{F} is a sheaf on X , then $\Gamma(\mathcal{F}) = \mathcal{F}(X)$ the global sections.

Definition 5.5 (Sheaf Cohomology). *The sheaf cohomology $H^q(X, \mathcal{F}) = (R^q \Gamma)(\mathcal{F})$.*

Γ -acyclic=flasque.

Fact $\pi_* = \text{direct image}$ is right adjoint to π^* , so left exact.

$\text{hom}(\pi^* A, B) \simeq \text{hom}(A, \pi_* B)$, also π_* (injective) is flasque.

We get some spectral sequence $E_2^{pq} = H^p(X, R^q \pi_* \mathcal{F}) \Rightarrow H^{p+q}(\tilde{X}, \mathcal{F})$, the “Leray Spectral Sequence”.

BACK TO GROUPS:

$H \trianglelefteq G$, and $M \mapsto M^H$ is a map from G -mod to G/H -mod.

If G acts on M , then G acts on $M_H = M/\text{subgroup generated by } \{h(m) - m\}$.

Coinvariants, because $g(hm - m) = ghm - gm = ghg^{-1}gm - gm = g'm' - m'$ for $g' = gh^{-1}h \in H$ and $m' = gm$.

Also, $H_*(G, M) = L_*(-_G)(M)$, the derived functors of $M \mapsto M_G$.

Then the following triangle satisfies the hypotheses of the theorem above

$$\begin{array}{ccc}
 G\text{-mod} & \xrightarrow{M \mapsto M_H} & G/H\text{-mod} \\
 \swarrow \scriptstyle{()_G} & & \searrow \scriptstyle{()_{G/H}} \\
 & \mathbf{Ab} &
 \end{array}$$

So, $\mathbb{Z}G$ is a projective G -module, and $(\mathbb{Z}G)_H \simeq \mathbb{Z}[G/H]$ is a projective G/H -module, so $E_{pq}^2 = H_p(G/H; H_q(H; M)) \Rightarrow H_{p+q}(G, M)$, with the right side the cohomology of M with coefficients in H .

This is the Lydon-Hochschild-Serre Spectral Sequence.

Application: Let α be an automorphism of a group H . We can form the semidirect product $G = H \rtimes \mathbb{Z}$.

Then we have $H_0(\mathbb{Z}; M) = M_\alpha$ and $H_1(\mathbb{Z}; M) = M^\alpha$.

So $H_0(G, M) = H_0(H, M)_\alpha$.