1 Preliminaries

You should know the basics of category theory, schemes, varieties, morphisms, proper, flat, Hilbert Polynomial, and the genus of a curve.

References: Harris and Morrison "Moduli of Curves" (Ch 1,§A), Harris "Algebraic Geometry" (Lecture 21), Eisenbud and Harris "Geometry of Schemes" (Section VI), Kollar "Rational Curves on Algebraic Varieties"

General Overview: People want to classify objects, so pick a set of properties and attempt to make the set of objects of that sort into a variety. We want to say carefully "What is a moduli problem?" and w"What does it mean that a particular variety (scheme, stack) solves this moduli problem?"

The answer consists of two parts.

1. What is M?
2. What can we say about the geometry of M?

For part 2, these are the types of questions we ask about M:

1. Is the moduli space proper? If not, does it have a modular compactification? Is the moduli space projective?
2. What is the dimension? Is the moduli space connected? Is M irreducible? What kinds of singularities does it have?
3. What is the cohomology ring/Chow ring of the moduli space?
4. What is the Picard group of M? If M is projective, can one describe the ample divisors? The effective divisors?
5. Can the moduli space be rationally parametrized? What is its Kodaira dimension?

What makes a moduli problem?

1. A collection A of algebro-geometric objects
   (a) For a fixed variety or scheme X, let A be the collection of configurations of n distinct points on X.
   (b) A collection of smooth curves of genus g
   (c) Morphisms $\mathbb{P}^1 \to \mathbb{P}^n$
   (d) Hypersurfaces of degree d in $\mathbb{P}^n$.
2. An equivalence relation $\sim$ on A with M the underlying set of points of $A/\sim$ and the geometry of M reflecting how objects move in families.
   (a) $\sim$ can be trivial or if $X = \mathbb{P}^r$, $A/\sim$ can be the configurations of points up to projective equivalence
   (b) $\sim$ is up to isomorphism.

(c) $\sim$ is up to isomorphism of maps (commuting diagrams as follows:)

$$\begin{array}{ccc} \mathbb{P}^n & \xrightarrow{\varphi} & \mathbb{P}^n \\ \uparrow f & \nearrow g & \\ \mathbb{P}^1 & & \end{array}$$

(d) $\sim$ is up to projective transformation

3. Notion of an equivalence class of families of $A/\sim$.

   (a) If X is a scheme and A is a configuration of n distinct points on X with $\sim$ relation, then an equivalence class of families is an equivalence class of diagrams $B \times X \xrightarrow{\pi_1} B$ with n sections $B \to B \times X$ such that for $b \in B$ closed point, $\pi_1^{-1}(b) = b \times X \simeq X$ and so $\sigma_1(b), \ldots, \sigma_n(b)$ gives n distinct points of X.

   (b) A family parametrized by B of smooth curves of genus g up to isomorphism is a flat morphism $X \xrightarrow{\pi} B$ where for each $b \in B$, $\pi^{-1}(b)$ is an isomorphism class of smooth curves of genus g.

   (c) A family of isomorphism classes of morphisms $f : \mathbb{P}^1 \to \mathbb{P}^r$ is a

$$\begin{array}{ccc} X & \xrightarrow{\mu} & \mathbb{P}^r \\ \downarrow \pi & & \\ B & & \end{array}$$

   diagram for each $b \in B$ a closed point, $\pi^{-1}(b) \simeq \mathbb{P}^1$ and $\mu|_{\pi^{-1}(b)} : \mathbb{P}^1 \to \mathbb{P}^r$

   (d) A family of hypersurfaces of degree d in $\mathbb{P}^r$ is a diagram

$$\begin{array}{ccc} X & \dashrightarrow & B \times \mathbb{P}^r \\ \downarrow \pi & \swarrow \pi_1 & \\ B & & \end{array}$$

   such that for all $b \in B$ closed points, $\pi^{-1}(b) = H_b \to b \times \mathbb{P}^r$ is a hypersurface of degree d

We will now look at part II, families, in more depth:

Definition 1.1 (Family of Objects)**.** *Let A be a collection of algebro-geometric objects and $\sim$ an equivalence relation on A. A family of objects of $A/\sim$ parametrized by a scheme (or variety) B is a morphism $\pi : X \to B$ satisfying three properties:*

1. *If $B = \mathrm{Spec}(k)$, then X consists of a single element of $A/\sim$*

2. *We can define an equivalence relation $\sim$ on $X \to B$ which restricts to the original equivalence relation if $B = \mathrm{Spec}(k)$*

3. *Families pull back to families functorially: if $\pi : X \to B$ and $f : B' \to B$, then*

$$\begin{array}{ccc} f^*X = B' \times_B X & \longrightarrow & X \\ \downarrow & & \downarrow \\ B' & \xrightarrow{f} & B \end{array}$$

This pull back operation satisfies the following:

   (a) $(f \circ f')^*X = (f')^*f^*X$

   (b) *If the family is* $\mathrm{id}_B : B \to B$*, then we get* $f^*B = B'$ *and the pullback family is* $\mathrm{id}_{B'} : f^*B \to B$*.*

   (c) *If* $X \to B$ *and* $X' \to B$ *are families and* $X \sim X'$*, then* $f^*X \sim f^*X'$*.*

We will fix some notation: If $X \to B$ is a family, we will write $f^*X = B' \times_B X = X_{B'}$, so if we have $b \to B$ an inclusion of a point, then X_b is the fiber of the family over $b \in B$.

Suppose that $\mathscr{M}$ is a scheme whose underlying set of points is $A/\sim$. Then if $X \to B$ is a family of elements of $A/\sim$, we get a "classifying map" $\eta_X : B \to \mathscr{M}$ which will take a closed point b to $[X_b]$.

If $\mathscr{M}$ is any sort of moduli space, then at minimum we require that this map be a morphism. Ideally η_X should define a bijective correspondence between equivalence classes of families $X \to B$ and morphisms $B \to \mathscr{M}$.

We begin by defining a contravariant functor $F : \{\text{Schemes}\} \to \{\text{Sets}\}$ by $B \mapsto F(B) = \{\text{equivalence classes of families parameterized by } B\}$. If we have $f \in \mathrm{Mor}(B, B')$, then $F(B) \subset F(B')$ and take the morphism to be $X \to B$ maps to $f^*X \to B'$.

We want to say what $\mathscr{M}$ (a scheme whose underlying points are $A/\sim$) has to satisfy in order to be the answer to the problem posed by this functor F. We consider the functor of points $\hom(*, \mathscr{M}) : \{\text{schemes}\} \to \{\text{sets}\}$ which takes X to $\hom(X, \mathscr{M})$. We define $\phi : F \to \hom(*, \mathscr{M})$ by putting $B \in Obj(\{\text{Sch}\})$, $\phi(B) : F(B) \to \hom(B, \mathscr{M})$ by $X \to B$ is sent to $\eta_X : B \to M$. If $f \in \mathrm{Mor}(B, B')$ for B, B' schemes, we get a map $B' \xrightarrow{f} B \xrightarrow{\eta_X} \mathscr{M}$ which we can compose to get a map $B' \to \mathscr{M}$.

We say that $\mathscr{M}$ solves the problem posed by F if φ is a natural isomorphism, that is, $(\mathscr{M}, \phi)$ represents F.

Definition 1.2 (Fine Moduli Space)**.** *A fine moduli space for a given moduli problem described by a functor F is the pair $(\mathscr{M}, \phi)$ that represents F.*

Notice:

1. $\phi(\mathrm{Spec}\, k) : F(\mathrm{Spec}\, k) = A/\sim \to \mathrm{Mor}(\mathrm{Spec}\, k, \mathscr{M}) \simeq \mathscr{M}$ is a bijection.

2. $\phi(\mathscr{M}) : F(\mathscr{M}) \to \mathrm{Mor}(\mathscr{M}, \mathscr{M})$. The latter contains $\mathrm{id}_{\mathscr{M}}$ which corresponds to a unique family $\mathscr{U} \to \mathscr{M}$ (the Universal Family), such that any family is a pullback of this family. Why is this? If $X \to B$ is sent to $B \overset{\eta_X}{\to} \mathscr{M} \overset{\mathrm{id}}{\to} \mathscr{M}$ composes to $\mathrm{id} \circ \eta_X$, so $X \simeq (\mathrm{id} \circ \eta_X)^* \mathscr{U}$.

Definition 1.3 (Fine Moduli Space (alternate)). *A fine moduli space consists of a scheme (or variety or stack) $\mathscr{M}$ and a family $\mathscr{U} \to \mathscr{M}$ called the universal family such that for every $X \to B$ there exists a unique morphism (the classifying morphism) $\eta_X : B \to \mathscr{M}$ for which $X = \eta_X^* \mathscr{U}$*

Example 1.1. *Take $X = \mathbb{P}^1$, and $F(\mathbb{P}^1, n)$ that we are looking for is the moduli space of configurations of n distinct points on $\mathbb{P}^1$ with the trivial equivalence relation.*

$F(\mathbb{P}^1, n) = \mathbb{P}^1 \times \ldots \times \mathbb{P}^1 \setminus \cup$diagonals, where there are n $\mathbb{P}^1$'s and the diagonals are the subloci where points coincide. Let $\mathscr{U} = F(\mathbb{P}^1, n) \times \mathbb{P}^1$ with map $\pi : \mathscr{U} \to F(\mathbb{P}^1, n)$ the first projection. It comes with n sections σ_i where $\sigma_i = p_i \circ \mathrm{id}$ where $p_i : \mathbb{P}^1 \times \ldots \mathbb{P}^1 \setminus \Delta \to \mathbb{P}^1$ projects to the i^{th} copy of $\mathbb{P}^1$.

2 Lecture 2

$X \simeq \mathbb{P}^1$ (more generally, a scheme over S and sometimes even a stack)

Agenda: $F(X, n)$ for $n \in \mathbb{N}$ is a fine moduli space that was studied by topologists originally (n points on X), and it was given a compactification by Fulton and MacPherson, $X[n]$.

But for now, we will look at $G(k, n)$, the Grassmanian, which we will use to study Chow Varieties $G(k, d, n)$, which will be used to construct $\bar{\mathscr{M}}_{0,n}$, the moduli space of n-pointed stable curves of genus zero.

Recall that solving a moduli problem has two stages, we are going to discuss when we can expect to have a moduli space and when they can be constructed.

Definition 2.1 (Fine Moduli Space). *Let F be a contravariant functor $F : \{\mathrm{Schemes}/S\} \to \{\mathrm{Sets}\}$ we say a scheme $X(F)$ and $\mathscr{U}(F) \in F(X(F))$ represents the functor finely if for every scheme Y the map $\phi(Y) : \mathrm{hom}(Y, X(F)) \to F(Y)$ given by $g : Y \to X(F)$ maps to the square*

$$\begin{array}{ccc} g^*\mathscr{U}(F) & \longrightarrow & \mathscr{U}(F) \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\;g\;} & X(F) \end{array}$$

is an isomorphism.

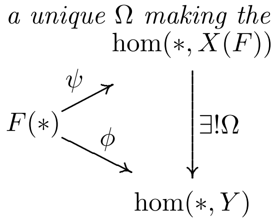
Note, there simply may not be such a pair $(X(F), \mathscr{U}(F))$. A rule of thumb for evaluating whether a fine moduli space exists is that if $E \in A/\sim$ has $\mathrm{Aut}(E) \neq \mathrm{id}$, then there is no chance.

Why? Suppose that there is an E with nontrivial automorphism group. Then one can use $\mathrm{Aut}(E)$ to define a nontrivial family $\pi : X \to B$ such that every fiber $X_0 \simeq E$. If there were a fine moduli space $X(F)$ for families of $A/\sim$, then we could show that $X \to B$ is trivial.

Indeed: If $(X(F), \mathscr{U}(F))$ is a fine moduli space, then $X \simeq \eta_X \mathscr{U}(F)$, where $\eta_X : B \to X(F)$ is the defining map. That is, $X \simeq B \times_{pt} \mathscr{U}(F)$, which is trivial, and so assuming such a nontrivial family can be constructed, there can be no such moduli space.

There are two ways to deal with this. One way is to enlarge the category to, say, Stacks. Another way is to ask less of the moduli space.

Definition 2.2 (Coarse Moduli Space). *Given a contravariant functor F : $\{Scheme\} \to \{Sets\}$, we say that the scheme $X(F)/S$ coarsely represents the functor if there is natural transformation $\phi : F \to \hom(-, X(F))$ such that $\phi(\operatorname{Spec} k) : F(\operatorname{Spec} k) = A/\sim \to \hom(\operatorname{Spec} k, X(F)) \simeq X(F)$ is bijective and for any S-scheme Y and any natural transformation $\psi : F \to \hom(-, Y)$ we get a unique Ω making the following diagram commute:*

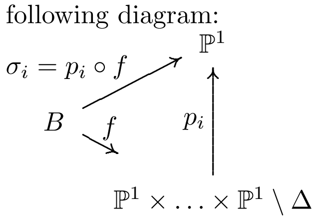


We are now going to look at the moduli space $F(\mathbb{P}^1, n)$, the moduli space of n-points on $\mathbb{P}^1$. It is $\mathbb{P}^1 \times \dots \mathbb{P}^1 \setminus \Delta$ where Δ is the locus where two or more points coincide.

We will show that this represents the functor $F : \{Var\} \to \{Sets\}$ that takes a variety B to $B \times \mathbb{P}^1 \to B$ the projection onto B with n sections $\sigma_1, \dots, \sigma_n$: $B \to B \times \mathbb{P}^1$ with disjoint images.

If $b \in B$ is a point then $\pi^{-1}(b) = b \times \mathbb{P}^1 \simeq \mathbb{P}^1$ and $\sigma_1(b), \dots, \sigma_n(b) \in \mathbb{P}^1$ are n disjoint points. Lets sketch a natural isomorphism $F \to \hom(*, F(\mathbb{P}^1, n))$.

Suppose that $B \in Obj(\{Var\})$ and $f \in \hom(B, F(\mathbb{P}^1, n))$. We have the following diagram:



If $B \times \mathbb{P}^1 \to B$ with $\sigma_1, \dots, \sigma_n$ is an element of $F(B)$, then $\sigma : B \to F(\mathbb{P}^1, n) = \mathbb{P}^1 \times \dots \times \mathbb{P}^1 \setminus \Delta$ by $b \mapsto (\sigma_1(b), \dots, \sigma_n(b))$.

Part II: $G(k, n)$

Notation: The underlying set of points of $G(k, n)$ correspond to the k-dimensional vector subspaces of a fixed n-dimensional vector space.

$$w \in G(k, n) \leftrightarrow W^k \subset V^n \leftrightarrow \mathbb{P}(W^k) \subset \mathbb{P}(V^n) \simeq \mathbb{P}^{n-1}$$

. So points in $G(k, n)$ also correspond to $(k-1)$ dimensional projective subspaces of $n-1$ dimensional projective space, so some point denote the Grassmanian by $\mathbb{G}(k-1,n-1)$. Note that $\mathbb{P}^{n-1}=G(1,n)$.

Set $V=\mathbb{A}^n$ and choose any decomposition into coordinate subspaces $\mathbb{A}^n = \mathbb{A}^k \oplus \mathbb{A}^{n-k}$, and so any linear operator $A:\mathbb{A}^k \to \mathbb{A}^{n-k}$ has $Graph(A)$ a k-dimensional subspace of $\mathbb{A}^n$ and so corresponds to a point $\Gamma_A \in G(k,n)$.

Taking all points of $G(k,n)$ obtained in this way, we get an open subset $U \subset G(k,n)$ which is just an isomorphism $\hom(\mathbb{A}^k,\mathbb{A}^{n-k})$ where eU is an $(n-k)k$ dimensional space. We check that the transition functions between decompositions are ok, and that this makes sense. This generalizes the way in which we give plücker coordinates to $\mathbb{P}^n$.

The Plücker coordinates come from the classical Plücker embedding which is a map $G(k,n)\to\mathbb{P}(\Lambda^k V)$ by $W\subset V \mapsto \Lambda^k W$. For a basis $b_1,\ldots,b_n$ of V, any basis $\beta_1,\ldots,\beta_k$ of W be be expressed $\beta_i=\sum_{j=1}^n c_i^j b_j$ where the c_i^j form a rectangular k by n matrix, and we take the coordinates $p_{i_1,\ldots,i_k}$ to be the determinants of the minors.

So then we get a mapping into $\binom{n}{k}-1$ dimensional projective space. There is an inequality $\binom{n}{k}-1 \geq k(n-k)$, and if $k\geq 2$, this is strict. So we get relations, called the Plücker relations.

To write down the Plücker relations, we'll just assume that the coordinates $p_{i_1,\ldots,i_k}$ are defined for any distinct indices $i_1,\ldots,i_k$ and that changing the order of indices changes the sign of the coordinates once for each transposition.

Theorem 2.1. *1. For any two sequences $1\leq i_1<\ldots<i_{k-1}\leq n$ and $1\leq j_1<\ldots<j_{k+1}\leq n$, the Plücker coordinates on $G(k,n)$ satisfy $\sum_{a=1}^{k+1}(-1)^a p_{i_1,\ldots,i_k,j_a}p_{j_1,\ldots,\hat{j}_a,\ldots,j_{k+1}}=0$ and any vector $(p_{i_1,\ldots,i_k})\in\Lambda^k V^n$ satisfying such relations is the Plücker coordinates of some k-dimensional subspaces of V.*

2. Moreover, the graded ideal of all polynomials in $p_{i_1,\ldots,i_k}$ vanishing on the image of $G(k,n)$ is generated by these "Plücker polynomials."

This generalizes to Chow Varieties $G(k,d,n)=k$ dimensional vector subspaces of degree d in a fixed vector space of dimension n. $G(k,n)=G(k,1,n)$.

Part 1 is proved by Griffiths and Harris and part 2 is proved by Hodge and Pedoe.

Say V is an n-dimensional vector space and $e_1,\ldots,e_n$ is the standard basis, a set of Plücker coordinates $\{p_{i_1,\ldots,i_k}\}_{1\leq i_1<\ldots<i_k\leq n}$ represents a point $W\in Gr(k,n)$ iff $R=\sum_{1\leq i_1<\ldots<i_k\leq n}p_{i_1,\ldots,i_k}e_{i_1}\wedge\ldots\wedge e_{i_k}\in\Lambda^k V$ is decomposable, $R=v_1\wedge\ldots\wedge v_k$ for $v_i\in V$.

The coordinate ring of $G(k,n)$ in the Plücker embedding is $B=\oplus_d B_d$ is the quotient ring of the polynomial ring by the Plücker ideal: $k[p_{i_1,\ldots,i_k}]_{1\leq i_1<\ldots<i_k\leq n}$ modulo the (Plücker Relations), these are often referred to as bracket polynomials, with the "bracket" being $[i_1,\ldots,i_k]$.

3 Lecture 3

The two main sources for Grassmanians and Chow Varieties these are GKZ "Discriminants, Resultants and Multidimensional Determinants" and Kollar's "Rational Curves on Algebraic Varieties."

Cayley, published a new analytic representation of curves in projective space in the quarterly journal of mathematics volume 3, 225-234 and volume 5 81-86 in 1860 and 1862.

What Cayley did

Given a curve D on $\mathbb{P}^3$, let $C_a(D) \subset G(2,4) = \{\text{lines in } \mathbb{P}^3\}$ be the set of lines in $\mathbb{P}^3$ that meet D. Cayley proves that $C_a(D)$ is a divisor on $G(2,4)$ and so any curve $D \subset \mathbb{P}^3$ we can associate a graded ring $B = \oplus B_d$ which is factorial and, in particular, codimension 1 subvarieties of $G(k,n)$ are determined by f up to a constant multiple. So to each curve $D \subset \mathbb{P}^3$, we can define a degree d "Cayley Form" on $G(2,4)$.

This is the basis of the definition of Chow varieties, written by Chow and v.d.Waerden to generalize the approach invented by Cayley.

Hodge-Pedoe "Methods of AG" and Samuel "Méthods d'Algébre Abstaite en Géométrie Algébrique" published by Springer in 1955.

We should think of the Chow Varieties and Hilbert Schemes as different compactifications of the same space, but the Hilbert Schemes are easier to get your hands on.

Add to references: Chow and v.d. Waerden "Zur Algebraischen Geometrie, ix" in Math Annalen 113, 692-704.

Task: To construct $G(k,d,n)$ the Chow Variety of $k-1$ dimensional projective subvarieties of $\mathbb{P}^{n-1}$ of degree d.

If we want to construct $G(n-1,d,n)$, what do we do? If $X \subset \mathbb{P}^{n-1}$ is a hypersurface of degree d, then X is determined by a homogeneous polynomial of degree d and can take a vector space of all such and projectivize it.

Associate to $X \subset \mathbb{P}^{n-1}$ of dim $k-1$ and degree d a hypersurface $Z(X) \subset G(n-k,n)$ of degree d. As $G(1,n) = \mathbb{P}^{n-1}$, take $H \subset G(1,n)$. What is the degree of H? It is the intersection number of H with a general line. We can compute $\deg H$ by intersecting H with a generic pencil p_{NM} defined as follows: $N^{k-2} \subset M^k \subset \mathbb{P}^{n-1}$ and N, M are projective subspaces of dimension $k-2, k$. Then $P_{NM} = \{P \in G(k,n) | N \subset P \subset M\}$ Why is this one dimensional? Because it is $\mathbb{P}(V^{k-1}) \subset \mathbb{P}(V^k) \subset \mathbb{P}(V^{k+2})$, so $L = V^k/V^{k-1}$ is $\mathbb{P}^1$ contained in $\mathbb{P}^2$.

Recall that $B = \oplus B_d$ is the coordinate ring of $G(k,n)$.

Proposition 3.1. *1. B is factorial (ie, each element $f \in B$ has a decomposition into irreducible factors which is unique up to a constant multiple and a permutation of the factors)*

2. If $Z \subset G(k,n)$ is an irreducible hypersurface of degree d, then there exists $f \in B_p$ such that Z is given by $f = 0$.

We will not prove this, see pages 98-99 of GKZ

Let $X \subset \mathbb{P}^{n-1}$ be a fixed $k-1$ dimensional degree d subvariety. We'll define $Z(X)$ the associated hypersurface as a set $Z(X) = \{L \in G(n-k,n) | L \cap X \neq \emptyset\}$.

Proposition 3.2. *$Z(X)$ is an irreducible hypersurface in $G(n-k,n)$ of degree d.*

Proof. Put $B(X) = \{(x,L) | x \in X, L \in G(n-k,n)$ and $x \in L\}$. We have a projection p to X and a map q to $Z(X)$ which is birational. Why? For $L \in Z(X)$, $q^{-1}(L)$ is generically a point.

So to prove that $Z(X)$ is irreducible, we can show that $B(X)$ is irreducible.

The map p is a Grassmanian fibration, $x \in X$ has $p^{-1}(x) \simeq \{L \in GL(n-k,n) | x \in L\} = G(n-k-1,n-1)$. So $B(X)$ is irreducible and so $Z(X)$ is irreducible. We need to intersect $Z(X)$ with a generic pencil p_{NM} in $G(n-k,n)$ to show it has degree d.

$N^{n-k-2} \subset M^{n-k} \subset \mathbb{P}^{n-1}$ and count the number of elements $L \in Z(X)$ such that $N \subset L \subset M$. As $\dim M = n-k$ and $\dim X = k-1$, So in $\mathbb{P}^{n-1}$, $\dim(X \cap M) = 0$. In fact, since $\deg X = d$, $X \cap M = \{x_1, \ldots, x_d\}$ and so any such L is the projective space of $N \notin x_i$, and so there are d such L.

We know that $Z(X)$ is defined by the vanishing of some element $R_X \in B_d$ which is unique up to a constant factor. ☐

Notations/Definitions: $Z(X)$ is the associated hypersurface, R_X is the Chow form of X, after fixing a basis for B_d, can write R_X in terms of coordinates which we call Chow coordinates.

Facts:

1. X can be recovered from its Chow coordinates.
2. Can use Plücker coordinates in the case $d = 1$ to write R_X as a bracket polynomial.

By a $(k-1)$-dimensional algebraic cycle in $\mathbb{P}^{n-1}$, we mean a formal finite linear combination $X = \sum n_i X_i$ with nonnegative integer coefficients and where $X_i \subset \mathbb{P}^{n-1}$ are irreducible closed subvarieties of dimension $k-1$. $\deg X = \sum m_i \deg X_i$

$G(k,d,n)$ = the set of all $(k-1)$-dimensional algebraic cycles on $\mathbb{P}^{n-1}$ of degree d.

If X is a $(k-1)$ cycle of degree d, then its Chow form is $R_X = \prod R_{X_i}^{m_i} \in B_d$.

Theorem 3.3 (Chow-van der Waerden)**.** *The map $X \mapsto R_X$ defines an embedding of the set $G(k,d,n)$ into $\mathbb{P}(B_d)$ as a closed algebraic variety.*

The variety $G(k,d,n)$ with the structure induced from this embedding is called the Chow Variety and the embedding is called the Chow Embedding.

$G(2,2,4)$ is the set of 1 dimensional varieties in $\mathbb{P}^3$ of degree 2. These are all plane quadrics, because if we take $X \subset \mathbb{P}^3$ an irreducible curve of degree 3, x, y, z to be three non-collinear points in X, then x, y, z span a plane containing X, because the plane intersects the curve in three points, which is greater than the degree of the curve, and so the curve must be in the plane.

Claim: $G(2,2,4) = C \cup D$ and describe $C \cap D$ for some C, D.

Now we see that all the 1-cycles in $\mathbb{P}^3$ of degree 2 are unions of two lines or irreducible plane quadrics. Define C to be the set of plane quadrics and D to be the pairs of lines.

Monday: Examples and a proof of Chow-vdWaerden Theorem by Wednesday.

4 Lecture 4

Recall: $G(k,d,n)$ is the moduli space parameterizing dimension $k-1$ cycles of degree d in $\mathbb{P}^{n-1}$. $G(k,1,n) = G(k,n)$, the Grassmanian.

Reminder of the Chow embedding: $G(k,n) \to G(n-k,n)$ my taking χ to $X(\chi)$ where X is the associated hypersurface operator.

$X(\chi) = \{L = \mathbb{P}(W^{n-k}) \in G(n-k,n) | L \cap \chi \neq \emptyset\} \subset G(n-k,n)$.

L is then $n-k-1$ dimensional and it is guaranteed to intersect any $\mathbb{P}(V^{k+1}) \subset \mathbb{P}^n$.

Let $B = \oplus B_d$ be the coordinate ring of $G(n-k,n)$ then $X(\chi) = Z(R_\chi)$ where $R_\chi \in B_d$. This defines the Chow embedding, $G(k,d,n) \to \mathbb{P}(B_d)$ by $X \mapsto [R_X]$ with R_X the Chow form.

Then $[R_X] \in \mathbb{P}(B_d)$ is given by chow coordinates.

Intermezzo: Construction of associated hypersurface is an analog or generalization of the construction of a dual variety.

$\mathbb{P} = \mathbb{P}^n$ and $\mathbb{P}^*$ the set of hypersurfaces in $\mathbb{P}^n$, and then $\mathbb{P}^n = G(1,n+1) \mapsto G(n,n+1) = (\mathbb{P}^n)^*$.

The construction gives us a way of identifying $\mathbb{P}$ with $(\mathbb{P}^*)^*$.

$G(1,1,n+1)$ is then the degree 1, dimension 0 subvarieties of $\mathbb{P}^n$, or the points.

For $\mathbb{P}^2$, the dualization map takes $p \mapsto X(p) = \{L \in G(n,n+1) : p \subset L\}$.

In general, $p^\vee$ is a projective hypersurface of $\mathbb{P}^*$ so this gives $\mathbb{P} \to (\mathbb{P}^*)^*$ by $p \mapsto p^\vee$

Elements $X = \sum m_i X_i \in G(k,d,n)$ have $\deg X = \sum m_i \deg X_i = d$ and X_i are irreducible dimension $(k-1)$ projective subvarieties of degree di. $R_X = \prod_{i=1}^n R_{X_i}^{m_i}$.

So then $X(X_i) \subset G(n-k,n)$ is codimension 1. R_{X_i} is a polynomial whose coeffs are given by polynomials in k linear forms $f_1, \ldots, f_k$. $R_{X_i} \in B_{d_i}$ is a degree d_i form that vanishes when X_i intersects the hypersurface.

$L \in G(n-k,n)$ are codimension k, linear subspaces of $\mathbb{P}(V^n)$, ie, $L = \cap_{i=1}^k Z(f_i)$ where the f_i are linear forms on $\mathbb{C}^n \simeq V^n$

So we think of the R_{X_i} as a polynomial whose coefficients are polynomials in k indeterminate linear forms $f_1, \ldots, f_k$, so we think of $R_{X_i}(f_1, \ldots, f_k)$.

Then $\mathbb{P}^{k-1} = \mathbb{P}(\mathbb{C}^k) \to \mathbb{P}(S^d\mathbb{C}^k) = \mathbb{P}^{n-1}$ where $n = \binom{k+d-1}{d}$ by the Veronese map.

So we have $y_1 - y_0 = x_0^{d-1}x_1 - x_0^d x$.

Part II: Zero Cycles. $G(1,d,n)$ degree d zero-cycles on $\mathbb{P}^{n-1}$.

Weil: Proved over $\mathbb{C}$ that $\mathrm{Sym}^d(\mathbb{P}^{n-1}) \simeq G(1,d,n)$ and note that Neeman in "0-Cycles in $\mathbb{P}^n$" shows that this is false in positive characteristic, in Advances in Math, 89, 1991, 217-227

Recall the definition of symmetric products: If X is a quasiprojective variety, then $\mathrm{Sym}^d(X)$ is informally the quotient of X^d by the action of S_d.

Suppose that X is affine, and R the affine coordinate ring. Then $R^{\otimes d} = R \otimes \ldots \otimes R$ is the coordinate ring of X^d. The coordinate ring of $\mathrm{Sym}^d(X)$ is the set of S^d-invariants of R^d. These are regular functions $f(x_1, \ldots, x_d)$ with $x_i \in X$ such that permuting x_i doesn't change f.

We want that $\gamma : \mathrm{Sym}^d(\mathbb{P}^{n-1}) \to G(1,d,n)$ by $\{x_1, \ldots, x_d\} \mapsto \sum x_i$ is an isomorphism over $\mathbb{C}$

First: this is a set-theoretically a bijection. An affine open subset $\mathbb{C}^{n-1} \subset \mathbb{P}^{n-1}$ is given by $x_n = 1$. Then we compare $\mathrm{Sym}^d(\mathbb{C}^{n-1})$ with the image of $\gamma|_{\mathrm{Sym}^d(\mathbb{C}^{n-1})}$. The coordinate ring of $\mathrm{Sym}^d(\mathbb{C}^{n-1})$ is $S(d, n-1)$ consisting of regular functions $f(\vec{x}_1, \ldots, \vec{x}_d)$ where $\vec{x}_i = (x_{i,1}, \ldots, x_{i,n-1})$ that are symmetric.

For d scalar variables $x_1, \ldots, x_d$, the symmetric functions can be expressed in terms of elementary symmetric functions given by the equation $e_k(x_1, \ldots, x_d) = \sum_{1 \le i_1 \le \ldots \le i_k \le d} x_{i_1} \ldots x_{i_k}$ satisfying $1 + \sum_{i \ge 1} e_i(x_1, \ldots, x_d) t^i = \prod_{i=1}^d (1 + x_i t)$.

Now we take $t_1, \ldots, t_{n-1}$, and look at the product $\prod +i = 1^d (1 + x_{i,1} t_1 + x_{i,2} t_2 + \ldots + x_{i,n-1} t_{n-1})$ and this gives a polynomial in $t_{i_1}, \ldots, t_{i_k}$ whose coefficients are symmetric. These coefficients are the elementary symmetric polynomials in vector variables.

So we have $1 + \sum e_{k_1, \ldots, k_{n-1}}(\vec{x}_1, \ldots, \vec{x}_d) t_1^{k_1} \ldots t_{n-1}^{k_{n-1}}$.

Note that for any d vectors in $\mathbb{C}^{n-1} \subset \mathbb{P}^{n-1}$, computing these symmetric functions gives the Chow coordinates for the cycle $\sum \vec{x}_i = X$, $[R_X] \in \mathbb{P}(B_d)$.

Use $t_1, \ldots, t_n$ as coordinates on $\mathbb{P}^{n-1}$ and $G(1,d,n) \to G(n-1,n) \simeq \mathbb{P}^{n-1}$ by $X \mapsto \chi(X) = Z(R_X)$ $R_X(t_1, \ldots, t_n) = \prod_{i=1}^d (x_{i,1} T_1 + \ldots + t_{i,n-1} t_{n-1} + 1 t_n)$.

Proposition 4.1 (2.3 in GKZ, page 134)**.** *Let $Z^d(\mathbb{C}^{n-1})$ be the open subset of $G(1,d,n)$ consisting of cycles $X = \sum X_i$ with $X_i \in \mathbb{C}^{n-1}$. The ring of regular functions on $Z^d(\mathbb{C}^{n-1})$ is the subring of $S(d, n-1)$ generated by by elementary symmetric functions.*

We have that $A(Z^d(\mathbb{C}^{n-1})) \subset S(d, n-1) = A(\mathrm{Sym}^d \mathbb{C}^{n-1})$. And that $\mathrm{Sym}^d(\mathbb{C}^{n-1}) \to Z^d(\mathbb{C}^{n-1}) \subset G(k,d,n)$.

Now we use the fundamental theorem for symmetric polynomials in vector variables:

Theorem 4.2 (Fundamental Theorem for Symmetric Polynomials)**.** *Any symmetric polynomial in vector variables $\vec{x}_1, \ldots, \vec{x}_d \in \mathbb{C}^{n-1}$ can be expressed as a polynomial in the elementary symmetric polynomials. This expression is generally not unique.*

Facts about $G(1,d,n)$:

1. $\mathrm{Sym}^d(\mathbb{P}^1) \simeq G(1,d,2) \simeq \mathbb{P}^d = \mathbb{P}(S^d\mathbb{C}^2)$
2. $\mathrm{Sym}^d(\mathbb{P}^{n-1}) \simeq G(1,d,n)$ rational.

The subscheme in $\mathbb{P}(S^d\mathbb{C}^n)$ defined by these equations is not reduced (proved by Weyman).

Tropical: Speyer Theorems, Sturmfels and Speyer

5 Lecture 5

Today we will start to prove the Chow-vd Waerden Embedding Theorem, and to do it we will need more information on Resultants and Stiefel Coordinates on Grassmanians

Let $W^k \subset V^n$ and given a basis $e_1, \ldots, e_n$ for V and a basis $b_1, \ldots, b_k$ for W, with $b_i = \sum_{j=1}^n c_{ij} e_j$, so we can map W to the matrix $[c_{ik}] = M$. M has rank k. So if $g \in GL(k)$, we have $R(gM) = R(M)$ and the Stiefel coordinates $\|c_{ij}\|$ are not unique.

Let $S(k,n)$ denote the Stiefel variety of all $(k \times n)$ matrices of rank k. $G(k,n) = S(k,n)/GL(k)$, and so $\mathbb{P}^{n-1} = G(1,n) = S(1,n)/GL(1) = \mathbb{C}^n \setminus \{0\}/\mathbb{C}^*$.

Recall the resultants setup: $G(k,d,n) \to G(n-k,d,n)$ by $X \mapsto Z(X) = Z(R_X)$ and $X = \mathbb{P}(W^k) \subset \mathbb{P}(V^n) = \mathbb{P}^{n-1}$, if $\deg X = d$, $\deg R_X = d$ and $R_X \in B_d$, where $B = \oplus_{i \geq 0} B_d = A(G(n-k,d,n))$.

So $Z(X) = \{G \in G(n-k,d,n) | H \cap X \neq \emptyset\} \subset G(n-k,d,n)$ is a hypersurface, and $H = \cap_{i=1}^k Z(f_i) \subseteq \mathbb{P}^{n-1}$ where $f_i \in \hom(\mathbb{C}^n, \mathbb{C})$.

The elements of $G(k,d,n)$ are cycles $\sum m_i X_i$ with $m_i \geq 0$, X_i irreducible dimension $k-1$ projective subvarieties of $\mathbb{P}^{n-1}$ of degree d.

$S(n-k,n) \to S(n-k,n)/GL(n-k) \simeq G(n-k,n) \supset Z(X)$, and then $\bar{Z}(X) = P^*(Z(X)) \subset M_{n-k,n}$. So now we know that $\bar{Z}(X) \subset M_{n-k,n}$ is a hypersurface, and so $\bar{Z}(X) = Z(\tilde{R}_X)$. Here, R_X is just the d-form on $G(n-k,n) = S(n-k,n)/GL(n-k)$ that cuts out $Z(X)$ and $\tilde{R}_X$ is the lift of the form. This takes our matrix $[c_{ij}]$ to the matrix $\begin{bmatrix} 1 & 0 & \ldots & 0 & a_{1,k+1} & \ldots & a_{1,n} \\ 0 & 1 & \ldots & 0 & \vdots & & \vdots \\ 0 & \ldots & 1 & 0 & \vdots & & \vdots \\ 0 & 0 & \ldots & 1 & a_{k,k+1} & \ldots & a_{k,n} \end{bmatrix}$, where the a_{ij} are the Stiefel Coordinates for $\tilde{R}_X$.

So now $Z(X) = \{H \in G(n-k,n) | H \cap X \neq \emptyset\}$, $Z(X) = Z(R_X)$ and $H = \cap_{i=1}^k Z(f_i)$ where f_i are linear forms.

$f_i = \sum_{j=1}^n a_{ij} x_j$, and we need our field to not be of characteristic two.

How to recover X from R_X?

Fact: A $(k-1)$-dimensional irreducible subvariety $X \subset \mathbb{P}^{n-1}$ is uniquely determined by its associated hypersurface $Z(X)$. More precisely, $p \in \mathbb{P}^{n-1}$ lies in X iff every $(n-k-1)$-dimensional plane containing p belong to $Z(X)$.

So let $x \in \mathbb{P}^{n-1}$ be given.

Recall that a skew symmetric form on a vector space V over a field k is a bilinear form $S : V \times V \to k$ $(v,w) \mapsto S(v,w)$ with $S(v,w) = -S(w,v)$. If $v \in \mathbb{C}^n$, $\mathbb{P}(v) = x \in \mathbb{P}^{n-1}$, $Z(S(v,-)) \ni x$, as $S(v,v) = -S(v,v)$, so $2S(v,v) = 0$, and so $S(v,v) = 0$ as k is not of char 2.

So now if $x \in \mathbb{P}^{n-1}$, $x = \mathbb{P}(\vec{x})$, and take $i_X = S(x,-) : \mathbb{C}^n \to \mathbb{C}$ by $y \mapsto S(x,y)$ is a one-form on $\mathbb{P}^{n-1}$. $Z(i_X) \subset \mathbb{P}^{n-1}$ is a hyperplane passing through x.

Corollary 5.1 (2.6 p 102 GKZ)**.** *Let $X^{k-1} \subset \mathbb{P}^{n-1}$ be an irreducible subvariety and $\tilde{R}_X(f_1,\ldots,f_k)$ the X-resultant. Let us consider k indeterminate skew-symmetric forms $S_1(x,-),\ldots,S_k(x,-)$ which are given by the equations $S_i(x,y) = \sum_{j,r=1}^n s_{jr}^{(i)} x_j y_r$ where for each i, $[s_{jr}^{(i)}]$ is skew symmetric matrix of otherwise independent variables. For any $x \in \mathbb{C}^n$, consider the following polynomials in coeffs $s_{jr}^{(i)}$ of all forms $p(\vec{x},(s_{jr}^{(i)})) = \tilde{R}_X(i_X(S_1),\ldots,i_X(S_r))$. Then the coefficients of p are polynomials in $\vec{x}$ which form a system of equations of degree d that cut out X set theoretically.*

This result was known to vd Waerden.

Proposition 5.2 (Catanese, 1991)**.** *These equations in fact cut out X scheme theoretically.*

Comment: Could you use these equations to define $Trop(X)$ for any variety $X \subset \mathbb{P}^{n-1}$?

The goal is to give algebraic conditions which, if satisfied by $F \in B_d$, then imply that $F = R_X$ for some $X \in G(k,d,n)$. Let's see what we know about $R_X \in B_d$.

WLOG, we can assume X is irreducible. Let $f_1,\ldots,f_{k-1}$ be any $k-1$ 1-forms, then $\Pi = \cap_{i=1}^{k-1} Z(f_i) \subset \mathbb{P}^{n-1}$ and $n-1-(k-1) = n-k$, and so X intersects Π.

If $X \cap \Pi = \{x^1,\ldots,x^\ell\}$, where $x^i \in \mathbb{C}^n$, then for any $f_k \in \hom(\mathbb{C}^n,\mathbb{C})$, $R_X(f_1,\ldots,f_k) = 0$ iff $x \in Z(f_k)$ for some $x \in \{x^1,\ldots,x^\ell\}$.

So we have $R_X(f_1,\ldots,f_{k-1},-)$ taking $f_k \mapsto R_X(f_1,\ldots,f_k)$, as X has degree d, then $R_X(f_1,\ldots,f_k)$ factors into d linear forms depending on f_j and $x^1,\ldots,x^d$.

So $R_X(f_1,\ldots,f_k) = (f_k,x^1)\ldots(f_k,x^d)$. As each $x^i = x^i(f_1,\ldots,f_{k-1})$, we know that $f_i(x^i) = 0$ for $1 \le i \le k-1$, and that if $S_1(x,y),\ldots,S_k(x,y)$ are any k indeterminate skew-symmetric forms on $\mathbb{C}^n$, then $R_X(i_X(s_1),\ldots,i_X(s_k)) = 0$. We will refer to them by the numbers:

1. $R_X \in B_d$
2. Factors into linear forms $R_X(f_1,\ldots,f_k) = (f_k,x^1)\ldots(f_k,x^d)$
3. $f_i(x^i) = 0$ for $1 \le i \le k-1$
4. If $s_1(x,y),\ldots,s_k(x,y)$ are k indeterminate skew-symmetric forms on $\mathbb{C}^n$, then $R_X(i_x(s_1),\ldots,i_x(s_k)) = 0$

This proves one direction of the following proposition:

Proposition 5.3. *A polynomial $F(f_1,\ldots,f_k)$ of degree dk is the Chow form of some cycle from $G(k,d,n)$ iff it satisfies the following:*

1. *$F \in B_d$*

2. *For any fixed $f_1, \dots, f_{k-1} \in (\mathbb{C}^n)^*$, the polynomial $F(f_1, \dots, f_{k-1}, -) : f_k \mapsto F(f_1, \dots, f_k)$ decomposes into d linear factors $(f_k, x^1) \dots (f_k, x^d)$ for $x^i \in \mathbb{C}^n$. Furthermore, if $F(f_1, \dots, f_{k-1}, -) \not\equiv 0$, then the points $x^i = x^i(f_1, \dots, f_{k-1})$ satisfy the following two conditions:*

3. *$f_i(x^i) = 0$ for $1 \le i \le k-1$*

4. *If $s_1(x, y), \dots, s_k(x, y)$ be any k skew-symmetric forms. Then we have that $F(i_{x^i}(s_1), \dots, i_{x^i}(s_k)) = 0$ for all i.*

Miknalkin vs Speyer-Sturmfels Tropical Geometry

6 Lecture 6

Next Wednesday, we will have a visitor talking about his thesis on birational geometry of $\bar{M}_{0,n}$.

We will forget about the rest of the proof of Chow-vdWaerden, and move on to moduli of curves.

Today we will introduce $\bar{M}_{g,n}$, and on Monday will construct the moduli space $\bar{M}_{0,n} = G(2, n)//^{ch}T^{n-1}$, (we will use Chow varieties rather than Hilbert Schemes).

Wednesday will be Matt Simpson, Monday fall break, and on the next Wednesday, we will do Hilbert Schemes.

$\bar{M}_g$. This will be a coarse moduli space, and $\bar{M}_g \supseteq M_g$ is a compactification called the Deligne-Mumford compactification.

M_g has closed points corresponding to isomorphism classes of smooth curves of genus g.

Old questions often could not be answered until this point of view was adopted, for instance, can you write down the "general" smooth curve of genus g in terms of equations? Rephrase in terms of M_g, the answer is yes for $g \le 14$, and no for $g \ge 22$.

$\bar{M}_g$ has closed points corresponding to isomorphism classes of stable curves of genus g.

The stable curves are the ones which have at worst nodal singularities and a finite number of automorphisms.

$\bar{M}_g \setminus M_g = \partial \bar{M}_g = \{\cup \Delta_i\}$, where for $i > 0$, $\Delta_i \subseteq \bar{M}_g$, consists of the closure of the locus of curves whose generic element is a nodal curve $C_i \cup C_{g-i}$.

Fact: The set of points in $\bar{M}_g$ corresponding to curves with k nodes has codimension k. For $g \ge 2$, $\Delta_0 =$ the closure of the set of $g-1$ genus curves with a single node.

Whenever $3g - 3 + n \ge 0$, we can construct a moduli space $M_{g,n}$ whose closed points are in correspondence with isomorphism classes $(n+1)$-tuples $(C, p_1, \dots, p_n)$, where C is a curve of genus g, and $p_1, \dots, p_n$ are n distinct labeled points on C where $(C, p_1, \dots, p_n) \cong (C', p'_1, \dots, p'_n)$ if there is $\varphi : C \to C'$ an isomorphism such that $\varphi(p_i) = p'_i$ for all i.

Then $\bar{M}_{g,n}$ is called the Deligne-Mumford-Knudsen compactification. The closed points will correspond to isomorphism classes of stable $(n+1)$-tuples $(C, p_1, \ldots, p_n)$ where C has at worst nodal singularities and $p_1, \ldots, p_n$ are distinct simple points on C. Stability requires that the $(n+1)$-tuples have finitely many automorphisms.

The boundary is $\bar{M}_{g,n} \setminus M_{g,n} = \cup \Delta_{I,i}$ where $\Delta_{I,i}$ is the closure of the locus where I is a subset of $\{p_1, \ldots, p_n\}$ on the branch of the curve near the node with genus i, and the others are on the other branch.

A toric variety is a variety on which a torus acts.

Suppose we have a toric variety X_Δ with a torus T. The set of torus invariant divisors defines a stratification of X_Δ that tells us a lot. $X_\Delta \supset S^1 = \cup D_i \supset S^2 = \cup(D_i \cap D_j) \supset \ldots$ where the D_i are the torus invariant divisors.

Then S^n is the set of torus invariant fixed points, S^{n-1} is the set of fixed curves, etc.

There is an analogous stratification for $\bar{M}_{g,n}$.

$S^1 = \partial \bar{M}_{g,n} = \cup \Delta_{I,i} \subset \bar{M}_{g,n}$. So then $S^2 = \cup(\Delta_{I,i} \cap \Delta_{J,j})$, etcetera. So S^{3g-4+n} is a union of curves and S^{3g-3+n} is a union of points.

This analogy is interesting because people ask questions about $\bar{M}_{g,n}$ that they know are true on toric varieties, related to the stratification.

In the case $g = 0$, Fulton studied it and $\bar{M}_{0,n}$ is a fine moduli space. $M_{0,n+1}$ can be thought of as $\mathbb{A}^1$ with n marked points. $F(X, n)$, the moduli space of n points on a scheme X was studied by Fulton and Macpherson, and they gave a compactification $X[n]$, which, in the case of $X = \mathbb{P}^1$ is $\bar{M}_{0,n}$.

Conjecture 6.1 (Fulton's Conjecture)**.** *On a toric variety, a cycle of codimension k can be expressed as an effective sum of components of S^k. Is this true for $\bar{M}_{0,n}$?*

Evidence that it is true: For 0-cycles, yes.

Seven years ago, Keel and Vermeire (thesis Harvard) showed that Fulton's conjecture is false for cycles of dimension $d \geq 2$.

Question open for $d = 1$ and known true up to $\bar{M}_{0,n}$ for $n \leq 7$.

Matt Simpson's Thesis gives support for this conjecture, Hacon and McKernan are working on this with Mori Theory, and Maclagan and Gibney are working on this from a different perspective.

We know that the cycle structure for X_Δ depends on the stratification, so Fulton conjecture is only for $\bar{M}_{0,n}$.

Mori Theory: $Nef(X_\Delta) = \cap_{\sigma \in S_0} C_\sigma$. If X is a projective scheme, then a divisor D on X is nef iff $D \cdot C \geq 0$ for all curves C on X. Then $\text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R} \supseteq Nef(X) = \{\text{cone generated by the nef divisors}\}$, and this is the set $Ample(X)$.

$f : X \to Y$ with Y projective then there exists an ample divisor A on Y, $D = f^*A$ is a divisor on X, and D is nef. To see this, let $C \subset X$ be any curve $f_*(D \cdot C) = f_*(f^*A \cdot C) = A \cdot f_*C$.

If X_Δ is a toric variety, then $Nef(X_\Delta) = \cap C_\sigma$ where σ is a torus fixed point. This is also equal to $\{D \in \text{Pic}(X_\Delta) \otimes_{\mathbb{Z}} \mathbb{R} | D \cdot C_\tau \geq 0, \forall \tau \in S_1\}$ and $C_\sigma = \{\sum a_i D_i$ where $a_i \in \mathbb{R}_{\geq 0}$ and D_i is a torus invariant divisor such that $D_i \notin S(\sigma)\}$.

$S(\sigma)$ then consists of all torus invariant divisors that one intersects to get σ. These are called splits of σ.

The analogy for $\bar{M}_{g,n}$. Faber did the case $n = 0$, $g = 2, 3$ and part of $g = 4$.

Call the irreducible components of the 1-dimensional part of the boundary stratification of $\bar{M}_{g,n}$ F-curves (for Faber). Then $F_{g,n} = \{D \in \mathrm{Pic}(\bar{M}_{g,n}) \otimes \mathbb{R} | D \cdot C \geq 0$ for all F-curves$\}$. The Nef cone sits inside this. Faber and Pondhaipanda $g = 4$.

Fulton's Conjecture for curves implies this for all g and is equivalence for $g = 0$.

7 Lecture 7

$M_{0,n}$ is a fine moduli space corresponding to $(C, p_1, \ldots, p_n)$ with $C \cong \mathbb{P}^1$ and $p_1, \ldots, p_n$ are distinct marked points.

$\bar{M}_{0,n}$ is the moduli space of isomorphism classes of stable n-pointed curves C trees of $\mathbb{P}^1$'s, each comp ≥ 3 markings.

One problem is to describe $Nef\bar{M}_{0,n}$.

The nef cone for toric varieties X:

A toric variety has a stratification $S_0 \subset S_1 \subset \ldots \subset S^2 \subset S^1$ where S^1 is the union of torus invariant divisors, S^2 is intersections of pairs of elements of S^1 and then S_1 is the dimension 1 stratum and S_0 is the set of fixed points. (lower index is dimension, upper is codimension)

Fact: For toric varieties, S_0, S_1 determine the Nef Cone. Given $\sigma \in S_0$ a torus fixed point, ie, $\sigma = \cap D_i$ for $D_i \in S(\sigma)$, the set of torus invariant divisors that one intersects to get σ. Define $C^\sigma = \{\sum a_i D_i | D_i \in S^1 \setminus S(\sigma), a_i \geq 0 \in \mathbb{R}\} \subset \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ and $C = \cap_{\sigma \in S_0} C^\sigma = Nef(X)$ if X is projective. Otherwise it is the globally generated divisors. This is also equal to $\{D \in \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R} | D \cdot C \geq 0, C$ are irreducible components of $S_1\}$.

$\bar{M}_{0,n}$ "feels" like a toric variety in the sense that $B^1 = \bar{M}_{0,n} \setminus M_{0,n} = \cup_{I \subset \{1,\ldots,n\}} \Delta_I$ such that $|I|, |I^c| \geq 2$. This is the set of curves with at least one node.

So we get a stratification with B^i's, where B^1 is the set of curves with at least one node, B^2 is the set of curves with at least two, etc. $B^1 = \cup \Delta_I$, $B^2 = \cup(\Delta_I \cap \Delta_J)$ and in general B^k the set of curves with at least k nodes.

Eventually, you get to B_1, the local of curves with $(n-4)$ nodes, and the components are intersections of $n-4$ boundary divisors. Then B_0 is the set of curves with $(n-3)$ nodes.

Now we look at $B_1 = \cup C_{(A,B,C,N \setminus (A \cup B \cup C))}$, that is, a curve is determined by a partition of $N = \{1, \ldots, n\}$.

We note that $\Delta_I \simeq \bar{M}_{0,|I|+1} \times \bar{M}_{0,|I^c|+1}$. And then we can see that $\Delta_I \cap \Delta_J \simeq \bar{M}_{0,|J^1|+1} \times \bar{M}_{0,|J^2|+1} \times \bar{M}_{0,|I|+1}$.

So $Nef\bar{M}_{0,n} \subset \{D \in \mathrm{Pic}(\bar{M}_{0,n}) | D \cdot C_{A,B,C} \geq 0$ where $A, B, C, N \setminus (A \cup B \cup C)$ is a partition of $N\}$.

If $\sigma \in B_0$ is a zero dimensional strata, that is, σ is the intersection of $n-3$ boundary divisors on $\bar{M}_{0,n}$, the elements of $S(\sigma)$, then define $C^\sigma_{0,n} =$

$\{\sum a_i\delta_I | a_i \geq 0 \in \mathbb{R}, \delta_I \notin S(\sigma)\}$ where $\delta_I = [\Delta_I]$ =the class of Δ_I in $\mathrm{Pic}(\bar{M}_{0,n})$

Let $C_{0,n} = \cap_{\sigma\in B_0} C^{\sigma}_{0,n}$.

Theorem 7.1 (Gibney-Maclagan). $C_{0,n} \subset Nef(\bar{M}_{0,n})$

So it is unknown if there are any equalities in $C_{0,n} \subseteq Nef(\bar{M}_{0,n}) \subset F_{0,n}$.

Conjecture 7.1 (F-Conjecture). $Nef(\bar{M}_{0,n}) = F_{0,n}$.

This is known for $n \leq 7$.

Conjecture 7.2 (C-Conjecture). $C_{0,n} = Nef\bar{M}_{0,n}$

This is true for $n \leq 6$.

How would one investigate this question? How could you tell if $F_{0,n} \subset C_{0,n}$?

Goal: to show that these are the same is for $D \in F_{0,n} = \{D \in \mathrm{Pic}\otimes\mathbb{R} | D \cdot C_{A,B,C} \geq 0\}$ if $\sigma \in S_0$, show $D \in C^{\sigma}_{0,n} = \{\sum a_I\delta_I | a_I \in \mathbb{R}^{\geq 0}, I \notin S(\sigma)\}$.

The dual graph of σ is then a trivalent tree with n labeled leaves. The graph corresponds to a curve, the vertices are connected components and half edges are marked points.

Given $D \in F_{0,n}$ and given $\sigma \in B_0$, we want to show that $D \in C^{\sigma}_{0,n}$ where $\sigma = \cap_{I\in S(\sigma)}\Delta_I$. We want to show that D is an effective sum of boundary divisors not supported on the $S(\sigma)$.

Each planar realization of Γ_σ gives a basis for the Picard group of $\bar{M}_{0,n}$ consisting of the boundary divisors not containing δ_I for $I \in S(\sigma)$.

Given $\sigma \in B_0$, each of the 2^{n-3} planar realizations of Γ_σ gives a good σ-compatible basis for $\mathrm{Pic}\,\bar{M}_{0,n}$.

$D = \sum_{I\notin S(\sigma)} a_I\delta_I$. So can we show that the a_I are nonnegative?

Given some Γ_σ, we can find a numbering of the vertices of an n-gon, and divide it into blocks and gaps (nonempty subsets containing only elements adjacent and with any two blocks separated by a gap) and then we get a basis $\delta_{B_1,\ldots,B_i}$ taking all of them over the i's.

So for $n = 5$, then $D = \sum(D \cdot C_{B_1,G_1,B_2,G_2})\delta_{B_1,B_2}$. So as a corollary, if $D \in F_{0,5}$, then $D \in C_{0,5}$.

For $n = 6$, we have a basis and the only possible three block sequence is $\delta_{1,4,6}$, so this is the only possible coefficient that can be bad. It is possible to show that if $C^1_{1,4,6}$ is negative, then in a different basis, $C^2_{1,3,6}$ is positive.

8 Lecture 8 - Matt Simpson

Algebraic Families of Pointed Spheres and Topological Invariants

Let $T \to C$ be a family of curves. If there exists a moduli space M, then this is the same as $C \to M$. We might want a somewhat more concrete classification, or at least be able to bound the fibers of these families with numerical properties.

These properties connect to subvarieties, cones, intersection theory, and the birational geometry of M.

We want to look at $M_{0,n}$m which is the collection of maps $X \to T$ with n sections that are flat with fibers isomorphic to $\mathbb{P}^1$ with distinct marked points.

Two pointed spheres C, C' are isomorphic if $f : C \to C'$ an isomorphism and $f(p_i) = p'_i$. So $M_{0,n} = (\mathbb{P}^1)^n \setminus \Delta / PGL(2)$ which is isomorphic to $(\mathbb{P}^1)^{n-1} \times \{0\} \times \{1\} \times \{\infty\} \setminus \Delta$ by taking $p_1 \to 0, p_2 \to 1, p_3 \to \infty$.

What we want to do next is compactifiy. We want it to still be a moduli space and we'd like the boundary to still tell us a lot. We also want it to be the least singular thing possible.

The "obvious" compactification is $(\mathbb{P}^1)^n / PGL(2)$, which just allows marked points to coincide. The problem with this space is that it is nonseparated. (there are strictly semistable points) Also, the boundary doesn't contain very much information.

The more standard version is the Knudsen-Grothendieck compactification $\bar{M}_{0,n}$ which is connected, compact, $g_a = 0$ curves with the number of marked points plus the number of nodes on a component is at least 3.

This moduli space is smooth and projective.

Example 8.1. *If $n = 4$, then $M_{0,4} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and so $\bar{M}_{0,4} = \mathbb{P}^1$, with the extra points given by $p \to 0$, $p \to 1$ and $p \to \infty$ giving curves with two components.*

The reason that geometers like it is because $\partial \bar{M}_{0,n}$ is a normall crossing divisor. That is, it is locally the intersection look like $s_1 = s_2 = \ldots = 0$.

The boundary is also a disjoint union of the points parameterized by curves with k nodes. Each of these gives a codimension k part of the boundary.

So then divisors: for $I \subset \{1, \ldots, n\}$, $2 \leq |I| \leq n-2$, define D_I to be the set of curves with one node such that I points are on one component and I^c points are on the other.

Fact: $D_I \simeq \bar{M}_{0,|I|+1} \times \bar{M}_{0,|I^c|+1}$, and there is inductively structure on the boundary.

Back to classifying families. Keel showed that the cohomology ring (which equals the Chow ring) is generated by D_I and H^{2k} is the codimension k strata.

Theorem 8.1 (Blowup Theorem of Kapranov)**.** *$\bar{M}_{0,n}$ is the blowup of $\mathbb{P}^{n-3}$ along linear subspaces.*

Proof. We will only sketch the proof.

Let $p_1 = (1, 0, \ldots, 0)$ etc through p_{n-2} and $p_{n-1} = (1, \ldots, 1)$.

Then $\bar{M}_{0,n}$ =blow-up along $p_1, \ldots, p_{n-1}$, along linear spans of pairs, etc through the linear spans of $n-3$ of the points. The first collection are like D_{in}, the second are D_{ijn} and so on. □

These are the exceptional divisors D_I, and so $\mathrm{Pic}(\bar{M}_{0,n})$ is generated by the D_I.

Exercise: $\bar{M}_{0,4} = Bl_{4pts}\mathbb{P}^2$ and $\mathbb{P}^2 \setminus$ lines blown up is $(\mathbb{P}^1 \setminus \{0, 1, \infty\})^2 \setminus \Delta = M_{0,4}$.

Problem: Any k-dimensional subvariety can then be written as $\sum a_i(k - strata)$, but what are the coefficients of a_i? We don't understand the subvarieties unless we know what the a_i can be.

Special case: 1 dimensional families are subcurves of $\bar{M}_{0,n}$.

An example of a numerical property is "How many fibers of some type can be in this family?"

If $n = 4$, then $\bar{M}_{0,n} = \mathbb{P}^1$, and so $C \to \mathbb{P}^1$ is constant or surjective. Thus, our family $T \to C$ has either constant fibers or else has every kind of fiber. In particular, if it isn't constant, there must be at least three singular fibers.

Fact about the 1-strata: This is the locus of curves with at least $n-4$ nodes. This must have one component isomorphic to $\bar{M}_{0,4} = \mathbb{P}^1$ with four special points and the rest of the components with three special points. Let A, B, C, D be the collections of special points separated by the four on the component with four special points. So this partitions $\{1, \ldots, n\}$ into $A \cup B \cup C \cup D$.

Definition 8.1 (Cone of Curves). *The cone of curves $\bar{NE}$ is the closure of the collection of $\sum a_i[C_i]$ where C_i is a curve and $a_i \geq 0$ is a subset of $H_2 \otimes \mathbb{R}$.*

Each of these can be written as some sum of 1-strata, not necessarily effective.

Conjecture 8.1 (Fulton). $\bar{NE} = \{\sum a_i(1 - strata)$ *where the* $a_i \geq 0\}$

In particular, this implies that it is a finite polyhedral cone.

Fulton's Conjecture is known to be true for $n \leq 7$, which was prove by Gibney, Keel, Morrison, McKernon.

There is also an S_n-equivariant version known for $n \leq 24$.

There are two established methods for trying to prove conjectures like this one:

1. Use the inductive structure of $\bar{M}_{0,n}$.
2. Contract $\bar{M}_{0,n} \to X$ and study the cone on X and the pullback map. (Dual cone curves are $\{\sum D \in H^2 \otimes \mathbb{R} | D \cdot C \geq 0 \forall C \in \bar{NE}\}$ is the Nef cone.)

Matt Simpson's Own Work:

Take $\rho : \bar{M}_{0,n} \to \bar{M}_{0,A}$ the weighted pointed spheres.

Definition 8.2 (Contraction). *Map $\rho : \bar{M}_{0,n} \to X$ with X projective and normal and ρ has connected fibers $\rho_*(\mathscr{O}_{\bar{M}_{0,n}}) = \mathscr{O}_X$.*

Any map is a composition of a contraction and a finite map.

We take ρ to be birational (in fact, an isomorphism on $M_{0,n}$) and the 1-strata contracted has $w(|A| + |B| + |C|) \leq 1$. So Fulton's conjecture for $M_{0,A}$ is just that the cone of curves generated by 1-strata not satisfying the above inequalities.

Theorem 8.2 (Simpson). *For "smallest weights" the nef cone conjectured by Fulton's conjectures for $M_{0,A}$ is the nef cone (in S_n-equivariant case)*

The method of proof is to construct $M_{0,A}$ by using GIT.

Conjecture: $K + \alpha D$ is ample on $M_{0,A}$ (for $\bar{M}_{0,A}$ it is true for $\alpha \in [0, 1/2]$ and $\rho^*(K + \alpha D)$ defines ρ.)

Fulton's conjecture implies this.

9 Lecture 9

We've looked at $\bar{M}_{0,n} \supset M_{0,n}$.

Kapronov's Two Good Ideas:

Theorem 9.1 (Castelnuevo)**.** *There is a unique rational normal curve in $\mathbb{P}^n$ passing through $m+3$ points in general prosition.*

Recall: A rational normal curve in $\mathbb{P}^n$ is a curve projectively equivalence to the Veronese embedding of $\mathbb{P}^1$ in $\mathbb{P}^n$.

So the first idea is that $M_{0,n}$ is the space of configurations of n points in $\mathbb{P}^{n-3}$ fixed in general position.

Kapranov says that points in $\bar{M}_{0,n}$ correspond to configurations of points in $\mathbb{P}^{n-3}$ via the "blow up construction"

Intermediate Step: Instead of looking at $n+1$ points in general position in $\mathbb{P}^{n-2}$, Kapranov considers the set of rational normal curves in $\mathbb{P}^{n-2}$ passing through n points in general position. (ie, no $n-1$ of thse points lies on a hyperplane) and this can be identified with $M_{0,n}$.

Theorem 9.2. *Take n points $p_1, \ldots, p_n \in \mathbb{P}^{n-2}$ in general position. Define $V_0(p_1, \ldots, p_n)$ be the set of Veronese curves in $\mathbb{P}^{n-2}$ passing through $p_1, \ldots, p_n$. Consider $V_0(p_1, \ldots, p_n) \subset \mathscr{H}$ the Hilbert scheme of dimension 1 and degree $n-2$ subvarieties of $\mathbb{P}^{n-2}$ or $V_0(p_1, \ldots, p_n) \subset Ch$ the Chow Variety $G(2, n-2, n-1)$.*

Then $V_H(p_1, \ldots, p_n)$ is the closure of $V_0(p_1, \ldots, p_n)$ in $\mathscr{H}$, we have that $V_H(p_1, \ldots, p_n) \simeq \bar{M}_{0,n}$.

$V_{Ch}(p_1, \ldots, p_n)$ is the closure of $V_0(p-1, \ldots, p_n)$ in Ch, and so we have $V_{Ch}(p_1, \ldots, p_n) \simeq \bar{M}_{0,n}$.

The first step is to identify $M_{0,n} \leftrightarrow V_0(p_1, \ldots, p_n)$ by $(C, x_1, \ldots, x_n)$ corresponds to $\varphi : C \to \mathbb{P}^{n-2}$ with $x_i \mapsto p_i$. and we want $\bar{M}_{0,n}$ ot be the closure of V_0.

The goal is to take $(C, x_1, \ldots, x_n) \in \bar{M}_{0,n}$ and define an embedding $C \to \mathbb{P}^{n-2}$ with $x_i \mapsto p_i$.

That is, we want a very ample line bundle with $n-1$ global sections. We will use the line bundle $\Omega_C(x_1 + x_2 + \ldots + x_n) \leftrightarrow \varphi$.

Today we will define this line bundle and prove the following lemma:

Lemma 9.3. *Let $(C, x_1, \ldots, x_n) \in \bar{M}_{0,n}$ and $\varphi : C \to \mathbb{P}^{n-2}$ the embedding defined by $\Omega_C(x_1 + \ldots + x_n)$. The the images of x_i are in general position.*

First we talk about Ω_C when $C \simeq \mathbb{P}^1$. In this case, $\Omega_C = \mathscr{O}_{\mathbb{P}^1}(-2)$.

Suppose that A is a k-algebra. We can form the module of universal differentials $(DA, d : A \to DA)$ which has the property that for any A-module M, $d_M : A \to M$ there is a unique homomorphism $DA \to M$ such that everything commutes.

In our context, $(X, \mathscr{O}_X)$ is a scheme over k and $U \subset X$ has $\mathscr{O}_X(U)$ a k-algebra. (Kapranov works over $\mathbb{C}$)

Then $D\Omega_X(U) = \{\sum f_n dg_n | f_n, g_n \in \mathscr{O}_X(U)\}$. This defines a sheaf that we denote by Ω^1_X or just Ω_X. If $A = \mathscr{O}_X(U) = k[x_1, \ldots, x_n]$, then $DA \simeq A^n$.

Claim 1: Elements of DA are of teh form $\sum_{i=1}^n f_i(\vec{x})dx_i$ (chain rule on dg_n).

We have $\delta : A^n \to DA$ by $(f_1, \ldots, f_n) \mapsto \sum f_i dx_i = 0$, we know that δ is onto. We can use the universal property of DA to show that it is injective. $A^n = M$ is an A-module, and $d_{M'} : A \to A^n = M$ by $f \mapsto \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$.

Then if $f \mapsto \sum \frac{\partial f}{\partial x_i} dx_i = 0$ then all the partials must be the zero polynomial, by the commutativity of the diagram in the definition. From this, we conclude injectivity.

So $Dk[x] \simeq k[x]$. We want to undersatand $\Omega^1\mathbb{P}^1$. As $\mathbb{P}^1$ is $\operatorname{Spec} k[x] \cup \operatorname{Spec} k[x^{-1}]$, the ideal begine $\Omega^1\mathbb{P}^1 = \mathscr{O}_{\mathbb{P}^1}(-2)$ is that DA satisfies the following property: $S \subset A$ a multiplicative set, then $DA[S^{-1}] \simeq DA \otimes_A A[S^{-1}]$ and $d(a/S) = (Sda - adS)/S^2$.

So $Dk[x, x^{-1}] = D(k[x] \otimes_{k[x]} k[x, x^{-1}]) = \{\sum f/x^n df\}$. Now $\mathbb{P}^1$ is defined to be $\operatorname{Proj} k[x, y]$. Then $B = \oplus_{d \geq 0} B_d$, and $M = B[-2] = B[-2]_0 \oplus B[-2]_1 \oplus B[-2]_2 \oplus \ldots$ and so $\mathscr{O}(-2) = \tilde{B}[-2]$.

So for $U \subset X$, $f \in \tilde{B}[-2](U)$, then for all $x \in U$, there exists $x \in V \subset U$ with $f|_V = g/h$ for $g \in B[-2]_d = B_{d-2}$ and $h \in B_d$.

So $U_x = \operatorname{Spec} k[y/x]$ and $U_y = \operatorname{Spec} k[x/y]$. Then $\mathscr{O}(-2)(U_x) = \{\omega = f(y/x)/x^2$ with $f \in \mathscr{O}(U)$ of degree $0\}$ and $\mathscr{O}(-2)(U_y)$ is the same, with the x and y interchanged.

So $C \simeq \mathbb{P}^1$ and $\Omega^1 C = \mathscr{O}(-2)$.

What is Ω^1_C if C is a free of $\mathbb{P}^1$'s?

If f is meromorphic, then on an open set $U \subset C$ for every $z_0 \in U$ w can write $f = \sum a_n(z - z_0)^n$, and $\operatorname{Res}_{z_0}(f) = a_{-1}$.

So a section ω of Ω_C when C is a tree of $\mathbb{P}^1$'s satisfies

1. ω is regular at the smooth points of C.
2. If x is a point of self intersection and if C_1 and C_2 are brances of C near x, then $\omega|_{C_1}$ and $\omega|_{C_2}$ has at worst simple poles and $\operatorname{Res}(\omega|_{C_1}) = \operatorname{Res}(\omega|_{C_1})$.

We want to talk about $\Omega_C(x_1 + \ldots + x_n)$. This is $\Omega_C \otimes_{\mathscr{O}_C} \mathscr{O}_C(x_1 + \ldots + x_n)$. If $C \simeq \mathbb{P}^1$, then $\mathscr{O}_{\mathbb{P}^1}(x_1 + \ldots + x_n) = \mathscr{O}_{\mathbb{P}^1}(n)$, and $\Omega^1 = \mathscr{O}(-2)$, and so the tensor product is $\mathscr{O}(n-2)$. The global sections of $\mathscr{O}(n-2)$ has basis $x^{n-2}, x^{n-3}y, \ldots, xy^{n-3}, y^{n-2}$.

This defines a map $\mathbb{P}^1 \to \mathbb{P}(\Gamma(\mathscr{O}_{\mathbb{P}^1}(n-2)))$ which is the Veronese Embedding.

If $f \in \mathscr{O}_C(x_1 + \ldots + x_n)$, $(f) + x_1 + \ldots + x_n \geq 0$ is effective. So if f has a single pole at x and $n = 1$, then $(f) + x_1 = \sum n_i y_i - x_1 + x_1 = \sum n_i y_i$ is effective implies that $f \in \mathscr{O}_C(x_1)$.

We're going to prove the lemma which says that $(C, p_1, \ldots, p_n) \in \bar{M}_{0,n}$ gives a map $\varphi_{|\Omega_C(x_1 + \ldots + x_n)|} : C \to \mathbb{P}^{n-2}$ with $x_i \mapsto p_i$ then the p_i are in general position.

Proof. By induction on hte number of irreducible components of C. The base case is $C \simeq \mathbb{P}^1$.

For contradiction, in the case $C \simeq \mathbb{P}^1$ assume some $n-1$ of the points are not in general position. WLOG, say $p_2, \ldots, p_n$ lie on a hyperplane $H \subset \mathbb{P}^{n-2}$, $H = V(S)$. Theat is, $S(p_i) = 0$ for $i \geq 2$. Then $\varphi^* S \in \Gamma(\Omega_C(x_1 + \ldots + x_n))$ and $0 = S(\varphi(x_i)) = (\varphi^* S)(x_i)$ and so $\varphi^* S \in \Gamma(\Omega_C(x_1 + \ldots + x_n))$ vanishes on $x_2, \ldots, x_n$. In fact, $\varphi^* S \in \Gamma(\Omega_C(x_1)) = \Omega_{\mathbb{P}^1}(-1)$ which has no global section, and so we get a contradiction.

Assume the result for $\Omega_C(x_1 + \ldots + x_n)$, fix $k \geq 1$, if C is a curve with $\leq k$ components.

Now let C be a curve with $k+1$ components. Then take C' to be the curve C with one fewer components, an the result follows. □

10 Lecture 10

Theorem 10.1 (.01 in Kapranov's Paper)**.** *Take n points $p_1, \ldots, p_n$ in $\mathbb{P}^{n-2}$ in general position (no $n-1$ lie on a hyperplane) and let $V_0(p_1, \ldots, p_n)$ be the space of all Veronese curves in $\mathbb{P}^{n-3}$ passing through $p_1, \ldots, p_n$. Consider $V_0(p_1, \ldots, p_n)$ as a subvariety of the Hilbert Scheme $\mathscr{H}$ parameterizing subschemes of $\mathbb{P}^{n-2}$. Then*

1. *$M_{0,n} \simeq V_0(p_1, \ldots, p_n)$*
2. *$\bar{M}_{0,n} \simeq V(p_1, \ldots, p_n)$ the closure of V_0 in $\mathscr{H}$.*
3. *The analogues of 1 and 2 hold when we take the Chow variety $G(1, n-2, n-2)$.*

10.1 Introduction to the Hilbert Scheme

References:

1. Moduli of Curves by Morrison and Harris
2. Mumford 1966 "Lectures on Curves on Algebraic Varieties"
3. Mumford and Fogarty GIT
4. Kollar "Rational Curves on Algebraic Varieties"
5. Viehweg, E "Quasiprojective Moudli for Polarized Manifolds"
6. Grothendieck "Techniques de Construction et Théorémes d'existence en geometrie algébrique IV"

We define $\mathscr{H} = \cup \mathscr{H}_{p,r}$ to be the Hilbert scheme, where $\mathscr{H}_p$ is the hilbert scheme parameterizing families of subschemes of $\mathbb{P}^r_k$ with the same hilbert polynomial p.

Given a closed subscheme $X \subset \mathbb{P}^r_k$, described by a saturated ideal $I(X) \subset S = k[x_0, \ldots, x_r] \simeq \oplus_{i \geq 0} S_i$.

Suppose $F_1, \dots, F_n$ homogeneous, and $R = S/I(X) = \oplus R_i$. The basic idea is to associate to X a function $H(X, -) : \mathbb{N} \to \mathbb{N}$ by $i \mapsto H(X, i) = \dim_k(R_i)$. More generally, if M is any finitely generated S-module, $H(M, -) : \mathbb{N} \to \mathbb{N}$ takes $i \mapsto \dim_k(M_i)$.

Theorem 10.2 (Hilbert 1890)**.** *There exists a unique polynomial $p(X)$ such that $p(X, i) = H(X, i)$ for $i >> 0$. More generally, this is true for finitely generated modules.*

Hilbert Functions and polynomials are important for many reasons:

1. Keeps track of geometric information
   (a) $\deg p(X) = \dim X$
   (b) If $\dim X = 0$ then $p(X) = \deg X$.
   (c) In general, define $\deg X$ for $X \subset \mathbb{P}^r$ with $\dim(X) > 0$ to be $n!$ times the lead coefficient.
2. A family of closed subschemes of projective space is flat iff every fiber in the family has the same hilbert polynomial, so this gives a geometric interpretation of flatness.

Fact: The set of all subvarieties of $\mathbb{P}^r$ having the same Hilbert polynomials p is a scheme $\mathscr{H}_p$ that is a fine moduli space for the Hilbert Functor.
This theorem is due to Grothendieck.
The Hilbert scheme has good properties with respect to families.

1. If $X \subset \mathbb{P}^r_k \times B \to B$ is any flat family with hilert polynomial P, then there is a morphism $\varphi_X : B \to \mathscr{H}_P$ by $b \mapsto [X_b]$.
2. Given any scheme B over k, then the set of flat families over B with Hilbert Polynomial P is naturally identified with $\hom(B, \mathscr{H}_P)$.
3. All works over $\mathrm{Spec}(\mathbb{Z})$.

Definition 10.1 (Hilbert Functor)**.** *The Hilbert Functor h_P "the functor of flat families n $\mathbb{P}^r_{\mathbb{Z}}$ with hilbert poylnomial P" is h_P : (Schemes) $\to$ (Sets) given by B maps to the set of flat families over B with hilbert polynomial P.*

Theorem 10.3 (Grothendieck)**.** *There exists a scheme $\mathscr{H}_P$ whose functor of points is naturally isomorphic to h_P.*

Theorem 10.4 (Mumford 1962)**.** *There are Hilbert Schemes that are nonreduced even at points that correspond to nonsingular irreducible projective varieties.*

Lemma 10.5. *Let $p_1, \dots, p_n$ be n points in general position on $\mathbb{P}^{n-2}$, and let $V_0(p_1, \dots, p_n)$ be the subset of $\mathscr{H}$ corresponding to the set of Veronese curves passing through $p_1, \dots, p_n$. The $M_{0,n} \simeq V_0(p_1, \dots, p_n) \subset \mathscr{H}$.*

Proof. The plan is that we want to define an injective morphism $h : M_{0,n} \to \mathscr{H}$ with image $V_0(p_1, \ldots, p_n)$.

Fix $(\mathbb{P}^1, x_1, \ldots, x_n) \in M_{0,n}$. We use the line bundle $L = \Omega_{\mathbb{P}^1}(x_1 + \ldots + x_n)$ to define an embedding $\phi_L : \mathbb{P}^1 \to \mathbb{P}^{n-2}$. Then $x_i \mapsto y_i$ for some y_i in general position.

It is a fact that there is a projective transformation $T : \mathbb{P}^{n-2} \to \mathbb{P}^{n-2}$ with $T(y_i) = p_i$ and the image $T(C)$ passing through the points $p_1, \ldots, p_n$, and so this gives an identification of $(\mathbb{P}^1, x_1, \ldots, x_n)$ with a point $C \in V_0(p_1, \ldots, p_n)$.

Why is this representation of $(\mathbb{P}^1, x_1, \ldots, x_n)$ unique? If there are two isomorphic Veronese curves, C, C' representing it, then we want to show that this map extends to $F : \mathbb{P}^{n-2} \to \mathbb{P}^{n-2}$ that fixes n points, which means it must be the identity.

As C and C' are two Veronese curves, we have an isomorphism $C \to \mathbb{P}^1 \to C'$ taking $p_i \mapsto x_i \mapsto p_i$.

We proceed by identifying $\mathbb{P}^{n-2} \simeq (\mathbb{P}^{n-2})^*$, its dual, which is $\mathrm{Sym}^{n-2}(C) \xrightarrow{f} \mathrm{Sym}^{n-2}(C') \simeq (\mathbb{P}^{n-2})^* \simeq \mathbb{P}^{n-2}$ by $H \mapsto H \cap C \mapsto H \cap C'$. So if $C \subset \mathbb{P}^{n-2}$ is a curve of degree $n2$, then $C \cap H$ has $n-2$ points. □

Lemma 10.6. *Let $V(p_1, \ldots, p_n)$ be the closure of $V_0(p_1, \ldots, p_n) \subset \mathscr{H}$. Then $V(p_1, \ldots, p_n) \simeq \bar{M}_{0,n}$.*

Proof. We must define a map $\bar{M}_{0,n} \to \mathscr{H}$ which restricts to the right map on $M_{0,n}$.

If $\pi : \mathcal{C} \to \mathcal{S}$, $s_i : \mathcal{S} \to \mathcal{C}$ is a family of curves with n points with for each closed point $p \in \mathcal{S}$, $(\pi^{-1}(p) = \mathcal{C}_p, s_1(p), \ldots, s_n(p)) \in \bar{M}_{0,n}$. We want to construct a map $S \to \mathscr{H}$.

Note that $\pi_*(\Omega_{\mathcal{C}/\mathcal{S}}(s_1 + \ldots + s_n))$ is a vector bundle on S, and let $p \in \mathcal{S}$ be a closed point. Then $\pi_*(\Omega_{C/S}(s_1 + \ldots + s_n)) = \Gamma(\Omega_{C_p}(s_1(p) + \ldots + s_n(p)))$, this line bundle has $n-1$ linearly independent section $\sigma_1, \ldots, \sigma_{n-1}$.

So for each $p \in \mathcal{S}$ a closed point, $C_p \to \mathbb{P}(\Gamma(\Omega_{C_p}(s_1(p) + \ldots + s_n(p)))^\vee)$ by $x \mapsto \ell_x \subset \Gamma(\Omega_{C_p}(s_1(p) + \ldots + s_n(p)))^\vee = \hom(\Gamma, k)$, $\sum \lambda_i \sigma_i(x) \neq 0$.

These give an embedding $\mathcal{C} \to \mathbb{P}((\pi_*(\Omega_{C/S}(s_1 + \ldots + s_n)))^\vee)$. We can trivialize the bundle and get a map $C \to \mathbb{P}^{n-2} \times S$ over S. □

11 Lecture 11

Last time, we continued the proof that if $p_1, \ldots, p_n \in \mathbb{P}^{n-2}$ in general position and if $V_0(p_1, \ldots, p_n) \subset \mathscr{H}$ parametrizes Veronese curves in $\mathbb{P}^{n-2}$ passing through $p_1, \ldots, p_n$, then $M_{0,n} \simeq V_0(p_1, \ldots, p_n)$.

To clarify, we assumed that $M_{0,n}$ is a fine moduli space with universal curve $M_{0,n+1} \to M_{0,n}$, and more generally, that $\bar{M}_{0,n}$ is a fine moduli space with universal family $\bar{M}_{0,n+1} \to \bar{M}_{0,n}$ with projection map forgetting the $n+1$st marked point (and possibly contracting a component) In fact, this map takes the boundary to the boundary.

Kapranov shows that there is a morphism $\phi : V_0(p_1, \ldots, p_n) \to M_{0,n}$ and then proves that it is a bijection.

To show that there is a map, we want to see that there is a flat family $\mathscr{F} \to V_0(p_1, \ldots, p_n)$ whose fibers are n-pointed smooth genus zero cuves. We take U_{V_0} to be $V_0 \times_{\mathscr{H}} U \to V_0 \times \mathbb{P}^{n-2}$, which means that if $p \in V_0$, then $\pi^{-1}(p) = C_p \subset \mathbb{P}^{n-3}$ and $\pi^{-1}(0)$ passes through $p_1, \ldots, \sigma_n(p) = p_n$.

And so we have a map $\phi : V_0(p_1, \ldots, p_n) \to M_{0,n}$. The rest of the proof is showing that it is a bijection.

Why is it surjective? If $(\mathbb{P}^1, x_1, \ldots, x_n) \in M_{0,n}$, then using $\Omega_{\mathbb{P}^1}(x_1 + \ldots + x_n)$, we get a Veronese embedding $\mathbb{P}^1 \to \mathbb{P}^{n-2}$.

For part (b), we have $V(p_1, \ldots, p_n)$ to be the closure in $\mathscr{H}$ of $V_0(p_0, \ldots, p_n)$. This will imply that $V(p_1, \ldots, p_n) \simeq \bar{M}_{0,n}$.

Outline: Define a map of points $\bar{M}_{0,n} \to V(p_1, \ldots, p_n)$ which is bijective on closed points. Using that $\bar{M}_{0,n}$ is smooth over $\mathbb{C}$ and the properties of the map we conclude that it is an isomorphism.

We'll proove that for every scheme S, there is a natural bijective map $\gamma_S : \hom(S, \bar{M}_{0,n}) \to \hom(S, V(p_1, \ldots, p_n))$, which will give a morphism $\gamma : \bar{M}_{0,n} \to V(p_1, \ldots, p_n)$.

We must now prove the existence of γ_S. Given $\phi : S \to \bar{M}_{0,n}$, we want to construct $S \to V(p_1, \ldots, p_n) \subset \mathscr{H}$.

The map gives us a family $\mathcal{C} \to S$ and using $\pi_* \Omega_{C/S}(s_1 + \ldots + s_n)$ (where $s_i = \phi^* \sigma_i$ are sections) we define a map $C \to \mathbb{P}((\pi_* \Omega_{C/S}(s_1 + \ldots + s_n))^*)$ which can be trivialized. So over $p \in S$, we have this restricting to $\Omega_{C_p/\operatorname{Spec} k}(s_1(p) + \ldots + s_n(p))$, and so we have a map $C_p \to \mathbb{P}^{n-2}$ sending $s_i(p) = p_i$, and so we have a map $S \to \mathscr{H}$ by $p \mapsto [\pi^{-1}(p)]$. If $S = M_{0,n}$, then $\pi^{-1}(p)$ is just $\mathbb{P}^1$ in its Veronese embedding, and so any scheme mapping to $M_{0,n}$ will map into $V_0(p_1, \ldots, p_n)$.

And so $\bar{M}_{0,n} \subset V(p_1, \ldots, p_n) \subset \mathscr{H}$.

The upshot is that $\gamma_S : \hom(S, \bar{M}_{0,n}) \to \hom(S, V(p_1, \ldots, p_n))$ by $\phi \mapsto \phi_{S, C = \phi^* \bar{M}_{0,n+1} = S \times_{\bar{M}_{0,n}} \bar{M}_{0,n+1}}$.

Injectivity follows by construction. Why is it surjective? We have $S \to V(p_1, \ldots, p_n) \to \mathscr{H}$ and a universal family over $\mathscr{H}$, pulling it back all the way, we have a classifying morphism for the family $S \to \bar{M}_{0,n}$.

Next: Let $W_0(p_1, \ldots, p_n)$ be the locus in $Ch = G(2, n-2, n-1)$ the cycles of dimension 1 and degree n-2 in $\mathbb{P}^{n-2}$. Then $C = \sum a_i C_i$ with $a_i \in \mathbb{Z}^{\geq 0}$ and C_i irreducible curves in $\mathbb{P}^{n-2}$. Then $\deg C = \sum a_i \deg C_i = n-2$. So W_0 corresponds to the veronese curves passing through n fixed points $p_1, \ldots, p_n \in \mathbb{P}^{n-2}$ in general position.

Claim: $M_{0,n} \simeq W_0(p_1, \ldots, p_n)$. If W is the closure in Ch, then we also claim that $\bar{M}_{0,n} \simeq W$. How do we do this?

Fact: Any component of the Hilbert Scheme maps to a corresponding component of the Chow Variety. If $C \in \mathscr{H}$, then C scheme maps to $\sum \operatorname{mult}(C_i) C_i)$ summed over irreducible components of C.

Let H_{ver} be the component of $\mathscr{H}_{\mathbb{P}^{n-2}} = \mathscr{H}$ containing $V(p_1, \ldots, p_n)$. Then we have $\Phi : \mathscr{H}_{ver} \to Ch$ by $C \mapsto \sum m_i C_i$.

Restricting this to the actual Veronese curves, we have $\phi : V(p_1, \ldots, p_n) \to Ch$. This ϕ is a bijection of sets from a smooth variety.

Lemma 11.1. *Let $f : X \to Y$ be a morphism of complex varieties which is bijective on $\mathbb{C}$-points. Suppose that X is smooth and for all $x \in X$, $df_x : T_x(X) \to T_{f(x)}(Y)$ is injective. Then f is an isomorphism.*

Proof in Kapranov.

Lemma 11.2. *Let $C \subset \mathbb{P}^{n-2}$ belonging to $V(p_1, \ldots, p_n)$ and let $\xi \in T_C\mathscr{H}$ be a nonzero tangent vector to $\mathscr{H}$ at C. Then $d_C\phi(\xi)$ is a nonzero tangent vector.*

Proof in Kapranov.

So for $i \in [n]$, we have projections $\pi_i : \bar{M}_{0,n} \to \bar{M}_{0,n-1}$ by forgetting i.

Reinterpreting these maps, $\mathbb{P}_i^{n-3}$ is the projective space of lineas through $p_i \in \mathbb{P}^{n-2}$. Take $\pi_i : (\mathbb{P}^{n-2} \setminus \{p_i\}) \to \mathbb{P}^{n-3}$. If $C \in V(p_1, \ldots, p_n)$, and $C_i = \pi_i(C \setminus p_i) \subset \mathbb{P}^{n-3}$ passing through $\pi_i(p_j) = q_j$ for $j \neq i$, check that C_i has degree $n-3$.

Basic line bundles on $\bar{M}_{0,n}$.

Let $i = 1, \ldots, n$. Let L_i be the line bundle on $\bar{M}_{0,n}$ such that over the point $(C, x_1, \ldots, x_n) \in \bar{M}_{0,n}$, it looks like $(T_{x_i}C)^*$.

$\gamma_{L_i} : X \to \mathbb{P}(\Gamma(X, L_i)^*)$ is regular at $x \in X$ as long as not all global sections of L_i vanish at x.

So $(C, x_1, \ldots, x_n) \in \bar{M}_{0,n} \simeq V(p_1, \ldots, p_n) \simeq W(p_1, \ldots, p_n)$.

$C \to \mathbb{P}^{n-3}$ passes through $p_1, \ldots, p_n$.

Consider $\sigma_i : \bar{M}_{0,n} \to \mathbb{P}_i^{n-3}$ taking $(C, x_1, \ldots, x_n) \mapsto \ell_i$ where ℓ_i is the embedded tangent line to C at p_i.

Proposition 11.3.

1. $L_i \simeq \sigma_i^* \mathscr{O}_{\mathbb{P}_i^{n-5}}(1)$
2. $\dim \Gamma(\bar{M}_{0,n}, L_i) = n-2$
3. γ_{L_i} *everywhere regular birational morphisms*

$$\gamma_{L_i} : \bar{M}_{0,n} \to \mathbb{P}(\Gamma(M_{0,n}, L_i)^*) = \mathbb{P}^{n-3}$$

.

Next, we will study these birational maps and sequences of blowups of $\mathbb{P}^{n-3}$.

12 Lecture 12

Gelfand-MacPherson "Geometry in Grassmannians and a Generalization of the dilogam theorem" in Advances 1982 number 44, pages 279-312

MacPherson "The combinatorial Formula of Gabrielov, Gelfand and Losik for first Pontrjagin Class" in Sem Bourbaki no 497 1976-1977

1. First define these quotients
2. Example $G(k,n)//^\star(\mathbb{C}^*)^{n-1}$ where $\star$ is the Hilbert or Chow quotient
3. $(\mathbb{P}^{k-1})^n //^\star \mathrm{GL}(k)$

4. $2 \simeq 3$ (the Gelfand-MacPherson correspondence extended to these quotients)

5. $k = 2$ gives $\bar{M}_{0,n}$.

What are Chow Quotients?

Introduced fora special case by Kapranov, Sturmfels and Zalevinsky, quotients of Toric varieties. In Math Annalen in 1991.

Similar to construction of Hilbert Quotients by Pyalynicki-Birula, Sommerse in "A conjecture about compact Quotients By Tori" Advanced Studies in Pure Math 8 (1986) 59-68

Let H be an algebraic group acting on a scheme X. For $x \in X$ let Hx be the H-orbit of x and $\bar{H}x$ the closure of the orbit Hx in X. Then $\bar{H}x \subseteq X$ is a subscheme. If the action is "nice enough" (ie, reductive) then there is an open $U \subset X$ with $\dim(\bar{H}x) = r$ for all $x \in U$, and the $\bar{H}x$ all represent the same calss $\delta \in H_{2r}(X, \mathbb{Z})$. We can also assume that $U \subseteq X$ is such that U is H-invariant, and U/H is a "nice geometric quotient"

If there is such a Zariski open set $U \subset X$, then $U/X \to C_r(X, \delta)$ is a map to the Chow variety of r-cycles of homology class δ taking $\bar{H}x$ to $\bar{H}x$.

Boutlet constructed $C_r(X, \delta)$ as a projective variety in "Espace Analytique Reduil Des Cycles Analytiques Complexes, Compacts" page 1-158 of LEcture Notes in Math 482 by Springer-Verlag in 1975

An element of $C_r(X, \delta)$ is a finite formal sum $Z = \sum m_i Z_i$ with $m_i \in \mathbb{Z}^{\geq 0}$ and Z_i irreducible r-dimensional closed algebriac subset of X.

Definition 12.1 (Chow Quotient)**.** *The Chow Quotient $X//^{Ch}H$ is the closure of U/H in $C_r(X, \delta)$ which is a projective (and hence compact) variety*

Aside, $X \to \mathbb{P}^d$ a projective variety and H acts on X and $\mathbb{P}^d$. Then $X//^{Cd}H \to \mathbb{P}^d//^{Ch}H$, and the latter is a not necessarily normal toric variety

Theorem 12.1. *Let H be a reductive group acting on a projective variety X and $\mathscr{L}$ an ample line bundle on X and α a linearization (an extension of the H action on X to the line bundle $\mathscr{L}$). Then there is a regular birational morphism $\Pi_{\mathscr{L},\alpha} : X//^{Ch}H \to (X/H)_{\mathscr{L},\alpha}$, the GIT Quotient.*

Recall that for a projective variety X there is a fine moduli space $\mathscr{H}_X$ parameterizing all subschemes in X.

From any connected component K of $\mathscr{H}_X$, there is a regular morphism to a corresponding Chow variety by $Z \in K$, gives $Cyc(Z) = \sum \text{mult}_{Z_i}(Z) Z_i$ where the sum is taken over the dimension r components of Z.

Then we take $K \to C_r(X, \delta_K)$ by $Z \mapsto Cyc(Z)$.

We're in the situation of having a group H acting on a projective variety X and $U \subseteq X$ on which $\dim(\bar{H}x) = r$ for all $x \in U$ and all represent the same homology class $\delta \in H_{2r}(X, \mathbb{Z})$, and so we get $U/H \to \mathscr{H}_X$.

Definition 12.2 (Hilbert Quotient)**.** *$X//^{\mathscr{H}}H$, the Hilbert Quotient, is the closure of U/H in $\mathscr{H}_X$.*

We have $\Pi| : X//^{\mathscr{H}}H \to X//^{Ch}H$ is a birational morphism (proved by Kapranov). In general, the Chow quotient is more complicated.

12.1 II Lie Complexes and Chow Quotients of Grassmanians

Look at $G(k,n)$ the Grassmanian. By choosing a basis of V^n, then we can represent a point $P \in G(k,n)$ by a $(k \times n)$-matrix. So $H^n = \{\text{diag}[\lambda_1, \ldots, \lambda_n] | \lambda_i \in \mathbb{C}^*\} \simeq (\mathbb{C}^*)^n$ acts on $G(k,n)$.

Take $\mathbb{C}^* \subset H^n$ given by $\lambda_1 = \ldots = \lambda_n$, this acts trivially on $G(k,n)$, and so $H = H^{n-1} = H^n/(\mathbb{C}^*)^n$ acts on $G(k,n)$.

So now we are interested in describing the moduli space $G(k,n)//^{Ch}H$ and $G(k,n)//^{\mathscr{H}}H$, which are isomorphic.

The first thing to do is $U = G^0(k,n)$.

Tke a basis $x_1, \ldots, x_n$ for V^n. Notation: for $I \subseteq \{1, \ldots, n\}$, denote by L_I the subspace $x_i = 0$ for $i \in I$. Then $\dim L_I = n - |I|$. Denote by C_I the subsapce of C^n spanned by x_i for $i \in I$. Then $\dim C_I = |I|$.

Definition 12.3. *Call a k-dimensional subspace L of $\mathbb{C}^n = V$ geneic if for any $I \subset \{1, \ldots, n\}$, $L_I \cap L = 0$.*

Definition 12.4. *$G^0(k,n)$ consists of all points corresponding to $L \in G(k,n)$ generic.*

We call $G^0(k,n)$ the generic stratum.

Classically, the $(k-1)$ dimensional families of subspaces of $\mathbb{P}^{k-1}$ were called complexes. eg, a set of points in $\mathbb{P}^{k-1}$ is a $(k-1)$-dimensional family of $\mathbb{P}^{k-1}$.

$G(k,n)$ has set of points of $k-1$ dimensional projective subspaces of $\mathbb{P}^{n-1}$. $x \in G^0(k,n)$ has $\bar{H}x \subset G(k,n)$ and $\dim \bar{H}x = n-1$. Kapranov calls these closures of generic orbits Lie Complexes.

Proposition 12.2 (Fulton and MacPherson 1991 (in Kapranov))**.** *Each Lie complex is an $(n-1)$ dimensional variety and has just $\binom{n}{k}$ singular points.*

Tetrahedral complexes were first constructed by Lie and Klein.

1. Baker "Princples of Algebraic Geometry" Vol 3-4 Columbia University 1925
2. Jessop "A Treatise on the Line Complex" 1903
3. Gelfand-MacPherson "Geometry of Grassmanians"

Let $[x_1, \ldots, x_4]$ be coordinates on $\mathbb{P}^3$. The L_i is the coordinate plane $x_i = 0$. The configuration of these four planes gives a tetrahedron T.

$\ell \in G(2,4)$ is a line in $\mathbb{P}^3$, and ℓ doesn't lie in the intersection of the edges of the tetrahedron.

For $\ell \in G^0(2,n)$, we have $\{\ell \cap L_i\} = \{P_i\}$ are four distinct points and $(\ell, p_1, \ldots, p_4)$ is a configuration of 4 points on the line ℓ.

The cross ratio of the configuration of 4 points gives a map $G^0(2,4) \to \mathbb{C} \setminus \{0,1\} = \mathbb{P}^1 \setminus \{0,1,\infty\}$ by $\ell \mapsto r(\ell \cap L_1, \ell \cap L_2, \ell \cap L_3, \ell \cap L_4)$.

Let $\lambda \in \mathbb{C} \setminus \{0,1\}$. Let K_λ be the closure of the set of all $\ell \in G^0(2,4)$ with cross ratio λ. Then $G(2,4) \to \mathbb{P}^{\binom{4}{2}-1} = \mathbb{P}^5$, and so $K_\lambda = Z(p_{12}p_{34} + \lambda p_{13}p_{24})$.

Klein and Lie defined this complex.

13 Lecture 13

We have an action $G(k,n) \times (\mathbb{C}^*)^n \to G(k,n)$ by $W^k \subset V^n$ with basis $w_i = \sum c_{ij} v_j$ with action given by $[c_{ij}]\,\mathrm{diag}[\lambda_1, \ldots, \lambda_n]$. This action doesn't change the column space of the matrix. So in fact, $H^{n-1} = (\mathbb{C}^*)^n/\mathbb{C}^*$ acts on $G(k,n)$

$G^0(k,n)$ is the "generic stratum"$=\{L \in G(k,n) | \forall I \subset \{1, \ldots, n\}, |I| = k, L_I \cap L = 0\}$ where $L_I = Z(x_i | x \in I)$. So in fact $G^0(k,n)/H^{n-1}$ is a nice geometric quotient.

First we'll see that if $x \in G^0(k,n)$ then xH is an $(n-1)$ dimensional subset of $G(k,n)$. That is, xH is an $n-1$ dimensional family of elements of $G^0(k,n)$, that is, an $(n-1)$ dimensional family of projective subspaces of $\mathbb{P}^{n-1}$ is an example of a complex.

Kapranov calls $\bar{xH}$ a Lie complex because of work by Sophus Lie.

Klyachko gave an explicit formula for the $2(n-1)$ dimensional homology class δ of these Lie complexes in terms of Schubert cycles and for all $x \in G^0(k,n)$, $\bar{xH}$ represents some δ.

Paper: Orbits of the maximal torus on the flag space "Functional Analysis" 19 number 1, 1985, 77-78

Upshot is that we can define an embedding $G^0(k,n)/H^{n-1} \to Ch_{k-1}(n-1, \delta)$ the Chow variety of $k-1$ dimensional cycles in $\mathbb{P}^{n-1}$ of homology class δ.

Then, using Kapranov's definition, we get that $G(k,n)//^{Ch} H^{n-1}$ is equal to the closure of the image of $G^0(k,n)/H^{n-1}$ in $Ch_{k-1}(n-1, \delta)$. Kapranov calls the cycles in the boundary the generalized Lie complexes.

$G^0(2,4)/H^3 \simeq \mathbb{P}^1 \setminus \{0, 1, \infty\} = \mathbb{C} \setminus \{0, 1\}$. And so the closure is $\mathbb{P}^1$.

In the boundary, there are three generalized complexes.

Kapranov proves the following:

Theorem 13.1. *Let $Z = \sum c_i Z_i$ be a cycle from $G(k,n)//^{Ch} H^{n-1}$. Then C_i is either 1 or 0 for all i.*

To emphasize this, Kapranov refers to these cycles as $Z = \cup Z_i$.

Definition 13.1 (Configuration)**.** *An ordered collection $\vec{x} = (x_1, \ldots, x_n)$ of points $x_i \in \mathbb{P}^{k-1}$ is called a configuration.*

The set of all n configurations is $(\mathbb{P}^{k-1})^n$.

A configuration of points on $\mathbb{P}^{k-1}$ corresponds to a configurations of n hyperplanes on $(\mathbb{P}^{k-1})^\vee$.

We'll form the Chow quotient $(\mathbb{P}^{k-1})^n//^{Ch} GL(k)$ and compare it to $G(k,n)//^{Ch} H^{n-1}$, showing that they are isomorphic.

$GL(k)$ acts on $\mathbb{P}^{k-1}$ by matrix multiplication, and so this induces an action of $GL(k)$ on $(\mathbb{P}^{k-1})^n$.

Definition 13.2. *Duppose that $(x_1, \ldots, x_n) \in (\mathbb{P}^{k-1})^n$ is a configuration of points. We say that $\vec{x}$ is generic if any subset of i of them spans and $(i-1)$-dimensional subspace of $\mathbb{P}^{k-1}$*

$(\mathbb{P}^{k-1})^n_{gen}$ is the set of generic configurations.

If $n \leq k+1$, then the $GL(k)$ orbits of points $\vec{x}, \vec{y} \in (\mathbb{P}^{k-1})^n_{gen}$ then $GL(k)\vec{x} = GL(k)\vec{y}$, that is, the action is transitive on the generic points.

Assume that $n \geq k+2$.

In this case, for $\vec{x} \in (\mathbb{P}^{k-1})^n$, we have the dimension of the orbit is $k^2 - 1$.

In general, for $n \geq k+1$, the dimension of the stabilizer of $\vec{x}$ is 1.

Proposition 13.2. *The homology class of the closure of any $GL(k)$ orbit of a point in $(\mathbb{P}^{k-1})^n_{gen}$ is given by an explicit formula.*

All are the same.

So we get an embedding $(\mathbb{P}^{k-1})^n_{gen}/GL(k) \to Ch_{k^2-1}((\mathbb{P}^{k-1})^n, \delta)$.

Definition 13.3. $(\mathbb{P}^{k-1})^n//^{Ch}GL(k)$ *is the closure of the image of* $(\mathbb{P}^{k-1})^n_{gen}/GL(k)$ *in* $Ch_{k^2-1}((\mathbb{P}^{k-1})^n, \delta)$.

We'll first show that there are open sets $G(k,n)_{max} \subset G(k,n)$ and $(\mathbb{P}^{k-1})^n_{\max} \subset (\mathbb{P}^{k-1})^n$ such that the coset spaces (which are not in general varieties) have a bijection (the Gelfand-MacPherson correspondence)

Kapranov proves that this correspondence extends to an isomorphism $G(k,n)//^{Ch}H^{n-1} \simeq (\mathbb{P}^{k-1})^n//^{Ch}GL(k)$, where $G(k,n)_{\max} = \{L \in G(k,n) | \{L \cap H_i\}$ is a configuration of n hyperplanes on L with dimension $H_i \cap L = k-1\}$.

We want that the class of projective isomorphisms of configurations of n hyperplanes in $\mathbb{P}(L) = \mathbb{P}^{k-1}$ is equivalent to a $GL(k)$ orbit of a point in $(\mathbb{P}^{k-1})^n$.

$(\mathbb{P}^{k-1})^n_{\max} = \{\pi = (\pi_1, \ldots, \pi_n) \in (\mathbb{P}^{k-1})^n | \dim(GL(k)\pi) = k^2 - 1\}$.

We make various definitions now:

1. $M(k,n)$ is the set of $k \times n$ matrices
2. $M^0(k,n)$ the subset of $M(k,n)$ with rank k.
3. $M'(k,n)$ the matrices with nonzero columns.

And now note that $G(k,n) = M^0(k,n)/GL(k)$, $(\mathbb{P}^{k-1})^n = M'(k,n)/(\mathbb{C}^*)^n$.

Next time, we will consider the action of $GL(k) \times (\mathbb{C}^*)^n$ on $M(k,n)$, and compare things.

14 Lecture 14

What is the Gelfand-Macpherson Correspondence?

Let $L \in G^0(k,n)$, $L^k \subset V^n$. If $H_1, \ldots, H_n$ are the coordinate hyperplanes of V, $L \not\subseteq H_i$ for any H_i, then $\{L \cap H_i\}_{i=1}^n$ is a collection of n hyperplanes on L.

$L \cap H_i$ corresponds to a line in L^k and hence a point in $\mathbb{P}(L^*)$ giving a configuration of n points in $\mathbb{P}(L^*) \simeq \mathbb{P}^{k-1}$.

These H_i are given by a basis for $V^* = \hom(V, \mathbb{C})$. Say the coordinate basis is $f_i : V \to \mathbb{C}$. Then $H_i = \ker f_i = \{v \in V | f_i(v) = 0\}$.

$L \cap H_i = \ker(f_i|_L)$, $f_i|_L : K \to \mathbb{C}$, $f_i|_L \in L^* = \hom(L, \mathbb{C})$. Then $([f_1|_L], [f_2|_L], \ldots, [f_n|_L]) \in (\mathbb{P}(L^*))^n \simeq (\mathbb{P}^{k-1})^n$.

Check that $([f_1]|_L, \ldots, [f_n|_L])$ is in fact in the generic subset.

Then $(y_1, \ldots, y_n) \in (\mathbb{P}^{k-1})^n_{gen}$ if $\vec{y_1}, \ldots, \vec{y_n} \in \mathbb{C}^k$ there exists $L^k \subset V$ such that $L^k \cap H_i = \vec{y_i}$.

On open sets, the correspondence is due to G-M.

Kapranov proves that this extends to an isomorphiosm $G(k,n)//^{Ch}H \to (\mathbb{P}^{k-1})^n//^{Ch}Gl(k)$, where $H = (\mathbb{C}^*)^n$.

We will outline Kapranov's approach.

Write $G(k,n) \simeq M_0(k,n)/GL(k)$. Then $p : M_0(k,n) \to M_0(k,n)/GL(k)$ and $(\mathbb{P}^{k-1})^n \simeq M'(k,n)/H$

We define $\rho : M'(k,n) \to M'(k,n)/H = (\mathbb{P}^{k-1})^n$.

$M(k,n)$ vs $GL(k) \times H$ acts on $\mathbb{P}(M(k,n))$. Kapranov defines maps $\alpha : G(k,n)//^{Ch}H \to \mathbb{P}(M(k,n))//^{Ch}GL(k) \times H$ and $\beta : (\mathbb{P}^{k-1})^n//^{Ch}GL(k) \to \mathbb{P}(M(k,n))//^{Ch}GL(k) \times H$.

With α taking $Z = \sum Z_i \mapsto \sum \overline{p^{-1}(Z_i)}$, and β taking $W = \sum W_i \mapsto \sum \overline{\rho^{-1}(W_i)}$.

Kapranov uses Bartlet's Criterion to show that these are morphisms and argues that α^{-1} and β^{-1} exist and are morphisms.

There's a classical duality called "the association" by ABCOBLE Algebraic Geometry and Theta Functions, AMS Coll Pub Vol 10 1928. 1969 omits 3rd.

Read about this also in Dolgachev and Ortland "Point Sets in Projective Spaces and Theta Functions" in Asterisque 165, 1988

Kapranov shows there is an isomorphism of Chow quotient $A_{k,n} : (\mathbb{P}^{k-1})^n//^{Ch}\,\mathrm{GL}(k) \to (\mathbb{P}^{n-k-1})^n//^{Ch}GL(k)$. The codomain is isomorphic to $G(n-k,n)//^{Ch}H$ and the domain to $G(k,n)//^{Ch}H$.

For $n = 2k$, the source and target are the same, but the map is not the identity.

Definition 14.1. *If $x \in (\mathbb{P}^{k-1})^n$ and $y \in (\mathbb{P}^{n-k-1})^n$ are two configurations of points, then we say that x is associated to y is both of their $GL(k)$ orbits are maximal dimensional and x is taken to y by $A_{k,n}$ or y is taken to x by $A_{n-k,n}$.*

Special case: $n = 2k$, then $A_{k,2k} : (\mathbb{P}^{n-1})^{2k}//^{Ch}GL(k) \to (\mathbb{P}^{k-1})^{2k}//^{Ch}GL(k)$ and one can give criteria for when a configuration x is self-associated, via Matroid Theory.

For $k = 2$, $A_{2,n} : (\mathbb{P}^1)^n//^{Ch}GL2) \to (\mathbb{P}^{n-3})^n//^{Ch}GL(2)$. Now we note that $\bar{M}_{0,n} \simeq G(2,n)//^{Ch}H$.

This gives a second way to relate $\bar{M}_{0,n}$ with Veronese Curves.

Now we define an isomorphism $G(k,n) \to G(n-k,n)$ by taking $L \in G(k,n)$, $L^k \subset V^n$ and mapping it to $L^\perp \in G(n-k,n)$, the subset of V^* given by $f \in V^*$ with $f|_L = 0$.

H acts on $G(k,n)$ and induces an action on $G(n-k,n)$. If $h \in H$, then $h(L^\perp) = h(\{f : V \to \mathbb{C} | f|_L = 0\}) \in G(n-k,n)$. The action is $h(f) : V \to \mathbb{C}$ is given by $v \mapsto f(h^{-1}(v))$.

If $g, h \in H$, then $(gh)(f) = g(h(f))$ is eay to see, so it is an action.

Let's concretely describe how to associate a configuration $y \in (\mathbb{P}^{n-k-1})^n_{\max}$ to a given configuration $x \in (\mathbb{P}^{k-1})^n_{\max}$ where the max refers to the set of points whose $GL(k)$ orbit is of maximal dimension.

The game plan is that we seek a k dimensional vector space $L \subset V^n$ and an identification $C^k \xrightarrow{\phi} L^*$ such that $L \cap H_i \subset L$ hyperplanes in L which are dual to lines in L^* and hence points in $\mathbb{P}(L^*) \simeq \mathbb{P}(\mathbb{C}^k)$ by taking $[\ell_i] \to x_i$.

Then to get the associated configuration y, we have $L^\perp$ and $\mathscr{H}_i$ the coordinate hypelanes in V^*, and so $L^\perp \cap \mathscr{H}_i$ gives a configuration of n hyperplanes in $L^\perp$, which corresponds to a configuration of n lines in $(L^\perp)^*$, and hence n points in $\mathbb{P}((L^\perp)^*) \simeq \mathbb{P}^{n-k-1}$.

Supposeing for now we have $L^k \subset V^n$, let $f_1, \ldots, f_n$ be a basis for V^*. Then $H_i = \ker f_i$, and $L^k \cap H_i = \ker(f_i|_L) \subset L$. We seek an isomorphism $\mathbb{C}^k \to L^\perp$ taking x_i to $f_i|_L$.

15 Lecture 15

Today's class will have three parts

1. Association in general
2. $k = 2$ relating $\bar{M}_{0,n}$ to Veronese curves in $\mathbb{P}^{n-3}$ which will link the two Veronese pictures
3. Fat points and moduli of fat pointed rational curves.

The classical association identifies maximal $Gl(k)$ orbits $(\mathbb{P}^{k-1})^n_{\max}/Gl(k)$ with maximal $Gl(n-k)$ orbits $(\mathbb{P}^{n-l-1})^n/Gl(n-k)$.

Kapranov extends this correspondence to Chow quotients $(\mathbb{P}^{k-1})^n//^{Ch}Gl(k) \xrightarrow{A_{k,n}} (\mathbb{P}^{n-k-1})^n//^{Ch}Gl(n-k)$. By the G-M correspondence, this is the same as $G(k,n)//^{Ch}H \to G(n-k,n)//H^n$. Let us recall how one associates to a generaic configuration $x \in (\mathbb{P}^{k-1})^n_{\max}/Gl(k)$ a configuration $y \in (\mathbb{P}^{n-k-1})^n_{\max}/Gl(n-k)$.

We need a $L^k \subset V^n$ such that if $H_1, \ldots, H_n$ are the coordinate hyperplanes on V^n then $H_i \cap L \subseteq L$ are hyberplanes then dual to these are lines $\ell_i \subset L^* = \hom(L, \mathbb{C})$ and $\mathbb{P}(\ell_i) = p_i \in \mathbb{P}(L^*) \simeq \mathbb{P}^{k-1}$ are identified with the x_i.

Given $L^k \subset V^n$ we also want an identification of $\mathbb{P}(L^*) \simeq \mathbb{P}^{k-1}$ taking $\mathbb{P}(\ell_i)$ to the original x_i.

The H_i came from a basis of functions on V, that is, a basis $f_1, \ldots, f_n$ of $V^* = \hom(V, \mathbb{C})$ and $H_i = \ker(f_i)$. BY intersecting $H_i \cap L^k = Z(f_i|_L)$.

We have $L \subset \mathbb{C}^n \simeq V$. Fix a basis $e_1, \ldots, e_n$ of $\mathbb{C}^n$ and $f_1, \ldots, f_n$ of $(\mathbb{C}^n)^*$.

Then $L^\perp = L^\vee = \hom(V/L, \mathbb{C}) = \{f : V \to \mathbb{C} : f|_L = 0\}$. We want to use $L^\perp$ to get the associated configuration y. The basis $e_1, \ldots, e_n$ for V is a basis of linear functions V^*, $e_i : V^* \to \mathbb{C}$ for each i. Define $\mathscr{H}_i = \ker e_i \subseteq V^*$. This is a hyperplane.

The intersections $\mathscr{H}_i \cap L^\perp \subset L^\perp$ are hyperplanes, and hence correspond to line $\mathscr{L}_i$ in $(L^\perp)^* = (\hom(V/L, \mathbb{C}))^* \simeq V/L$. So $\mathbb{P}(\mathscr{L}_i)$ are points in $\mathbb{P}(V^n/L^k) \simeq \mathbb{P}^{n-k-1}$.

And so $\ker(e_i) \cap L^\perp = \ker(e_i|_{L^\perp})$.

Consider the projection $V^n \to V^n/L^k$ $e_i \mapsto \bar{e}_i$.

Lines $\mathscr{L}_i$ are the lines in V^n/L^k spanned by $\bar{e}_i$.

Looking at $A_{2,n}$, we have a map $(\mathbb{P}^1)^n//^{Ch}Gl(2) \to (\mathbb{P}^{n-3})^n//^{Ch}Gl(n-2)$, which is a map $\bar{M}_{0,n} \simeq G(2,n)//^{Ch}H \simeq G(n-2,1)//^{Ch}H$ and we can interpret this association $A_{2,n}$ geometrically.

Take n distinct points on $\mathbb{P}^1$. x represents a maximal $Gl(2)$ orbit on $(\mathbb{P}^1)^n$, then the associated configuration y consists of n points in $\mathbb{P}^{n-3}$ in general position.

Remark 15.1 (Castelnuevo). *n points in $\mathbb{P}^{n-3}$ in general position corresponds to a unique veronese curve (the one passing through the points).*

Given these $x_1, \ldots, x_n$ on $\mathbb{P}^1$ define the Veronese map $\mathbb{P}^1 \to \mathbb{P}^{n-3}$ taking the x_i to n points of general position on $\mathbb{P}^{n-3}$.

We have a nontrivial check that these image points are actually the configuration y.

Next: How do the two Veronese descriptions of $\bar{M}_{0,n}$ relate?

Remember: We fix n points $p_1, \ldots, p_n$ in $\mathbb{P}^{n-2}$ in general position. Consider the sublocus $V_0(p_1, \ldots, p_n)$ of $\mathscr{H}$ the Hilbert Scheme of $\mathbb{P}^{n-2}$ consists of the set of Veronese curves in $\mathbb{P}^{n-2}$ passing through p_i. $M_{0,n} \simeq V_0(p_1, \ldots, p_n)$ and $\bar{M}_{0,n} = V(p_1, \ldots, p_n)$ the closure.

First for each p_i there is a natural hyperplane $\mathbb{P}_i^{n-3}$ consisting of all lines in $\mathbb{P}^{n-3}$ passing through p_i. There is a natural map $\sigma_i : \bar{M}_{0,n} = V(p_1, \ldots, p_n) \to \mathbb{P}_i^{n-3}$ by taking $C \mapsto [T_{p_i}C]$.

Recall: if X is a scheme and $\mathscr{L}$ is a line bundle on X, then $\varphi_{\mathscr{L}} : X \to \mathbb{P}((H^0(X, \mathscr{L}))^*)$. If $\varphi_{\mathscr{L}}$ is regular at $x \in X$ then $\varphi_{\mathscr{L}}(x) = \mathbb{P}(\ell_x)$ where ℓ_x is the line in $H^0(X, \mathscr{L})^*$ spanned by the map $x : H^0(X, \mathscr{L}) \to \mathbb{C}$, $\sigma \mapsto \sigma(x)$ as long $\{\sigma \in H^0(X, \mathscr{L}) | \sigma(x) = 0\} \subsetneq H^0(X, \mathscr{L})$ then $\varphi_{\mathscr{L}}$ is regular at x, and $\varphi_{\mathscr{L}}^* \mathscr{O}_{\mathbb{P}}(1) = \mathscr{L}$.

What is the $\mathscr{L}_i$ that defines σ_i?

To define $\mathscr{L}_i$, consider $\bar{M}_{0,n+1}$ as the universal curve over $\bar{M}_{0,n}$.

Then ω_π is the relative dualizing sheaf on $\bar{M}_{0,n+1}$. If $x = (C, x_1, \ldots, x_{n+1}) \in \bar{M}_{0,n+1}$, then $\omega_\pi|_x = (T_{x_{n+1}}C)^*$. And so $L_i = \tau_i^* \omega_\pi$ at a point $(C, x_1, \ldots, x_n) \in \bar{M}_{0,n}$ and $L_i|_x = (T_{x_i}C)^*$. In Gromov-Witten theorem, $\psi_i = c_1(L_i)$.

Claim: $\sigma_i^* \mathscr{O}_{\mathbb{P}}(1) = L_i$.

Plausibility argument that this is true: if $H \subseteq \mathbb{P}(T_{p_i}\mathbb{P}^{n-2})$ is a hyperplane then $H \cap \mathbb{P}(T_{p_i}C) = \mathbb{P}(\mathscr{H} \cap T_{p_i}C)$, and so $\mathscr{H} \cap T_{p_i}C \subseteq T_{p_i}C$ and so corresponds to a line $\mathscr{L}_i \subset (T_{p_i}C)^*$.

Proposition 15.1 (2.8 in Kapranov's Veronese paper).

1. *For any $i \in \{1, \ldots, n\}$ the space $H^0(\bar{M}_{0,n}, \mathscr{L}_i)$ has dimension $n-2$.*
2. *The corresponding morphism is everywhere regular and birational.*
3. *In the Veronese picture, $\mathbb{P}(H^0(\bar{M}_{0,n}, L_i)^*)$ is identified with $\mathbb{P}_i^{n-3}$ and $\varphi_{\mathscr{L}_i}$ is identified with σ_i.*

Proof. Outline:

We consider $\sigma_i : \bar{M}_{0,n} \to \mathbb{P}_i^{n-3} = \mathbb{P}(T_{p_i}\mathbb{P}^{n-3})$. Assme that $\sigma_i^* \mathscr{O}_{\mathbb{P}^{n-3}}(1) = L_i$, then we can use σ_i to embed the global sections of $\mathscr{O}(1)$ into the global sections of L_i. $\sigma_i^* : H^0(\mathbb{P}, \mathscr{O}(1)) \to H^0(\bar{M}_{0,n}, L_i)$.

If we can show this embedding is an isomorphism, then (a) follows. □

Proposition 15.2 (2.9)**.** *The map* $\sigma_i : \bar{M}_{0,n} \to \mathbb{P}_i^{n-3}$ *has degree* 1.

This follows from the more precise classical statement (WLOG $i = n$)

Proposition 15.3 (2.10)**.** *The correspondence* $V_0(p_1, \dots, p_n) \leftrightarrow \{$ *lines in* $\mathbb{P}^{n-3}$ *passing through* p_n *but not lying on any of the herperplanes determined by the* $p_i\} = S$ *by* $C \mapsto T_{p_n}C$ *is a bijection.*

WLOG, $p_1 = [1 : 0 : \dots : 0]$, $p_{n-1} = [0 : \dots : 0 : 1]$ and $p_n = [1 : \dots : 1]$.

Start with a line $\ell \in S$, and show that there is a veronese curve $(C, p_1, \dots, p_n)$ and $T_{p_n}C = \ell$.

Consider the Cremona inversion $\psi : \mathbb{P}^{n-3} \to \mathbb{P}^{n-3}$ given by $[z_0, \dots, z_{n-3}] \to [1/z_0 : \dots : 1/z_{n-3}]$, then $\psi(\ell)$ is a degree $n-3$ rational curve in $\mathbb{P}^{n-3}$ passing through $p_1, \dots, p_n$, so it is a Veronese curve. $\ell = T_{p_n}\psi(\ell)$.

$\ell \in S$, ℓ doesn't lie in any of the $\mathbb{P}_i^{n-3}$ for $i \neq n$, then $\{\ell \cap \mathbb{P}_i^{n-3}\}_{i=1}^{n-1} = \{q_i\}_{i=1}^{n-1}$ distinct points on the Veronese curve.

16 Lecture 16

16.1 Fine Moduli Space $M_{0,\{n_1,\dots,n_k\}}$

This space has closed points parameterizing smooth rational curves with k distinct points such that each point has embedded scheme structure.

I'll compactify and get $\bar{M}_{0,\{n_1,\dots,n_k\}}$ of stable multi-pointed rational curves.

For certain values n_i, these are known to be toric varieties, to which $\bar{M}_{0,n}$ degenerates in a flat family.

Moduli spaces of $(n_1, \dots, n_k)$ multi-pointed curves.

This is all current research by Gibney and Maclagan.

So what is a point on a scheme? It is a morphism $p : \operatorname{Spec}(k) \to X$. Two points p_1, p_2 coincide if there is a morphism $\operatorname{Spec}(k) \coprod \operatorname{Spec}(k) \overset{p_1 \coprod p_2}{\to} X$ which factors as $\operatorname{Spec}(k) \coprod \operatorname{Spec}(k) \overset{f}{\to} \operatorname{Spec} k \overset{p}{\to} X$.

Equivalently, take $p_1, p_2 : \operatorname{Spec} k \to X$, then $\operatorname{Spec} k \times_X \operatorname{Spec} k$ is either the empty scheme or $\operatorname{Spec} k$. We say they coincide if the fiber product is $\operatorname{Spec} k$.

A multipoint σ_n (or a point of multiplicity n) on X is a morphism $\sigma_n : \operatorname{Spec}(k[\epsilon]/\epsilon^2) = \mathbb{T}^{n-1} \to X$

Notice that σ_n has an underlying regular point iven by $k[\epsilon]/\epsilon^n \to k$ by $\epsilon \mapsto 0$. This unduces $\operatorname{Spec} k \to \operatorname{Spec} k[\epsilon]/\epsilon^n \to X$.

$\mathbb{T}^{n-1} \simeq \operatorname{Spec} k \times \mathbb{T}^{n-1} \overset{\sigma_n^0 \times \sigma_n}{\to} X$.

Definition 16.1 (Indistinct). *A multipoint $\sigma_n : \mathbb{T}^{n-1} \to X$ is indistinct if the above map factors through the underlying point σ_n^0. Otherwise, σ_n is self-distinct.*

Let's suppose that $\pi : X \to B$ is a flat family of schemes and X_b the scheme theoretic fiber over a point $b : \operatorname{Spec} k \to B$.

An n-multisection of $\pi : X \to B$ is a morphism $\sigma_n : B \times \mathbb{T}^{n-1} \to X$ such that $\pi \circ \sigma_n = \pi_1 : B \times \mathbb{T}^n \to B$.

Definition 16.2. *An n-multisection σ_n of π or a multisection σ_n of weight n, is self-distinct if $\sigma_n|_{X_b}$ is self distinct.*

Each multisetion has an underlying zero section $\sigma_n^0 : B \times \operatorname{Spec} k \to B \times \mathbb{T}^{n-1} \to X$.

Definition 16.3. *Let $\pi : X \to B$ be a flat family of semistable curves of genus 0. Given multisections $\sigma_{n_1}, \dots, \sigma_{n_k}$ of multiplicity $n_1, \dots, n_k$. We say that $(\pi : X \to B, \sigma_{n_1}, \dots, \sigma_{n_k})$ is stable if the σ_{n_i} are self-distinct and distinct, and if for each point $p : \operatorname{Spec} k \to X$ adn each irreducible component $C \subset X_b$, the component has at least 3 markeings where a martking is either an attaching point or a multi-point σ_{n_i} counted with multiplicity n_i.*

Also, attaching points are not the images of zero sections of multi-points.

As $M_{0,n} = (\mathbb{P}^1 \times \dots \times \mathbb{P}^1 \setminus \Delta)/Gl(2)$, we have $M_{0,\{n_1,\dots,n_k\}} = (J^{n_1-1}\mathbb{P}^1 \times \dots \times J^{n_k-1}\mathbb{P}^1 \setminus \Delta)/Gl(2)$.

For a scheme X, the nth jet functor $\underline{J^n X}$ is a functor from schemes to sets defined by $Y \mapsto \hom(Y \times \mathbb{T}^n, X)$, and this is represented by a scheme $J^n X$. For $\mathbb{P}^1$ it is a variety. It is naturally isomorphic to $\hom(-, J^n X)$.

That is, for all schemes Y, $\hom(Y \times \mathbb{T}^n, X) = \hom(Y, J^n X)$. If we have a multisection $\sigma_n : B \times \mathbb{T}^{n-1} \to X$ of a family $\pi : X \to B$, then $\sigma_n \in \underline{J^{n-1}X}(B)$ can be thought of as an element of a subscheme of $J^{n-1}X$ corresponding to B.

Want to define a locus $\Delta_n \subseteq J^{n-1}X$ such that elements $\sigma_n \in J^{n-1}X \setminus \Delta_n$ correspond to self-distinct multisections.

$\pi_1 : X \times \mathbb{T}^n \to X$, $\pi_1 \in \hom(X \times \mathbb{T}^n, X) = \underline{J^n X}(X) = \hom(X, J^n X)$, so π_1 corresponds to $i : X \to J^n X$, and $\operatorname{Im} i = \Delta_n$.

Proposition 16.1. *$\sigma_n : \operatorname{Spec} k \to J^n X \setminus \Delta_n$. Then σ_n gives a self-distinct multi-point on X.*

Proof. $\sigma \in \hom(\operatorname{Spec} k, J^n X) = \hom(\operatorname{Spec} k \times \mathbb{T}^{n-1}, X)$, and so $\sigma_n : \operatorname{Spec} k \times \mathbb{T}^{n-1} \simeq \mathbb{T}^{n-1} \to X$ doesn't factor through $\operatorname{Spec} k$, because it isn't in Δ_n. □

To a $(X, \sigma_1, \dots, \sigma_n)$, with $\sigma_i : \mathbb{T}^{n_i - 1} \to X$ selfdistinct, we can associate a point in $J^{n_1-1}X \times \dots \times J^{n_k-1}X \setminus \cup_{i=1}^k \pi_i^{-1}\Delta_{n_i-1}$ where π_i is the ith projection.

SWe'd like to ahve a sublocus $\Delta \subset J^{n_1-1,\dots,n_k-1}X$ so that the points in its complement correspond to self distinct and distinct collections.

There is a morphism $J^n X \to X$ as long as we know maps for all shcemes Y, $\hom(Y \times \mathbb{T}^n, X) = \hom(Y, J^n X) \to \hom(Y, X)$. Then take $Y = J^n X$, and $\mathrm{id}_n \in \hom(J^n X, J^n X)$ corresponds to $\mathrm{id}_n^0 \in \hom(J^0 X, X)$.

These morphisms define a morphism from the product $J^{n_1-1}X \times \ldots \times J^{n_k-1}X \to X$. So now we define $\Delta = (\cup_{i=1}^k \pi_k^{-1}(\Delta_{n_k-1})) \cup \Delta_\pi$.

Definition 16.4. $M_{0,\{n_1,\ldots,n_k\}} = (J^{n_1-1,\ldots,n_k-1}\mathbb{P}^1 \setminus \Delta)/Gl(2)$.

Next time we will define an action and show that this is a fine moduli space.

17 Lecture 17

Related to $\bar{M}_{0,n}$. Moduli spaces of Del Pezzo Surfaces.

X_n is $\mathbb{P}^2$ blown up at n points. Then $\bar{M}_{0,5}$ is X_4.

Definition 17.1. *A collection of $n \leq 8$ points in $\mathbb{P}^2$ are in general position if no three lie on a line, no six lie on a conic, and any cubic containing 8 of them has to be smooth at those points.*

Definition 17.2 (Del Pezzo)**.** *A del Pezzo surface X_n is the blowup of $\mathbb{P}^2$ at $n \leq 8$ general points. Degree $X_r = 9 - r$.*

Aut $\mathbb{P}^2$ takes any four points to any four, so for $n \leq 4$, X_n is unique, and $X_4 \cong \bar{M}_{0,4}$.

$\mathrm{Pic}(X_r) \cong \mathbb{Z}^{r+1}$.

We can take as a basis ℓ the pull back of teh class of a line in $\mathbb{P}^2$, e_i the exceptional divisors. The intersection form is $e_i \cdot e_j = -\delta_{ij}$, $\ell^2 = 1$ and $\ell \cdot e_j = 0$.

The canonical divisor $K_{X_r} = -3\ell + \sum e_i$, and inf act for $n \leq 6$, $-K_{X_n}$ defines an embedding of $X_n \to \mathbb{P}^{9-n}$. So $X_4 \to \mathbb{P}^5$ is a subvariety of degree 5. $X_5 \to \mathbb{P}^4$ is the intersection of quadrics, and for $n = 6$, we have $X_6 \to \mathbb{P}^3$, the cubic surfaces.

Definition 17.3 (-1 Curve)**.** *A -1 curve $C \subset S$ is a curve with $C^2 = -1$ and $C \cdot K_S < 0$.*

These X_4, X_5, X_6 have special (-1)-curves that we can use to build their moduli spaces.

The number of blown up points is equal to the number of exceptional divisors. THe number of lines through points is 6,10,15, and the number of conics is $0, 1, 6$. And so exceptional plus lines plus conics gives X_6 having 27 lines (assumeing these are all lines)

Define the moduli space. Fix $p_1, \ldots, p_n \in \mathbb{P}^2$ in general position and let X_n be the blowup of $\mathbb{P}^2$ at the p_i. Denote this object by $(X_n, p_1, \ldots, p_n)$.

Let Y^n be the modul space of smooth n-pointed del Pezzo surfaces, then the points look like $(X_n, p_1, \ldots, p_n)$.

For $1 \leq n \leq 6$. Let $B(X_n)$ be the union of all the -1 curves on the del Pezzo. $Y_X^n = \{(X_n, p_1, \ldots, p_n) | B(X_n)$ has normal crossings$\}$. This is an open subset of Y^n.

Definition 17.4 (Kollar-Shepherd)**.** *The moduli stack of stable surfaces with boundary $\bar{M} : \{Sch/k\} \to \{Sets\}$ with $T \mapsto \bar{M}(T) = \{(\mathscr{S}, \mathscr{B} \sum B_i)/T\}$ where $\mathscr{S} \to T$ is a flat family and B_i are closed fibers over T.*

Then $(S, B = \sum B_i)$ consists of a pair with semi-log canonical singularities. $\omega_S(B)$ is ample. $\bar{M}$ is coarsely represented by a scheme $\bar{M}$. One way to compactify Y^n_X is to take the closure of T^n_X in $\bar{M}$.

Theorem 17.1. *$\bar{Y}^n_{SS}$ obtained this way is a compactification. It has a universal family and it has normal crossings boundary. $\bar{Y}^n_{SS}$ is smooth projective.*

Definition 17.5 (Log Minimal)**.** *A smooth variety Y is log minimal if for some smooth compactification $\tilde{Y}$ with normal crossings boundary, then linear system $|N(K_{\tilde{Y}} + B)|$ defines an embedding of Y into a projective space for $N >> 0$.*

Such a variety Y is expected to have associated $R = \oplus\Gamma(m(K_{\tilde{Y}} + B))$, a log canonical compactification.

Theorem 17.2 (Hacking, Keel, Tevelev)**.** *Y^n is log minimal for $n \leq 6$ or $n = 7$ in characteristic not 2. It's log canonical cmpactifications $\bar{Y}^n_{lc}$ issmooth and the boundary is a union of smooth normal crossing divisors.*

Let $\pi : Y^{n+1} \to Y^n$ be the natural morphism given by dropping one of the points at which we blew up. Then the following diagram

$$\begin{array}{ccc} S & \longrightarrow & \bar{Y}^{n+1}_{lc} \\ \downarrow & & \downarrow \\ \bar{Y}^n_{SS} & \longrightarrow & \bar{Y}^n_{lc} \end{array}$$

with the horizontal arrows isomorphisms for $n \leq 5$ and for $n = 6$ the log crepont birational morphisms.

Tevelev's tropical compactification is used, for example, to construct many interesting moduli spaces.

Idea: if X is the space you want to compactify, and it is closed and irreducible, then if $X \subset \mathbb{T} = (\mathbb{C}^*)^d$, then X is "very affine". $X \cap \mathbb{T} := X_0$. We can form a fan Trop X which can be used to compactify.

More generally, if $X \subseteq X_\Delta$ (with X_Δ is a smooth/normal toric variety with torus $\mathbb{T}$), we consider the closure of $(X \cap \mathbb{T}) = X_0$ inside of X_Δ. We call this $\bar{X}^\Delta$.

Definition 17.6 (Tropical Compactification)**.** *$\bar{X}^\Delta$ is called a tropical compactification if*

1. *$\bar{X}^\Delta$ is complete*

2. *$\mathbb{T} \times \bar{X}^\Delta \to X_\Delta$ given by the torus action is surjective.*

Consequences: modular interpretation of $\bar{X}^\Delta$, and $|\Delta| = \text{Trop}\, X$.

$\bar{M}_{0,n}$ is a tropical compactification, where $\text{Trop}(\bar{M}_{0,n})$ is $\text{Trop}(G^0(2,n))/T^{n-1}$. The fan structure has the same combinatorial data as $\bar{M}_{0,n} \setminus M_{0,n}$.

This quotient can be either the Chow or Hilbert quotient. This is not a normal toric variety, however, and so we must normalize.