1 Vector Spaces

Definition 1.1 (Vector Space)**.** *Let V be a set, called the vectors, and F be a field, called the scalars.*

Let $+$ be a binary operation on V with respect to which V is a commutative Group.

Let $\cdot : F \times V \to V$ be an operation called scalar multiplication such that $(\forall c_1, c_2 \in F)(\forall x, y \in V)$

1. $(c_1 c_2)x = c_1(c_2 x)$
2. $(c_1 + c_2)x = c_1 x + c_2 x$
3. $c_1(x + y) = c_1 x + c_1 y$
4. $1 \cdot x = x$

Definition 1.2 (Characteristic of a field)**.** *The smallest positive integer k (if any exists) such that* $1 + 1 + \ldots + 1 = 0$ *with k 1's. If no such k exists, then we say the field has characteristic 0.*

Definition 1.3 (Subspaces)**.** *Let V be a vector space over a field F and let $W \subseteq V$. W is a subspace if W itself is a vector space under the same field F and the same operations.*

There are two sets of tests to see if W is a subspace of V.

The First set of tests is:

1. $W \neq \emptyset$
2. W is closed under addition
3. W is closed under scalar multiplication

Alternatively

1. $\vec{0} \in W$
2. W is closed under linear combinations

Note: A subspace is also closed under subtraction.

Theorem 1.1 (The Intersection Property)**.** *The intersection of subspaces of a vector space is itself a subspace.*

Theorem 1.2 (Intersection Theorem)**.** *Let V be a vector space over a field F. Let $\{W_\lambda\}$ be a nonempty family of subspaces of V.*

Then, the intersection

$$T = \bigcap_{\lambda \in \Lambda} W_\lambda$$

is also a subspace of V.

Proof.

1. Test 0: $T \neq \emptyset$
   $(\forall \lambda \in \Lambda)\vec{0} \in W_\lambda$, $T \ni \vec{0}$
2. Test 1: $(\forall x, y \in T) x + y \in T$
   Let $x, y \in T$
   $(\forall \lambda \in \Lambda) x, y \in W_\lambda$ The W_λs are subspaces, so $(\forall \lambda \in \Lambda) x + y \in W_\lambda$
3. Test 2: $(\forall x \in T)(\forall c \in F) cx \in T$
   Let $x \in T$
   Let $c \in F$
   $(\forall \lambda \in \Lambda) x \in W_\lambda$
   Since W_λs are subspaces, $(\forall x \in T)(\forall c \in F) cx \in T$

□

Definition 1.4 (Span)**.** *Let* $S \subseteq V$*. We define* $\operatorname{span} S$ *as the set of all linear combinations of some vectors in S. By convention,* $\operatorname{span} \emptyset = \{\vec{0}\}$

Theorem 1.3. *The span of a subset of V is a subspace of V.*

Lemma 1.4. *For any S,* $\operatorname{span} S \ni \vec{0}$

Theorem 1.5. *Let V be a vector space of F. Let* $S \subseteq V$*. The set* $T = \operatorname{span} S$ *is the smallest subspace containing S. That is:*

1. *T is a subspace*
2. $T \supseteq S$
3. *If W is any subspace containing S, then* $W \supseteq T$

Examples of specific vector spaces.
$P(F)$ is the polynomials of coefficients from F.
$P_n(F)$ are the polynomials with coefficients from F with degree of at most n
The vector space dealt with in calculus is $F(\mathbb{R}, \mathbb{R})$

Definition 1.5 (Spanning Set)**.** *Let* $S \subseteq V$*. We say S is a spanning set if* $\operatorname{span} S = V$

Lemma 1.6. *Let* $S \subseteq T \subseteq V$*, then* $\operatorname{span} S \subseteq \operatorname{span} T$ *Hence, a superset of a spanning set is also spanning.*

Lemma 1.7. *Let S be a spanning set and let* $A \subseteq V$ *such that* $\operatorname{span} A \supseteq S$*. Then A is a spanning set.*

Definition 1.6 (Dependence)**.** *Let I be a set of vectors. We say that I is dependent if* $(\exists x \in I) x \in \operatorname{span}(I - \{x\})$

Definition 1.7 (Independence)**.** *I is independent if* $(\forall x \in I) x \notin \operatorname{span}(I - \{x\})$

Also, S is dependent if there is a linear combination of some vectors in S that is equal to $\vec{0}$ such that the vectors involved are distinct and at least one of the coefficients is nonzero.

Definition 1.8 (Basis). *B is a basis if it is both independent and spanning.*

Theorem 1.8. *Let $S \subseteq V$.*

S is a spanning set if and only if every vector in V can be expressed as a linear combination of some vectors in S in at least one way.

S is independent iff every vector in V can be expressed as a linear combination of some vectors in S in at most one way.

S is a basis iff every vector in V can be expressed as a linear combination of some vectors in S in exactly one way.

$v \in \operatorname{span} S$ means that for distinct x_is

$(\exists n \in \mathbb{N})(\exists x_1, x_2, \ldots, x_n \in S)(\exists c_1, c_2, \ldots, c_n \in F) c_1 x_1 + c_2 x_2 + \ldots + c_n x_n = 0$

Proof. Let $v \in V$, $n \in \mathbb{N}$, $c_1, c_2, \ldots, c_n, d_1, d_2, \ldots, d_n \in F$ and $x_1, x_2, \ldots, x_n \in S$ Assume $v = \sum c_i x_i$ Assume $v = \sum d_i x_i$ Assume the x_is are distinct

$$\sum c_i x_i = \sum d_i x_i$$

Hence, $\sum (c_i - d_i) x_i = \vec{0}$, and since $(\forall i) x_i \in S$ and S was independent, $(\forall i) c_i = d_i$ □

Theorem 1.9. *B is a basis iff B is a minimal spanning set.*

Theorem 1.10. *B is a basis iff B is a maximal independent set.*

Let V be n-dimensional ($n \in \mathbb{N}$)

Theorem 1.11. *B is a basis iff B is independent and $|B| = n$*

Theorem 1.12. *B is a basis iff B is spanning and $|B| = n$*

Definition 1.9 (Minimal Spanning Set). *A set $S \subseteq V$ is a minimal spanning set if it is a spanning set and if $T \subseteq S$ is a spanning set, then $T = S$*

Lemma 1.13. *Let $S \subseteq S' \subseteq V$. Then $\operatorname{span} S \subseteq \operatorname{span} S'$*

Lemma 1.14. *Let S be a spanning set and let S' be such that $S \subseteq \operatorname{span} S'$ Then S' is a spanning set.*

Lemma 1.15. *A subset $S \subseteq V$ is a subspace iff $S = \operatorname{span} S$*

Lemma 1.16 (Dependence Lemma). *Let I be independent. Let $x \notin I$. Then $I \cup \{x\}$ is dependent iff x is in $\operatorname{span} I$*

Corollary 1.17 (Fundamental Inequality). *Let I be independent and let S be spanning, then $|I| \leq |S|$*

Lemma 1.18. *Consequently, every basis is the same size.*

Theorem 1.19 (Exchange Property)**.** *Let I be a linearly independent set and let S be a spanning set. Then $(\forall x \in I)(\exists y \in S)$ such that $y \notin I$ and $(I - \{x\} \cup \{y\})$ is independent. Consequently, $|I| \leq |S|$*

Definition 1.10 (Finite Dimensional)**.** *V is said to be finite dimensional if it has a finite spanning set.*

Theorem 1.20. *Let V be a finite dimensional space. Then, V has a basis. Furthermore, every independent set can be extended into a basis and every spanning set contains a basis.*

Theorem 1.21. *Let V be a finite dimensional vector space of a field F, and W a subspace of V. Then, W is also finite dimensional and indeed,* $\dim(W) \leq \dim(V)$*. Furthermore, if* $\dim(W) = \dim(V)$*, then W=V.*

Proof. Let I be a maximal independent set in W

Such a set exists and is finite because of the fundamental inequality.

I spans W, and so is a basis for W. This is due to the dependence lemma showing that $\operatorname{span} I = W$.

Therefore, W is finite dimensional, and by the fundamental inequality, $|I| \leq \dim(V)$

Since I is a basis for W, if $|I| = \dim(V)$, then I is a basis for all of V. □

Definition 1.11 (Sumset)**.** *Let $S_1, S_2 \subseteq V$ be nonempty. The sumset of S_1 and S_2 is defined as*

$$S_1 + S_2 = \{x + y : (x \in S_1)(y \in S_2)\}$$

Definition 1.12 (Direct Sum)**.** *Let W_1 and W_2 be subspaces of V. $W_1 + W_2$ is called a direct sum, denoted as $W_1 \oplus W_2$ if $W_1 \cap W_2 = \{\vec{0}\}$*

Theorem 1.22. *Let V be finite dimensional and W_1 and W_2 subspaces. If $W_1 \cap W_2 = \{\vec{0}\}$, then* $\dim(W_1 \oplus W_2) = \dim(W_1) + \dim(W_2)$

Theorem 1.23. *The sum of two subspaces is a subspace*

Lemma 1.24. $W_1 \cup W_2 \subseteq W_1 + W_2$

Theorem 1.25. *Let W_1 and W_2 be subspaces of a finite dimensional vector space V. Then* $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$

Proof. Let $k = \dim(W_1 \cap W_2)$ and $l = \dim(W_1)$ and $m = \dim(W_2)$

Let $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$ be a basis for $W_1 \cap W_2$

Let $\beta_1, \beta_2, \dots, \beta_{l-k} \in W_1$ be such that $\{\alpha_1, \alpha_2, \dots, \alpha_k, \beta_1, \beta_2, \dots, \beta_{l-k}\}$ is a basis for W_1

This is possible because $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$ is a subset of W_1, and is independent, so it can be extended to a basis of W_1.

Similarly, let $\gamma_1, \gamma_2, \dots, \gamma_{m-k}$ be such that $\{\alpha_1, \alpha_2, \dots, \alpha_k, \gamma_1, \gamma_2, \dots, \gamma_{m-k}\}$ is a basis for W_2.

claim - B=$\{\alpha_1, \alpha_2, \dots, \alpha_k, \beta_1, \beta_2, \dots, \beta_{l-k}, \gamma_1, \gamma_2, \dots, \gamma_{m-k}\}$ is a basis for $W_1 + W_2$

Proof. By lemma above, $B \subseteq W_1 + W_2$

Let $x \in W_1 + W_2$. So $(\exists y \in W_1)(\exists z \in W_2)x = y + z$.

Since $y \in \operatorname{span}(\alpha_1, \alpha_2, \ldots, \alpha_k, \beta_1, \beta_2, \ldots, \beta_{l-k}) \subseteq \operatorname{span} B$

Also $z \in \operatorname{span} B$, so then $x = y + z$ is also in $\operatorname{span} B$ because $\operatorname{span} B$ is a subspace. Thus, B spans $W_1 + W_2$

Let $c_1, \ldots, c_k, d_1, \ldots, d_{l-k}, e_1, \ldots, e_{m-k} \in F$

Assume $\sum c_i\alpha_i + \sum d_j\beta_j + \sum e_s\gamma_s = \vec{0}$

$\delta := \sum e_s\gamma_s = -\sum c_i\alpha_i - \sum d_j\beta_j$. Because $(\sum e_s\gamma_s) \in W_2$, $(\sum b_j\beta_j) \in W_1$ and $(\sum c_i\alpha_i) \in W_1 \cap W_2$, then $\delta \in W_1 \cap W_2$

αs and βs form a basis for W_1, so δ can be expressed from them in a unique way.

Because $\delta \in W_1 \cap W_2$, it can be expressed with αs alone, so $(\forall j)d_j = 0$. The same argument shows that $(\forall s)e_s = 0$, so

$\vec{0} = \sum c_i\alpha_i$, but as the α_is are a basis, $(\forall i)c_i = 0$

Thus, B is a basis for $W_1 + W_2$ □

Because B is a basis for $W_1 + W_2$, $\dim(W_1 + W_2) = |B| = k + l - k + m - k = l + m - k$ □

Theorem 1.26. *Let W_1, W_2 be subspaces of V. Assume $W_1 + W_2 = V$. Then $W_1 \oplus W_2 = V$ is equivalent to saying*

1. $W_1 \cap W_2 = \{\vec{0}\}$
2. $(\forall v \in V)(\exists! x \in W_1)(\exists! y \in W_2)v = x + y$
3. $(\forall x_1, x_2 \in W_1)(\forall y_1, y_2 \in W_2)$ *if* $x_1 + y_1 = x_2 + y_2$ *then* $x_1 = x_2$ *and* $y_1 = y_2$
4. $(\forall x \in W_1)(\forall y \in W_2)$ *if* $\vec{0} = x + y$ *then* $x = y = \vec{0}$

Theorem 1.27.

1. $W_1 + W_2 = V$ *iff every vector in V can be expressed in at least one way from vectors in $W_1 + W_2$*
2. $W_1 \cap W_2 = \{\vec{0}\}$ *iff every vector in V can be expressed in at most one way from vectors in $W_1 + W_2$*
3. $W_1 \oplus W_2 = V$ *iff every vector in V can be expressed in exactly one way from vectors in $W_1 + W_2$*

Proof. 1 and 3 are true by definition.

Assume $W_1 \cap W_2 = \{\vec{0}\}$

Let $v \in V$

Let $x, x' \in W_1$ and $y, y' \in W_2$ such that $v = x + y$ and $v = x' + y'$.

$x + y = x' + y' \Rightarrow W_1 \ni x - x' = y' - y \in W_2$, thus $x - x' \in W_1 \cap W_2$, so $x - x' = \vec{0} = y' - y$

Assume that each vector in V can be written in at most one way.

Let $\delta \in W_1 \cap W_2$.

$\delta = W_1 \ni \delta + \vec{0} = \vec{0} + \delta \in W_2$, hence $\delta = \vec{0}$ and $\vec{0} = \delta$ □

2 Linear Transformations

Definition 2.1 (Linear Transformation)**.** *Let V and W be vector spaces over the same field F.*

A function $T : V \to W$ is a linear transformation if
$(\forall x, y \in V)(\forall c, d \in F)T(cx + dy) = cT(x) + dT(y)$

Usually, we perform two tests, one for additivity and one for homogeneity.

Theorem 2.1. $T(\vec{0}_V) = \vec{0}_W$

Definition 2.2 (Range)**.** *The range of T is $T(V) := \{T(x) : x \in V\}$ and is denoted by R_T and $R(T)$*

Definition 2.3 (Rank)**.** *The Rank of T is $\dim R_T$ and is denoted as* $\operatorname{rank}(T)$

Definition 2.4 (Null Space)**.** *The null space (or kernel) of T is $\{x \in V : T(x) = \vec{0}\}$ and is denoted by N_T or $N(T)$*

Definition 2.5 (Nullity)**.** *The nullity of T is $\dim N_T$ and is denoted as* $\operatorname{null} T$

Theorem 2.2. *Let A be a subspace of V and B be a subspace of W.*

Then, $T(A)$ is a subspace of W and $T^{-1}(B)$ is a subspace of V.

Theorem 2.3. *Let V, W, Z be Vector Spaces over a field F.*

Let $T : V \to W$ and $U : W \to Z$ be linear transformations.

Then, the composition of T and U, namely $U \circ T$, will act from V to Z and is also a linear transformation.

Theorem 2.4. *The following are equivalent:*

Definition 2.6 (Singular)**.** *T is singular means that $(\exists x \in V)x \neq \vec{0}$ and $T(x) = \vec{0}$*

Definition 2.7 (Nonsingular)**.** *T is nonsingular means $(\forall x \in V)$ if $x \neq \vec{0}$ then $T(x) \neq \vec{0}$*

Definition 2.8 (One-to-one)**.** *T is one-to-one means $(\forall x, y \in V)$ if $x \neq y$ then $T(x) \neq T(y)$*

1. *T is nonsingular*
2. *T is one-to-one*
3. *T maps linearly independent vectors to linearly independent vectors*
4. *T maps any basis to an independent set*
5. *V has a basis which is mapped into an independent set*

Proof. Assume T is not one-to-one

Then, $(\exists x, y \in V) x \neq y$ but $Tx = Ty$.

Let $z = x - y$, then $z \neq 0$, but $Tz = Tx - Ty = 0$, so T is singular.

Thus, if T is nonsingular, then T is one-to-one

The other direction is trivial. □

Theorem 2.5. *If T is nonsingular and $I \subseteq V$ is independent, then $T(I)$ is independent. This includes the claim that for any two different vectors $x, y \in I$, then Tx and Ty are different.*

Proof. We'll prove the contrapositive, that if $\exists I \subset V$ such that I is independent and $T(I)$ is dependent, then T is singular.

Let $I \subset V$ be independent such that $T(I)$ is dependent.

There are $y_1, y_2, \ldots, y_n \in T(I)$ and $c_1, c_2, \ldots, c_n \in F$ such that $\sum c_i y_i = 0$.

Also, $y_i \in T(I) \Rightarrow (\exists x_i \in I) Tx_i = y_i$.

Let $\alpha = \sum c_i x_i$. By linearity of T, $T\alpha = \sum (Tc_i x_i) = \sum c_i Tx_i = \sum c_i y_i = 0$. But $\alpha \neq 0$ because the x_i's were independent. Thus, T is singular. □

Theorem 2.6. *Conversely, if T maps linearly independent sets to independent sets, then T is nonsingular.*

Proof. Again, we'll prove the contrapositive, that if T is singular, then $\exists I \subset V$ such that $T(I)$ is dependent and I is independent.

Assume T is singular. That is, $(\exists x \in V) x \neq 0$ such that $Tx = 0$.

Let $I = \{x\}$. Then I is independent, and $T(I)$ is dependent. □

Theorem 2.7. *If V has a basis B which is mapped into an independent set, then T is nonsingular, and also, if one basis is mapped into an independent set, then all bases are.*

Proof. Let $x \in V$ and $x \neq 0$.

B is a basis for V, so $\exists t_1, \ldots, t_n \in B$ distinct and $c_1, \ldots c_n \in F$ such that $x = \sum c_i t_i$

Since $x \neq 0$, at least one c_i is not 0. By linearity of T, $Tx = \sum c_i Tt_i$, since Tt_i are independent and not all c_i are 0, $\sum c_i Tt_i$ is not 0. □

Theorem 2.8. *T can be defined arbitrarily on a basis of V, and then T is uniquely defined if linear on all of V.*

Theorem 2.9. *If T is assigned arbitrary values on an independent set, then T can be extended, usually in many ways, to a linear transformation on all of V.*

Definition 2.9 (Matrix Transformation)**.** *Let A be an $m \times n$ matrix over F. Let $\underline{c_1}, \ldots, \underline{c_m}$ be its columns.*

$V = F^m$ and $W = F^n$. Then, the matrix transformation is $L_A : V \to W : x \to Ax$.

Theorem 2.10. *Let U be a subspace of V.*

Spanning sets of U are mapped into spanning sets of $T(U)$

In particular, bases of U are mapped into spanning sets of $T(U)$.

Consequently, if U is finite-dimensional, then $\dim T(U) \leq \dim U$

Proof. Let $S \subseteq U$ be a spanning set of U.

Let $y \in T(U)$, Thus, by definition, $(\exists x \in U)Tx = y$.

As S is a spanning set of U, there are $x_1, \ldots, x_n \in S$ and $c_1, \ldots, c_n \in F$ such that $x = \sum c_i x_i$.

By applying T, we get $y = \sum c_i T x_i$, and we have that $(\forall i) T x_i \in T(S)$, by definition.

Thus, y is a linear combination of vectors in $T(S)$, and so $T(S)$ spans $T(U)$.

Let $B \subset U$ be a basis of U, $B = \{b_1, \ldots, b_n\}$

Then, $\{Tb_1, \ldots, Tb_n\}$ spans $T(U)$, and hence $\dim R(T) \leq n = |B| = \dim U$ $\square$

Theorem 2.11. *If T is a linear transformation defined on a basis, then T is defined everywhere.*

Thus, if T, T' are two linear transformations from V to W, and they agree on a basis, then $T = T'$

Theorem 2.12 (Fundamental Theorem of Linear Algebra)**.** $\operatorname{rank} T + \operatorname{null} T = \dim V$

Theorem 2.13. *Let T be a linear transformation, and let T be invertible, then, T^{-1} is also a linear transformation from W to V.*

Proof. Let $y_1, y_2 \in W$

Because T is onto, $\exists x_1, x_2 \in V$ such that $Tx_1 = y_1$ and $Tx_2 = y_2$.

Because T is linear, $T(x_1 + x_2) = Tx_1 + Tx_2 = y_1 + y_2$, so $T^{-1}(y_1 + y_2) = x_1 + x_2 = T^{-1}y_1 + T^{-1}y_2$. $\square$

3 Linear Operators

Definition 3.1 (Linear Operator)**.** *A Linear Transformation such that $V = W$ is called a Linear Operator on V.*

Definition 3.2 (T-Invariant)**.** *Let T be a linear operator on V.*

A subspace U of V is T-invariant if $T(U) \subset U$. That is, $(\forall x \in U) Tx \in U$

Theorem 3.1. *V, $\{\vec{0}\}$, $N(T)$ and $R(T)$ are invariant under arbitrary V and T.*

Proof. V and $\{\vec{0}\}$ are trivial.

Let $x \in N(T)$, that is, $Tx = 0$, hence $Tx \in N(T)$, because $N(T)$ is a subspace, and so contains $\vec{0}$, $N(T)$ is T-invariant.

Let $y \in R(T)$. $Ty \in R(T)$ because all outputs of T are in $R(T)$. $\square$

Definition 3.3 (Idempotent)**.** *T is idempotent if $T^2 = T$*

Definition 3.4 (Nilpotent)**.** *T is nilpotent if* $(\exists k \in \mathbb{N})T^k = 0$

Definition 3.5 (Restriction)**.** *Let T be a linear operator on V and let U be a T-invariant subspace of V.*

Then, $T_U : U \to U : x \mapsto Tx$ is called the restriction of T to U.

Theorem 3.2. *T_U is a linear operator on U.*

Definition 3.6 (Projection)**.** *Let A and B be subspaces of V such that $V = A \oplus B$.*

The projection of A along B is defined as the operator P on V such that $(\forall \alpha \in A)(\forall \beta \in B)P(\alpha + \beta) = \alpha$

Theorem 3.3. *Let $V = A \oplus B$ and P be the projection on A along B. Then,*

1. *P is idempotent*
2. *$R(P) = A$ and $N(P) = B$*
3. *$R(P) = \{x \in V : Px = x\}$*

Proof.

1. Let $v = \alpha + \beta$, then $Pv = \alpha$.

   $P^2 v = P(Pv) = P(\alpha) = \alpha$, hence $P^2 = P$, so P is idempotent.

2. Let $y \in R(P)$. By the definition of range, $(\exists x \in V)y = Px$

   $x = \alpha + \beta$, so $Px = \alpha = y \in A$.

   Let $a \in A$, since $a \in A, Pa = a$, so $a \in R(P)$.

   Let $x \in N(P)$, $Px = 0$. As $x = \alpha + \beta$, and $Px = \alpha$, then $\alpha = 0$, so $x = \beta \in B$.

   Let $x \in B$, then $x = 0 + \beta$, so $Px = 0$.

3. Let $I = \{x \in V : Px = x\}$

   $I \subset R(P)$ is trivial.

   Let $y \in R(P)$, that is, $(\exists t \in V)y = Pt$, thus $Py = P(Pt) = P^2 t = Pt = y$

□

Theorem 3.4. *Let E be an idempotent operator on V. Then, E is a projection. In fact, E is the projection to $A = R(E)$ along $B = N(E)$.*

Proof. First, we must prove that $A \oplus B = V$, that is $R(E) \oplus N(E) = V$

Let $v \in V$. Let $x = Ev$, then $y = v - x$.

$Ey = E(v - x) = Ev - Ex = Ev - E(Ev) = Ev - E^2 v = 0$.

Let $\delta \in R(E) \cap N(E)$.

$\delta \in R(E)$ means that $\exists \gamma \in V$ such that $\delta = E\gamma$, but $\delta \in N(E)$ means that $E\delta = 0$.

So, $E\delta = E(E\gamma) = E\gamma = \delta = 0$.

Next, we need to show that E is a projection to A along B.

Let $v \in V$, and let $a \in A$ and $b \in B$ such that $v = a + b$.

$Ev = Ea + Eb$. Because $Eb \in N(E)$, $Eb = 0$, so $Ev = Ea$.

Because $a \in R(E)$, $\exists x \in V$ such that $Ex = a$, so $Ea = E^2x = Ex = a$. So, $Ev = a$ □

Definition 3.7 (Characteristic Polynomial)**.** *Given a square matrix A, we define the characteristic polynomial of A, $p(x)$, as* $\det(A - xI)$

Theorem 3.5. *The Following Are Equivalent*

1. *λ is an eigenvalue of T*
2. $(\exists x \neq 0)$ *such that* $Tx = \lambda x$
3. $(\exists x \neq 0)$ *such that* $(T - \lambda I)x = 0$
4. $T - \lambda I$ *is singular*
5. $N(T - \lambda I) \neq \{0\}$
6. $\operatorname{null}(T - \lambda I) \geq 1$

Definition 3.8 (Geometric Multiplicity of λ)**.** *The geometric multiplicity of λ is defined as* $\operatorname{null}(T - \lambda I) = \dim\{x : Tx = \lambda x\}$

Definition 3.9 (Determinant of an Operator)**.** *If U is an operator,* $\det U = \det M$ *when M is a matrix of U on a basis of U*

Theorem 3.6. *Assume* $\dim V < \infty$*. Then, λ is an eigenvalue iff* $\det(T - \lambda I) = 0$*, iff λ is a root of* $p(x) = \det(T - \lambda I)$*, iff $(x - \lambda)$ divides $p(x)$.*

Theorem 3.7. *Let U be a linear operator on a finite-dimensional vector space V. Let β be an ordered basis of B and let A be the matrix of U for basis β. Then,* $\det A$ *does not depend on β. In fact, the characteristic polynomial* $p(x) = \det(A - xI)$ *does not depend on β.*

Proof. Let β' be another basis and let P be the change of basis matrix from β to β'.

$\det([U]_{\beta'}) = \det(P^{-1}[U]_\beta P) = \det P^{-1} \det [U]_\beta \det P = \det [U]_\beta$ □

Definition 3.10 (Algebraic Multiplicity of λ)**.** $\max d \in \mathbb{N}$ *such that* $(x - \lambda)^d$ *divides $p(x)$ is called the algebraic multiplicity of λ*

Definition 3.11 (Diagonalizable Operator)**.** *A finite dimensional operator is diagonalizable means that $\exists b$ a basis of B such that the operator has a diagonal matrix.*

Definition 3.12 (Similar Matrices)**.** *A and B are similar $n \times n$ matrices if $\exists P$ nonsingular such that $B = P^{-1}AP$. They describe the same linear operator.*

Theorem 3.8. *Similar is an equivalence relation*

Theorem 3.9. *Let λ be an eigenvalue of T. Then, the algebraic multiplicity of λ is greater than or equal to the geometric multiplicity of λ.*

Proof. Geometric Multiplicity = $\text{null}(T-\lambda I) = \dim\{x : Tx = \lambda x\} = k$

Let $\alpha_1, \ldots, \alpha_k$ be a basis for $N(T-\lambda I)$. Add $n-k$ more independent vectors to get a basis B for V.

Let A be the matrix of T in basis B, then $p(x) = (\lambda - x)^k q(x)$ □

Definition 3.13 (Generalized Eigenvector). *Let λ be an arbitrary scalar. We see that a vector x is a generalized eigenvector with eigenvalue λ if $(\exists m \in \mathbb{N})(T-\lambda I)^m x = 0$ and $x \neq 0$*

Definition 3.14 (Generalized Eigenspace). $K_\lambda = \{x : (\exists m \in \mathbb{N})(T-\lambda I)^m x = 0\}$

Theorem 3.10. *$K_\lambda \neq \{0\}$ iff λ is an eigenvalue.*

Proof. $E_\lambda \subseteq K_\lambda$ is trivial.

Assume $K_\lambda \neq \{0\}$

Let $x \neq 0$ and $x \in K_\lambda$, thus $\exists m \in \mathbb{N}$ such that $(T-\lambda I)^m = 0$. Let m_0 be the smallest such m.

If $m_0 = 1$, then x is an eigenvector, thus λ is an eigenvalue.

If $m_0 > 1$, then let $y = (T-\lambda I)^{m_0-1}x$. Then $y \neq 0$ and $(T-\lambda I)y = 0$, so y is an eigenvector, thus λ is an eigenvalue. □

Theorem 3.11. *K_λ is a subspace.*

Theorem 3.12. *Let V be n-dimensional,then $(\exists m \in \mathbb{N})$ such that $R(T^m) \oplus N(T^m) = V$*

Proof. $N(T) \subseteq N(T^2) \subseteq \ldots$ and from a certain index on, they're all the same.

$R(T) \supseteq R(T^2) \supseteq \ldots$ and from a certain index on, they're all the same.

Then, $(\exists m \in \mathbb{N})$ such that $(\forall i \geq m) N(T^{i+1}) = N(T^i)$ and $R(T^{i+1}) = R(T^i)$.

Let $x \in N(T^m) \cap R(T^m)$. From $x \in N(T^m)$, we have that $T^m x = 0$

As $x \in R(T^m)$, $\exists v \in V$ such that $T^m(v) = x$.

$T^m(T^m(v)) = T^{2m}(v) = 0$, so $x = 0$. □

Definition 3.15 (Generalized Null Space). $K(U) := \{x \in V : (\exists i \in \mathbb{N}) U^i x = 0\} = \cup_{i \geq 1} N(U^i)$

Theorem 3.13. *$K(U)$ is a subspace of V*

Theorem 3.14. *Let V be finite dimensional. Then $\exists k in \mathbb{N}$ such that $K(U) = \bigcup_{i=1}^k N(U^i) = N(U^k)$. In fact, if $\dim V = n$, then $K(U) = N(U^n)$.*

Proof. $N(U) \subseteq N(U^2) \subseteq \ldots \subseteq V$ are subspaces of V.

For some i, $N(U^i) \neq N(U^{i+1})$ may happen, then by theorem 1.11, $\dim N(U^i) < \dim N(U^{i+1})$

Hence, $\exists k$ such that $\forall i \geq k$ we have $N(U^i) = N(U^{i+1})$

That is, $\exists k$ such that $K(U) = N(U^k)$

Let $x \in K(U) \neq 0$, and p be the smallest positive integer i for which $U^i x = 0$. Then $x, Ux, U^2x, \ldots, U^{p-1}x$ are all nonzero, but $U^p x = 0$

Let $c_0, \ldots, c_{p-1} \in F$ such that $\sum_{i=0}^{p-1} c_i U^i x = 0$.

Assume $\exists k$ such that $c_i = 0$ for all $i < k$ but $c_k \neq 0$. Then apply U^{p-k-1}, and we get $c_k U^{p-1}x = 0$, and we know that $U^{p-1}x \neq 0$, so then $c_k = 0$, a contradiction.

Thus, $x, Ux, \ldots, U^{p-1}x$ are independent. □

Theorem 3.15. *Let $F = \mathbb{C}$ and V be finite dimensional. Then any linear operator T can be split as $T = D + N$ where D is diagonalizable and N is nilpotent, and they commute, that is, $DN = ND$.*

Theorem 3.16 (Cayley-Hamilton)**.** *Let B be finite dimensional. Let T be a linear operator on V and let $p(x)$ be the characteristic polynomial of T.*

Then, $p(T) = 0$.

Proof. Let x be a nonzero vector.

Let $x, Tx, T^2x, \ldots, T^{k-1}x$ be independent, but $x, \ldots, T^k x$ be dependent. Then k is defined for this x.

By the dependence lemma, there are scalars $c_0, \ldots, c_{k-1} \in F$ such that

$$\sum_{i=0}^{k-1} c_i T^i x + T^k x = 0$$

Let $W = \text{span}\{x, Tx, \ldots, T^{k-1}x\}$

Then, W is T-invariant, because applying T to the basis gives $\{Tx, \ldots T^k x\}$ and $T^k x$ is in W, and the other vectors are all in the basis of W used to define W.

So, T_W is defined as Tx for $x \in W$ and is a linear operator on W.

Then, we find the matrix of T in a basis $\alpha_1, \ldots, \alpha_n$ such that $\alpha_1, \ldots, \alpha_k$ is a basis for W. Then the matrix looks like

$$[T] = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$$

Then, $\det(T - xI) = \det(A - xI)\det(C - xI)$, so the characteristic polynomial of T_W divides the characteristic polynomial of T.

Let $U = T_W$

Thus, $\exists q$ such that $p_T = p_U q$

Thus, $p_T(T)v = (p_U(U))(q(T))v$

$p_U(x) = (c_0 + c_1 x + c_2 x^2 + \ldots + c_{k-1}x^{k-1} + x^k) = 0$, then $p_U(T)v = c_0 v + c_1 Tv + \ldots c_{k-1}T^{k-1}v + T^k v = 0$. □

Definition 3.16 (Cyclic Subspace generated by v)**.** $W := \text{span}(v, Tv, \ldots)$ *is the cyclic subspace generated by v*

Definition 3.17 (Annihilating Polynomial)**.** *If $q(x)$ is a polynomial and $q(T) = 0$ then q is an annihilating polynomial.*

Theorem 3.17. *The set of all annihilating polynomials is closed under +, -, scalar multiplication, and multiplication by polynomials.*

Definition 3.18 (Minimum Polynomial)**.** *The minimum polynomial of T is the annihilating polynomial of least degree which is monic and not the zero polynomial*