1 Lecture

Syllabus

1. Basic stuff on Morse functions
   (a) Normal Form
   (b) Relationship between handle attachments
2. First applications
   (a) Calculations
   (b) h-cobordism theorem
   (c) Lefshetz hyperplane theorem
3. Morse-Smale Gradient Systems

Theorem 1.1 (h-cobordism Theorem)**.** *Let W be a manifold with two boundary components, M_1, M_2 such that W deformation retracts onto either one. If $\pi_1(W) = \pi_i(M_i) = 0$ then if $\dim W \geq 6$, then W is diffeomorphic to $M_i \times I$.*

In particular, this says M_1 and M_2 are diffeo.

If $\pi_1(W) \neq 0$, then there is $Wh(\pi_1 W)$, the Whitehead group such that it is isomorphic to $Wh(\pi_1 M)$ with elements $\tau(W)$, and which is an abelian group.

Conjecture 1.1. *If $\pi_1(M)$ is torsion free, then $Wh(\pi_1 M)$ is zero.*

This is known in many cases: if M is hyperbolic, for instance.

We're going to focus on the techniques of Morse Theory, because people seem to need to invent their own version of Morse theory for new applications.

We are working in the smooth category.

Let $f : M \to N$, we look at the differential $df : TM \to TM$. Then a critical point of f, $x \in M$ is one such that the rank $d_x f : T_x M \to T_f(x) M$ is less than $\min(\dim M, \dim N)$. A critical value is a point $y \in N$ such that $f^{-1}(y)$ contains a critical point.

Theorem 1.2 (Sard)**.** *the set of critical values of f has measure zero.*

For $f : M \to \mathbb{R}$, $df : TM \to T\mathbb{R}$, if $x \in M$ is a critical point, $d_x f = 0$, that is, $X(f)(x) = 0$ for all vectors $X \in T_x M$:

Lemma 1.3. *If $x \in M$ is a critical point for f and X_1, Y_1, X_2, Y_2 vector fields in a enighborhood of x such that $X_1(x) = X_2(x)$, $Y_1(x) = Y_2(x)$, then $X_1(Y_1 f)(x) = Y_1(X_1(f))(x) = \ldots$.*

Proof. $[X_1, Y_1]f(x) = d_x f([X_1, Y_1]) = 0$ because x is a critical point. So $X_1(Y_1(f(x))) - Y_1(X_1(f(x))) = 0$, so $X_1 Y_1 f(x) = Y_1 X_1 f(x)$. Etcetera. ☐

Definition 1.1 (Hessian). *Let f be a function and x a critical point. Define $H_x f = \tilde{X}(\tilde{Y} f)(x)$ for $X, Y \in T_x M$ where $\tilde{X}, \tilde{Y}$ are vector fields defined in a neighborhood of x such that $\tilde{X}(x) = X$ and $\tilde{Y}(x) = Y$. We call this the Hessian, and it is a symmetric bilinear form on $T_x M$.*

In local coordinates around x, we have $H_x f = \frac{\partial^2 f}{\partial x_i \partial x_j}(0)$.

Definition 1.2 (Morse Function). *A function f is a Morse function if all of its critical points are nondegenerate, where a nondegenerate critical points x is one where $H_x f$ is nondegenerate.*

Recall from linalg that for any nondegenerate symmetric binlinear form $B : V \times V \to \mathbb{R}$, there exists a basis such that, in this basis, B is diagonal with -1 and 1 as the diagonal entries. The number of -1's is calle dhte index of B.

For $x \in M$, a nondegenerate critical point, the index of f at x, $\lambda_x f$ is the index of $H_x(f)$.

Lemma 1.4 (Morse Lemma). *If $f : M \to \mathbb{R}$, $x \in M$ is a nondegenerate critical point, then there exist coordinates about x, $x_1, \ldots, x_n$, such that $f(x) = f(0) + \sum_{i=1}^{\lambda_x(0)} -x_i + \sum_{i=\lambda_x(f)+1}^{n} x_i^2$.*

Moser's Method/Homotopy Method:

Theorem 1.5 (Moser). *Let M^n be a compact oriented manifold and $\Omega_0, \Omega_1 \in \Omega^n M$ such that $\int_M \Omega_0 = \int_M \Omega_1$, then there exists a diffeo $\psi : M \to M$ such that $\psi^* \Omega_1 = \Omega_0$.*

Proof. The idea is to construct ψ_t diffeos such that $\psi_0 = id$ and $\psi_t^* \Omega_t = \Omega_0$ where $\Omega_t = (1-t)\Omega_0 + t\Omega_1$. Then $\psi_1^* \Omega_1 = \Omega_0$ and we're done. We will be constructing ψ_t as the flow of a vector field. That is, we must find X_t a time dependent vector field, such that $\frac{d}{dt}\psi_t = X_t \circ \psi_t$.

Now we take $\psi_t^* \Omega_t = \Omega_0$ and differentiate both sides. We get $\frac{d}{dt}(\psi_t^* \Omega_t) = \frac{d}{dt}(\Omega_0) = 0$, and the first is $\psi_t^* \left(\frac{d}{dt}\Omega_t + d i_{X_t} \Omega_t + i_{X_t} d\Omega_t \right)$ This the the generalization of Cartan's formula for a time dependent vector field. The last term is zero because Ω_t is in the top degree, and so we get $\psi_t^*(\Omega_1 - \Omega_0 + d i_{X_t} \Omega_t)$. Now $\int_M (\Omega_1 - \Omega_0) = 0$ and so $\Omega_1 - \Omega_0 = d\eta$ for some $\eta \in \Omega^{n-1} M$, and so $\psi_t^*(d\eta + d i_{X_t} \Omega_t) = 0$. Look at the equation $\eta + i_{X_t} \Omega_t = 0$. So $\Omega_t \neq 0$ anywhere, so we must solve $i_{X_t} \Omega_T = -\eta$ for X_t. Since M is compact, X_t integrates globally to a 1-parameter family of diffeomorphisms ψ_t. Then $\psi_t^*(d\eta + d i_{X_t} \Omega_t) = 0$ and so $\psi_t^* \Omega_t = \Omega_0$. □

Lemma 1.6. *$f : M \to \mathbb{R}$, $x_0 \in M$ a critical nondegenerate point. Then there exist coordinates such that $f(x) = f(0) + \frac{1}{2} H_{x_0} f(x, x)$.*

Proof. First, wlog, $f(0) = 0$. Siunce the statement is local, we can assume $x_0 = 0 \in V$ a vector space $f : V \to \mathbb{R}$. Set $f_t(x) = \frac{1}{2} H_0 f(x,x) + t(f(x) - \frac{1}{2} H_0 f(x,x))$. Then $f_0 = \frac{1}{2} H_0 f$ and $f_1 = f$.

We want to construct $\psi_t^* f_t = f_0$ with $\frac{d}{dt}(\psi_t^* f_t) = \frac{d}{dt} f_0 = 0$, so we need $\psi_t^*(\frac{d}{dt} f_t + X_t f_t) = \psi_t^*(f_1 - f_0 + X_t f_t)$. We need to find X_t making this true. That is, $df_t(X_t) = f_0 - f_1$.

For x near 0 and $v \in V$, we have $d_x f_t(v) = \int_0^1 \frac{d}{ds} d_{sx} f_t(v) dx$.

So $d_x f_t(v) = \int_0^1 \frac{d}{ds} d_{sx} f_t(v) ds = \int_0^1 \frac{d}{ds} d_{sx}(v(f_t)) ds = \int_0^1 x(v f_t) ds = B_{x,t}(x, v)$ where $B_{x,t}(u, v)$ is the bilinear furm defined by $B_{x,t}(u, v) = \int_0^1 u(v(f_t))_{sx} ds$, and so $B_{x,t} = B_{x,0} + t(B_{x,1} - B_{x,0})$ and $B_{x,0} = H_0 f$. So then $B_{0,t} = H_0 f$ as well, and so $B_{0,t}$ is nonsingular for all t. Thus, there exists a neighborhood of 0 where $B_{x,t}$ is nonsingular for all $t \in [0, 1]$, and so $df_t(X_t) = f_0 - f_1$ becomes $B_{x,t}(x, X_t) = f_0 - f_1$.

So then $(f_0 - f_1)(0) = 0$ and $(df_0 - df_1)(0) = 0$, so now set $g = f_0 - f_1$. Then $g = \int_0^1 \frac{d}{ds} g(sx) ds = \int_0^1 d_{sx} g ds$. This is $\int_0^1 \int_0^1 d^2_{rsx} g(sx, x) ds dr = C_x(x, x)$ where C_x is the bilinear form given by $C_x(u, v) = \int_0^1 \int_0^1 d^2_{rsx} g(su, v) dr ds$.

So let X_t be the vector field satisfying $C_x(u, x) = B_{x,t}(u, X_t)$ for all u. Since B is nongenerated in a neighborhood of zero, X_t exists and is unique. This constructs X_t, and we can integrate (maybe in a smaller neighborhood) to get ψ_t. □

Corollary 1.7. *The set of critical points for a Morse function is discrete.*

Definition 1.3 (Symplectic Manifold)**.** *A symplectic manifold (M, ω) is a manifold together with $\omega \in \Omega^2 M$ where ω is nondegenerate and $d\omega = 0$.*

Exercise 1.1. *Use Moser's method to rpove that if $[\omega_0] = [\omega_1] \in H^2(M)$ with ω_0, ω_1 symplectic forms, then there exists $\psi : M \to M$ such that $\psi^* \omega_1 = \omega_0$.*

Theorem 1.8 (Darboux)**.** *(M, ω) symplectic, $x \in M$ then there exists coordinates $p_1, \ldots, p_n, q_1, \ldots, q_n$ such that $\omega = \sum dp_i \wedge dq_i$.*

Next time: manifolds have Morse functions (sketch of proof, even get Morse functions generic), then handlebodies.

2 Lecture

Refences 1) Morse Theory - Milnor 2) Lectures in Morse Theory - Bott 3) Morse Theory Indomitable - Bott (IHES publications) 4) Lectures on the h-Cobordism Theorem - Milnor 5) Invitation to Morse Theory - Nicolaescu 7) Differential Manifolds - Kosinski (this book checks that we can ignore proving that things can be made differentiable) (it is a dover)

Theorem 2.1 (Alexander's Trick)**.** *Suppose I have D^k and a diffeomorphism $\psi : \partial D^k \to \partial D^k$. Then you can extend ϕ to a homeomorphism $D^k \to D^k$*

Proof. Use polar coordinates to rewrite the disc as $[0, 1] \times \partial D^k / \sim$. Let $\tilde{\phi}(t, x) = (t, \phi(x))$. □

It can even be shown that there may not be a diffeomorphism of the disc. (not smooth at the center)

Proposition 2.2. *If $M^n \subseteq \mathbb{R}^k$ then for almost all $\vec{a} = (a_1, \ldots, a_k)$, $f(x) = a_1x_1 + \ldots + a_kx_k$ is a Morse function.*

Corollary 2.3. *Every manifold has lots of Morse functions.*

Definition 2.1 (Concrete Cobordism). *A concrete cobordism is a triple (W, M_0, M_1) where W is a manifold, $\partial W = M_0 \coprod M_1$. Which part of ∂W you call M_0 and which is M_1 is part of the data.*

If (W, M_0, M_1) and (W', M_0', M_1') are two concrete cobordisms and $\psi : M_1 \to M_0'$ is a diffeomorphism, then I can glue $(W \cup_\psi W', M_0, M_1')$ is a concrete cobordism. This includes the fact that on $W \cup_\psi W'$ there exists a unique differentiable structure restricting to those of W and W'.

Definition 2.2 (Cobordism). *A cobordism between M_0 and M_1 consists of a 3-tuple $((W, N_0, N_1), \psi_0, \psi_1)$ where (W, N_0, N_1) is a concrete cobordism and $\psi_0 : N_0 \to M_0$ and $\psi_1 : N_1 \to M_1$ are diffeos.*

We call two cobordisms equivalent if there exists a diffeomorphism $\psi : W \to W'$ with $\psi(N_i) = N_i'$ and that $\psi_i' \circ \psi|_{N_i} = \psi_i$.

Surgery

Let M^n be a manifold and $h : S^k \to M^n$ an embedding which has a trivializable normal bundle $\nu(h)$. A framing of $\nu(h)$ is a bundle isomorphism $\nu(h) \cong_\phi \mathbb{R}^{n-k} \times S^k$. This gives a tubular neighborhood $U = D^{n-k} \times S^k$. Drill out $\circ D^{n-k} \times S^k$ ($\circ D$ is the interior of the disc). Glue back in $S^{n-k-1} \times D^{k+1}$.

The result of this operation is called surgery on the k-sphere $M(S^k, h, \phi)$ and only depends on the isotyop type of h and the regular homtopy type of the framing. (only allow homotopies through immersions)

Example 2.1. *Surgery on a 0-sphere is connected sum.*

Cell attachment: D^k. We have $h : \partial D^k \to X$ and have $X \cup_h D^k$.

Handle Attachment

Let $H_{k,n} = D^k \times D^{n-k}$ be a k-handle in n-dimensions.

Let W^n be a manifold with boundary. To attach a k-handle $H_{k,n}$ we need an embedding $h : S^{k-1} \to \partial W$ with a trivializable nromal bundle $\nu(h)$ and a framing for $\nu(h)$ which gives an embedding of $S^{k-1} \times D^{n-k}$. Then we attach $H_{k,n}$ to ∂W be gluing in $D^k \times D^{n-k}$, and then we smooth.

Remark 2.1. *If I attach a k-handle to W using the data h, ϕ, then I've changed the boundary by performing surgery with the data.*

Remark 2.2. *If I do surgery on M with data, (S^k, h, ϕ) the $M(S^k, h, \phi)$ is cobordant to M.*

Use the data h, ϕ to attach a $k+1$ handle to the top part of the boundary of $M \times I$. We call this cobordism the trace of the surgery.

Exercise 2.1. *Show that every manifold is cobordant to a simply-connected manifold.*

Theorem 2.4 (First Fundamental Theorem of Morse Theory)**.** *Let $f : M \to \mathbb{R}$ be a Morse function which is exhaustive: $M^c = \{x \in M | f(x) \leq c\}$ is compact. If c_1, c_2 are regular values and there are no critical values between c_1 and c_2 then $M^{c_1} \cong M^{c_2}$.*

We will always use $\cong$ for diffeo here.

Theorem 2.5 (Second Fundamental Theorem of Morse Theory)**.** *If c_1, c_2 are regular and $c \in (c_1, c_2)$ is a critical value corresponding to only one critical point of index λ, then $M_{c_2} \cong M_{c_1} \cup H_{\lambda,n}$.*

Definition 2.3 (Gradient Like Vector Field)**.** *Let f be an exhaustive Morse function. A gradient like vector field X is one such that $Xf(x) > 0$ for non-critical x and such that for p critical, there exist coordinates about p such that $X = -2\sum_{i=1}^{\lambda_p(f)} x_i \frac{\partial}{\partial x_i} + 2\sum_{i=\lambda_p(f)+1}^{n} x_i \frac{\partial}{\partial x_i}$.*

Lemma 2.6. *Gradient like vector fields always exist.*

Proof. Let g be a Riemannian metric such that at every critical point p, there exist coordinates $f = f(p) - \sum_{i=1}^{\lambda_p} x_i^2 + \sum_{i=\lambda_p+1}^{n} x_i^2$ and $g = \sum dx_i^2$.

Let $X = \nabla_g f$. It is the gradient $g(\nabla_g f, Y) = Y(f)$ for all Y. This is gradient like. $\square$

The flow of a gradient like vector field in a neighborhood of p looks like $\gamma_t(x_1, \ldots, x_n) = e^{2tx_+} + e^{-2t}x_-$.

3 Lecture

Theorem 3.1. *Let $f : M \to \mathbb{R}$ be a smoth exhaustive Morse function. Suppose that $a < b$ such that $[a, b]$ contains no critical values. Then $M^a \cong M^b$ and M^a is a deformation retract of M^b.*

Proof. Since M^c are all compact, there exists $\epsilon > 0$ such that $f^{-1}((a-\epsilon, b+\epsilon))$ contains no critical points. Let X be a gradient like vector field. Let $\rho(x)$ be a smooth function such that $\rho(x) = \begin{cases} 0 & x \notin f^{-1}(a-\epsilon, b+\epsilon) \\ \frac{1}{|Xf(x)|} & x \in f^{-1}([a,b]) \end{cases}$.

Consider the fector field $-\rho(x)X(x)$. Let $\Phi : \mathbb{R} \times M \to M$ be the flow of $-\rho X$ and take $\Phi_t(x) = \Phi(t, x)$.

For all t and all $x \notin f^{-1}(a-\epsilon, b+\epsilon)$, we have $\Phi_t(x) = x$.

Calculate $\gamma(t) = \Phi_t(x)$ for fixed x. Then $\frac{df}{dt}(\gamma(t)) = df(\dot\gamma(t)) = \dot\gamma(t)f$, which is $-\rho(\gamma(t))X(\gamma(t))f(\gamma(t)) = -1$ for $x \in f^{-1}([a, b])$, that is, $\Phi_t(x)$ travels at one unit per second.

So $\Phi_{b-a}(M^b) = M^a$, and it's inverse is $\Phi_{a-b}(M^a) = M^b$. Let $H : I \times M \to M$ be defined by $H(t, x) = \Phi_{t(f(x)-a)^+}(x)$ where $c^+ = \max\{c, 0\}$. This is a deformation retraction. $\square$

Theorem 3.2. *Let $f : M \to \mathbb{R}$ be a Morse fnction, exhaustive. Let $[a, b]$ contain one critical value $c \in (a, b)$ such that $f^{-1}(c) = p$ contains one critical point. Then $M^b \cong M^a \cup_{\partial M^a} H^{\lambda,n}$ where $\lambda = \lambda_p(f)$.*

Proof. We sketch the proof here.

Let $\epsilon > 0$ be small, $[c-\epsilon, c+\epsilon] \subset (a, b)$. By the previous theorem, $M^a \cong M^{c-\epsilon}$ and $M^b \cong M^{c+\epsilon}$. So we only need to show the theorem for intervals of the form $[c - \epsilon, c + \epsilon]$ with $\epsilon > 0$ small.

Choose Morse coordinates for f so that $f(x) = c - \sum_{i=1}^{\lambda} x_i^2 + \sum_{i=\lambda+1}^{n} x_i^2$ and call these x_- and x_+.

Let U be the neighborhood $M^{c+\epsilon} \cap U = \{(x_-, x_+) | -x_-^2 + x_+^2 \leq \epsilon\}$ and $M^{c-\epsilon} \cap U = \{(x_-, x_+) | -x_-^2 + x_+^2 \leq -\epsilon\}$.

Let $\delta > 0$ be small, $\delta < \epsilon^2$. Set $H = M^{c+\epsilon} \cap \{x_-^2 < \delta\}$. Then $H \cong D^\lambda \times D^{n-\lambda}$.

Look at the closure of $M \setminus H$. This is diffeo to $M^{c-\epsilon}$, and so we're done. □

Corollary 3.3. $M^{c+\epsilon} \simeq M^{c-\epsilon} \cup e^\lambda$

Corollary 3.4. *If M is a manifold with an exhaustive Morse function, then M has the homotopy type of a CW complex with one cell of dimension λ, e_p^λ, for every critical point p.*

Let $P(t), Q(t) \in \mathbb{Z}[[t, t^{-1}]]$. Say that $P \geq Q$ if there exists $R \in \mathbb{Z}[[t, t^{-1}]]$ such that $R = \sum r_n t^n$ with $r_n \geq 0$ and $P - Q = (1 + t)R$.

Now, divide through by $1 + t$, then $R = (1 + t)^{-1}P - (1 + t)^{-1}Q$. So if $P = \sum_{-N}^{\infty} p_n t^n$ and $Q = \sum_{-M}^{\infty} q_m t^m$, Then this gives us $\sum_{i=0}^{\infty}(-1)^i p_{n-i} - \sum_{i=0}^{\infty}(-1)^i q_{n-i} = r_n \geq 0$, and so $\sum(-1)^i p_{n-i} \geq \sum(-1)^i q_{n-i}$, the abstract Morse inequalities

Let $\mathbb{F}$ be a field and let (C_*, ∂) be a chain complex over $\mathbb{F}$. Then let $P_C = \sum \dim_{\mathbb{F}} H_i(C_*, \partial)t^i$. $M_C = \sum \dim_{\mathbb{F}} C_i t^i$. Assume that $M_C \in \mathbb{Z}[[t, t^{-1}]]$.

Proposition 3.5. *$M_C \geq P_C$.*

Proof. $M_C - P_C = (1 + t)R$ with positive coefs.

There exist shor exact sequences $0 \to B_i \to C_i \to H_i \to 0$ and $0 \to Z_i \to C_i \to B_{i-1} \to 0$, and so $M_C - P_C = \sum(\dim C_i - \dim H_i)t^i$, which is $\sum(\dim B_{i-1} + \dim B_i)t^i = (1 + t)\sum \dim B_i t^i$. □

Apply this to the following situation:

$f : M \to \mathbb{R}$ a Morse function, f compact. This gives a cell decomposition, and thus a chain complex $C_*^{CW}(M, f)$ where $\dim C_k^{CW}(M, f) = \mu_k(f)$ =the number of critical points of index k. Then $M_{C^{CW}(M,f)} = M_f(t) = \sum \mu_k(f)t^k$.

Define $P(M) = \sum \dim H_k(M; \mathbb{F})t^k$.

Morse Inequalities:

$M_f(t) \geq P(M)(t)$, that is, there exists R with positive coefs with $M_f(t) - P(M)(t) = (1 + t)R(t)$.

Definition 3.1 (Perfect). *A Morse function is called perfect if $M_f(t) = P(M)(t)$.*

Theorem 3.6 (Laccunary Principle)**.** *Suppose that f is a Morse function such that if f has a critical point of index k, then it doesn't have one of indedx $k+1$. Then f is perfect.*

Proof. $C_*^{CW}(M,f)$ with have zeros on either side of a nonzero group. □

Theorem 3.7 (Reeb)**.** *If f is a Morse function on a compact manifold and has only two critical points, then M is homeomorphic to a sphere.*

Proof. Let p, q be the two critical points where p is the min and q is the max of f. Then $\lambda_p(f) = 0$ and $\lambda_q(f) = n$. If $c \in (f(p), f(q))$, then $M^c \cong H^{0,n} = D^0 \times D^n \cong D^n$, when we pass $f(q)$ we attach an n handle D^n. □

Aside: the Reeb Foliation:

Let $S^3 = \{(z_0, z_1) | |z_0|^2 + |z_1|^2 = 1\}$ Set $T_1 = \{(z_0, z_1) \in S^3 | |z_0|^{1/2} \leq 1/2\}$ and $T_2 = \{(z_0, z_1) \in S^3 | |z_0|^{1/2} \geq 1/2\}$. Then $T_1 \cap T_2 = \{(z_0, z_1) | |z_0|^2 = 1/2, |z_1|^2 = 1/2\}$, which is a torus.

Flag Varieties

A full flag in a complex vector space V is a sequence of subspaces $0 = V_0 \subset V_1 \subset \ldots \subset V_n = V$ where $\dim V_i = i$. The space of full flags in V forms a manifold X. $U(n)$ acts on V and $g \in U(n)$ sends a flag to a flag. This action is trasitive, and this can be shown by choosing coordinates.

What's the isotropy subgroup? For F_0, it is $ge_1 = \theta_1 e_1$. Etc, so we get the diagonal elements. So the Flag Variety X is diffeomorphic to $U(n)/T$ the maximal torus.

More generally, let $\hat{p} = p_1, \ldots, p_k$ be a partition of n, and we define a $\hat{p}$ flag to be $V_1 \supset \ldots \subset V_k$ where $\dim V_1 = p_1$ and $\dim V_i = \dim V_{i-1} + p_i$. These flags also form a manifold, $X_{\hat{p}}$.

Exercise 3.1. *Show*

1. *$U(n)$ acts transitively on these flags*
2. *Then $X_{\hat{p}} \cong U(n)/U(p_1) \times \ldots \times U(p_k)$*

In general, take a compact Lie group $T \subset G$ any torus, then $G/C(T)$ is called a generalized flag variety.

4 Lecture

We want to show that $M^{c+\epsilon} \cong M^{c-\epsilon} \cup H^{\lambda,n}$. The place where $M^{c+\epsilon}$ and $M^{c-\epsilon}$ differ is not contained in U. We want to find a manifold contained in $M^{c+\epsilon}$ but such that the difference between it and $M^{c-\epsilon}$ is contained in U.

Let $\phi(t)$ be a function that satisfies $\phi(t) \geq 0$, $\phi(0) > \epsilon$, $\phi(t) = 0$ for $t > 2\epsilon$ and $-1 < \phi'(t) \leq 0$

Set $F(x) = f(x) - \phi(x_-^2 + 2x_+^2)$ where $f(x) = c - x_-^2 + x_+^2$ in U and $F(x) = f(x)$ outside of U.

Claim: $M^{c+\epsilon}$ is diffeomorphic to $F^{-1}(-\infty, c-\epsilon]$.

First: $M^{c+\epsilon} = F^{-1}(-\infty, c+\epsilon]$. We check this by noting that $x \in M^{c+\epsilon}$, $f(x) \leq c+\epsilon$, so $F(x) \leq c+\epsilon$. Thus $M^{c+\epsilon} \subseteq F^{-1}(-\infty, c+\epsilon]$. If $x \in F^{-1}(-\infty, c+\epsilon]$, then $c+\epsilon \geq F(x) = f(x) - \phi(x_-^2 + 2x_+^2) = c - x_-^2 + x_+^2 - \phi(x_-^2 + 2x_+^2)$. If $\phi(x_-^2 + 2x_+^2) > 0$, then $x_-^2 + 2x_+^2 < 2\epsilon$, and so $f(x) = c - x_-^2 + x_+^2 \leq c + \frac{1}{2}x_-^2 + x_+^2 < c + \epsilon$. So $M^{c+\epsilon} = F^{-1}(-\infty, c+\epsilon]$.

Second: $dF(x) = df(x)$. For $x \notin U$, this is trivial. For $x \in U$, we have $dF = -2x_- dx_- + 2x_+ dx_+ - \phi'(x_-^2 + 2x_+^2)(2x_- dx_- + 4x_+ dx_+)$. This simplifies to $-2x_-(1+\phi')dx_- + 2x_+(1-2\phi')dx_+$. So dF has only one trivial point at $(x_-, x_+) = 0$. In fact, nondegerate of the same index, as $F(p) = f(p) - \phi(0) = c - \phi(0) < c - \epsilon$. Since F has no critical points in $F^{-1}[c-\epsilon, c+\epsilon]$, we see that $F^{-1}(-\infty, c+\epsilon] = M^{c+\epsilon} \cong F^{-1}(-\infty, c-\epsilon]$.

Lemma 4.1. *Let $q \in \mathbb{R}^\lambda$. Then the intersection of $F^{-1}(-\infty, c-\epsilon]$ with the plane $x_- = q$ is diffeomrophic to a disc of radius $r(q)$ where the function $r(q)$ is smooth and if $q^2 > 2\epsilon$ then $r(q) = (q^2 - \epsilon)^{1/2}$.*

Proof. The intersection consists of point $x = (x_-, x_+)$ where $x_- = q$, $c - \epsilon \geq F(x) = f(x) - \phi(x_-^2 + 2x_+^2) = c - q^2 + x_+^2 - \phi(q^2 + 2x_+^2)$ (*) or $-q^2 + x_+^2 - \phi(q^2 + 2x_+^2) \leq -\epsilon$ (**).

Set $t = q^2 + 2x_+^2$. Then (**) becomes $\frac{1}{2}t - \frac{3}{2}q^2 - \phi(t) \leq -\epsilon$ or $\phi(t) \geq \epsilon + \frac{t}{2} - \frac{3}{2}q^2$.

Now, $\phi(t) - t/2$ is monotone decreasing ($\phi'(t) - 1/2 < 0$) so $\phi(t) \geq \epsilon + t/2 - 3/2q^2$ for all $t \leq t_0$ where t_0 is such that $\phi(t_0) = \epsilon + t_0/2 - 3/2q^2$. The inverse function theorem implies that t_0 is a smooth function of q.

So (*) is satisfied for $t \leq t_0$. Now $t = q^2 + 2x_+^2$, so $x = (q, x_+)$ where $q^2 + 2x_+^2 \leq t_0$ or $x_+^2 \leq \frac{t_0 - q^2}{2}$. By the properties of ϕ, we have $t + \phi(t)$ monotone increasing so $t + \phi(t) > \phi(0)$ for $t > 0$, so $\phi(t) - \phi(0) > -t$. Thus $t_0/2 + \epsilon - 3q^2/2 = \phi(t_0) > \phi(0) - t_0 > \epsilon - t_0$.

So $3/2t_0 > 3/2q^2$, so $t_0 > q^2$. Now $x_+ \leq \left(\frac{t_0 - q^2}{2}\right)^{1/2} = r(q)$, and $r(q)$ is smooth since $t_0 > q^2$ and t_0 is a smooth function of q.

Now we need only show that $F^{-1}(-\infty, c-\epsilon) \cong M^{c-\epsilon} \cup H^{\lambda,n}$.

The handle is attached along an $S^{\lambda-1} \subset H^{\lambda,n} = D^\lambda \times D^{n-\lambda}$ to $S = \{x_+ = 0, x_-^2 = \epsilon\}$.

Define the attaching diffeomorphism $h(x_-, x_+) = \sqrt{2\epsilon}\left(\frac{x_-}{|x_-|}\left(\frac{3}{2} - x_-^2\right)^{1/2}, x_+\right)$. This is a diffeo of a tubular beighborhood $T(\epsilon) = S^{\lambda-1} \subset D^\lambda \times D^{n-\lambda}$ to a tubular neighborhood $T'(\epsilon) = \{x \in M^{c-\epsilon} | x_-^2 < 2\epsilon\}$ of S in $M^{c-\epsilon}$.

Then $M^{c-\epsilon} \cup H^{\lambda,n}$ is $(M^{c-\epsilon} \setminus S) \cup (H^{\lambda,n} \setminus S^{\lambda-1})/\sim$ where $(x_-, x_+) \sim \sqrt{2\epsilon}\left(x_-, x_+\left(\frac{x_-^2 - 1/2}{1 - x_-^2}\right)^{1/2}\right)$.

Define $g : M^{c-\epsilon} \cup H^{\lambda,n}/\sim \longrightarrow F^{-1}(-\infty, c-\epsilon]$ by $g(x) = \begin{cases} (x_-, x_+ \frac{r(x_-)}{\sqrt{x_-^2 - \epsilon}} & x \in M^{c-\epsilon} \setminus S \\ (x_- \sqrt{2\epsilon}, x_+ \frac{r(\sqrt{2\epsilon}x_-}{\sqrt{1 - x_-^2}} & x \in H^{\lambda,n} \end{cases}$.

This all works out. □

5 Lecture

Let W be a cobordism. It is trivial if it is diffeomorphic to $M \times I$. It is called elementary if it has a Morse function $f : W \to [0,1]$ such that $f(\partial_0 W) = 1$, $f(\partial_1 W) = 1$ with one critical point. Then $W = (\partial_0 W \times I) \cup H^{\lambda,n}$ and thus we know that $\partial_1 W = S(\partial_0 W, S^{\lambda-1})$.

Theorem 5.1. *Any cobordism can be obtained by composing elementary cobordisms.*

Proof. The proof consists of constructing a Morse function such that every critical value has one critical point. We can always do this by perturbing a Morse function. □

Corollary 5.2. *If $M_0 = \partial_0 W$ and $M_1 = \partial_1 W$, then M_0 can be turned into M_1 via surgery.*

Theorem 5.3 (Gromov-Lawson)**.** *Let M be a manifold with Riemannian metric g whose scalar curvature is positive, and we do surgery in codimension ≥ 3. Then the new manifold has a metric of positive scalar curvature.*

Fact: If M_0 and M_1 are compact and simply-connected, then being Spin cobordant is equivalent to going from M_0 to M_1 by a sequence of surgeries of codimension ≥ 3.

Corollary 5.4. *If M_0, M_1 are simply connnected and spin, and spin cobordant, then M_0 has a metric of positive sectional curvature iff M_1 does.*

$\Omega_*^{Spin} \to \mathbb{Z}$, $\hat{A}$-genus is the index of the Dirac operator.

Theorem 5.5 (Lichnerowisz, Stolz)**.** *$\pi_1 M = 0$, spin, then M has a metric of PSC iff $\hat{A}M = 0$.*

Now we turn to the Homology of Generalized Flag Varieties

Let G be a compact Lie group. and $T \subset G$ a torus, not necessarily maximal. Then look at $G/C(T)$. These are called generalized flag manifolds.

Some examples:

Define $X_p = U(n)/U(p_1) \times \ldots \times U(p_n)$ be as defined before. Then $p = (1, n-1)$ ggives $X_p = \mathbb{CP}^{n-1}$, more generally, $p = (k, n-k)$ gives $Gr_k(\mathbb{C}^n)$, and $p = (1, \ldots, 1)$ gives the classic flag varieties.

Let G be a compact Lie group and $\mathfrak{g}_0$ be it's Lie algebra over $\mathbb{R}$, and $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$.

Theorem 5.6. *Any compact Lie group has a maximal torus. Any two are conjugate by an element of G.*

Let $\mathfrak{t}_0$ be the Lie algebra, $\mathfrak{t} = \mathfrak{t}_0 \otimes \mathbb{C}$. ad is an action of $\mathfrak{g}$ on itself by $ad_X(Y) = [X, Y]$. Restricting to $\mathfrak{t}$, we get a map from $\mathfrak{t}$ to $End\mathfrak{g}$.

A map $\alpha : \mathfrak{t} \to \mathbb{C}$ is called a root of $\mathfrak{g}$. If $ad_H(X) = \alpha(H)X$ for all $H \in \mathfrak{t}$ for some nonzero X. Call Φ the collection of all roots.

Define $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} | ad_H X = \alpha(H)X \text{ for all } H \in \mathfrak{t}\}$.

Theorem 5.7 (Root Space Decomposition). $\mathfrak{g} = \mathfrak{t} \oplus \sum_{\alpha\in\Phi} \mathfrak{g}_\alpha$.

We define the Weyl group to be $W = N(T)/T$. Then W acts on $\mathfrak{t}$ and $\mathfrak{t}^*$. For α a root, $\ker\alpha \subseteq \mathfrak{t}$ a root hyperplane, this decomposes $\mathfrak{t}$ into chambers. Pick one and call it C.

Define a root to be positive if $\langle\alpha, H\rangle > 0$ for $H \in C$. Call these Φ_+.

Definition 5.1 (Simple Root). *A root $\alpha \in \Phi_+$ is called simple if it cant' be written as $\lambda_1\alpha_1 + \ldots + \lambda_k\alpha_k$ for $\lambda_i > 0$ and $k > 1$ and $\alpha_i \in \Phi_+$.*

Let Δ be the simple roots. These form a basis for $\mathfrak{t}^*$, if the group is semisimple.

Classication (up to conjugation) of tori.

Pick an element $H_0 \in \bar{C}$.

If $H_0 \in C$, then $C(H_0) = \{g \in G | Ad_g H_0 = H_0\} = T$. For any subset $S \subset \Delta$, assume that $H_0 \in \ker\{\alpha|\alpha \in S\}$. Then $C(H_0)$ is the centralizer of a torus, the torus generated by H_0.

For any subset of the roots, we get a flag variety.

Theorem 5.8. *Let $S \subset \Delta$ and X_S the corresponding flag variety. Then there exists a Morse function on X_S such that it has indices given by the following prescription:*

Choose $\xi_0 \in C$ and $H_0 \in \{\ker\alpha|\alpha \in S\}$. Look at the Weyl group orbit of H_0. wH_0 will contribute a critical point whose index istwice the number of root hyperplanes you need to cross to march from wH_0 back to ξ_0.

$G/C(T')$. $C(T') = C(H_0)$. Define a map $G \to \mathfrak{g}_0$ by $g \mapsto Ad_g H_0$. The orbit under Ad of H_0 is $G/isotropy$, and the isotropy is $C(H_0) = C(T')$.

So this embeds X_S in $\mathfrak{g}_0$, and $\mathfrak{g}_0$ is naturally a Euclidean space.

For instance, on $U(n)$, we have $\langle X, Y\rangle = -Tr(XY)$.

This behooves us to study submanifolds of Euclidean space, $M^n \subset M^{n+k}$.

Let $p \in \mathbb{R}^{n+k} \setminus M$. Then define $\ell_p : M \to \mathbb{R}$ by $\ell_p(x) = \|x - p\|$.

The critical points of ℓ_p are the points of M where T_xM is perpendicular to $\bar{px}$.

If q is any point on $\bar{px}$, thn x is also a critical point of ℓ_q.

Without loss of generality, we can assume that $x = 0$ and that $x_{n+1} = \ldots = x_{n+k} = 0$ defines T_0M.

Then $x_1, \ldots, x_n$ will be coordinates for M. Moreover, M will be locally defined near 0 by $x_{n+1} = g_1(x_1, \ldots, x_n)$ throught $x_{n+k} = g_k(x_1, \ldots, x_n)$. Define $p = (0, \ldots, 0, p_1, \ldots, p_k)$.

Let $t > 0$ and look at tp. We want to compute $H\ell_{tp}(0)$. $\ell_{tp}(x) = \|x - tp\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2 + \sum_{i=1}^k (g_i(x_1, \ldots, x_n) - tp_i)^2}$.

We will work with $\ell_{tp}^2(x)$. We will use the fact that $H\ell_{tp}(0) = \frac{H\ell_{tp}^2(0)}{2\ell_{tp}(0)}$.

Now note that $\frac{\partial}{\partial x_j}\ell_{tp}^2(x) = 2x_j + 2\sum_{i=1}^k (g_i - tp_i)\frac{\partial g_i}{\partial x_j}$, and then $\frac{\partial^2}{\partial x_m \partial x_j} = 2\delta_{mj} + 2\sum_{i=1}^k \frac{\partial g_i}{\partial x_m}\frac{\partial g_i}{\partial x_j} + (g_i - tp_i)\frac{\partial^2 g_i}{\partial x_m \partial x_j}$.

Next we evaluate at zero and get $2\delta_{mj} - 2\sum_{i=1}^k tp_i \frac{\partial^2 g_i}{\partial x_m \partial x_j}$.

So then $H\ell_{tp}(0) = \frac{I_n}{t\|p\|} - \sum_{i=1}^k \frac{p_i}{\|p_i\|} \frac{\partial^2 g_{ij}}{\partial x_i \partial x_j}$ as $\ell_{tp}(0) = t\|p\|$.

6 Lecture

Let G be a compact Lie group and Lie g_0 its Lie algbera. Then G acts on Lie g_0 by $Ad : G \to End(\text{Lie}\, g_0)$. (We can embed G in $U(n)$ and then Lie $g_0 \subset$ Lie $u(n)$ =skew Hermitian matrices, so Lie g_0 is $X \in \text{Lie}\, u(n)$ with $\exp(tX) \in G$ for all t.)

The action is given by conjugation and if $X \in \text{Lie}\, g_0$, then its orbit is $\mathscr{O}_X = G \cdot X = \{Ad_g X = gXg^{-1} | g \in G\}$.

Fact: $\mathscr{O}_X$ is a generalized flag variety and all generalized flag varieties are of this form.

Now, Lie g_0 can be made a Euclidean space (not uniquely, and canonically if G is semisimple) by embedding in Lie $u(n)$ and using its inner product of $-Tr(XY)$. This is a nondegenerate positive definite symmetric bilinear form over $\mathbb{R}$. So Lie g_0 interits this inner prodcut, and it is preserved by teh adjoint action.

so this is an invariant inner product which translates, that is, $\langle [X,Y], Z\rangle + \langle Y, [X,Z]\rangle = 0$.

General situation:

We have $M^n \subset \mathbb{R}^{n+k}$ a compact submanifold, and for $p \in \mathbb{R}^{n+k}$ but not in M, we define $\ell_p : M \to \mathbb{R}$ by $\ell_p(m) = d(p,m) = \sqrt{\|p-m\|^2}$ and $L_p(m) = \ell_p^2(m)$.

Facts:

1. $x \in M$ is a critical point for ℓ_p (and L_p) iff $\bar{xp}$ is perpendicular to T_xM.
2. Assume, wlog, that $x_0 = 0$. Then $H_0 L_{tp} = \sum_\ell -tp_\ell \frac{\partial^2 g_\ell}{\partial x_i \partial x_j} + 2I_n$ and $H_0 \ell_{tp} = \frac{1}{t\|p\|} I_n - \sum_\ell \frac{p_\ell}{\|p\|} \frac{\partial^2 g_\ell}{\partial x_i \partial x_j}$.

We can simultaneously diagonalize I_n and $\frac{\partial^2 g_\ell}{\partial x_i \partial x_j}$. Then $H_{\ell_{tp}}(0)$ has diagonal $a_{ii} + 1/t$, and from this we can see that the coefficients are strictly decreasing in t, that for only a finite number of values is the matrix singular, for small t, this is positive definite, and the index is increasing with jumps at the t_i where it is singular and of size the nullity of $H_{\ell_{t_i p}}$.

Theorem 6.1. *Let $x_0 \in M$ be a critical point. Then the index of $H_{\ell_p}(x_0$ is $\sum_{0<t<1} \nu(H_{(1-t)p+tx_0})$ where $\nu(H)$ is the nullity of H.*

Let $N(M)$ be the normal bundle of M^n in $\mathbb{R}^{n+k}$. That is, $\{(x,v) | x \in M, v \perp T_xM\}$ in $\mathbb{R}^{n+k} \times \mathbb{R}^{n+k}$.

Define $E : N(M) \to \mathbb{R}^{n+k}$ by $E(x,v) = x + v$.

Define a point p in $\mathbb{R}^{n+k}$ to be a focal point for $M \subset \mathbb{R}^{n+k}$ at x if the Jacobian of E at (x,v) where $E(x,v) = p$ is singular and the nullity is called the multiplicity of the focal point. (We will write the Jacobian is dE)

Proposition 6.2. *The point $x_0 \in M$ is a critical point for ℓ_p iff p is a focal point at x_0 and then index of x_0 as a critical point is the multiplicity of the focal point.*

Theorem 6.3 (Morse Index Theorem)**.** *The index of ℓ_p at a nondegenerate critical point is equal to the number of focal points on the line $(1-t)p + tx_0$ for $0 \le t \le 1$ counted with multiplicity.*

The proof is prop+theorem.

Let $x_1, \dots, x_n$ be coordinates about x_0 for M. Then there exist $u_1(x_1, \dots, x_n)$ through u_{n+k} which are coordinates of $M^n \subset \mathbb{R}^{n+k}$.

Define $g_{ij} = \frac{\partial u}{\partial x_i} \cdot \frac{\partial u}{\partial x_j}$. This gives an $n \times n$ matrix, the first fundamental form of the embedding. The second fundamental form is a symmetric matrix of vectors defines as follows: $\frac{\partial u^2}{\partial x_i \partial x_j}$ is a vector at $x \in M$. Then $\bar{\ell}_{ij}$ =normal component of this vector relative to M. So $\ell_{ij} \in N_x(M)$. Now, for any unit vector in N_xM, we have $v \cdot \ell_{ij}$, and this is the second fundamental form of M at x in the direction v.

WLOG, assume that $g_{ij}(x_0) = I_n$. The eigenvalues of $\ell_{ij} \cdot v$, $\kappa_1, \dots, \kappa_n$, are called the principal curvatures of M at x_0 in the direction v. Then κ_i^{-1} are the radii of curvature.

Consdier the line $\ell(t) = x_0 + tv$.

Lemma 6.4. *The focal points of M at x_0 along $\ell(t)$ are the points $x_0 + \kappa_i^{-1}v$.*

Proof. Choose a k vector fields $\xi_1, \dots, \xi_k$ along M such that the ξ_i form an orthonormal frame for $N(M)$ near x_0. Then we form coordinates for $N(M)$ in a neighborhood of $(x_0, 0)$.

$(x_1, \dots, x_n, t_1, \dots, t_n) \mapsto (u, \sum t_a lpha \xi_\alpha)$. Then $E : N \to \mathbb{R}^{n+k}$ is written in coordinates as $u + \sum t_\alpha \xi_\alpha$. Take the gradient, and then dot with $u_{x_1}, \dots, u_{x_n}, \xi_1, \dots, \xi_k$. Then we get $\frac{\partial u}{\partial x_i} \cdot \frac{\partial u}{\partial x_j} + \sum t_\alpha \frac{\partial \xi_\alpha}{x_i} \cdot \frac{\partial u}{\partial x_j}$, $\sum t_\alpha \frac{\partial \xi_\alpha}{\partial x_i} \cdot \xi_\beta$, 0 and I as the four blocks making up the matrix. The rank of this matrix is the rank of the Jacobian, which is the rank of the first block.

Now, we know that $0 = \frac{\partial}{\partial x_i}\left(\xi_\alpha \cdot \frac{\partial u}{\partial x_j}\right) = \frac{\partial \xi_\alpha}{\partial x_i} \cdot \frac{\partial u}{\partial x_j} + \xi_\alpha \cdot \frac{\partial^2 u}{\partial x_i \partial x_j}$.

So $x_0 + tv$ is a focal point of multiplicity μ iff the matrix $g_{ij} - t_\alpha \xi_\alpha \ell_{ij}$ is singular with nullity μ. Now suppose that $g_{ij}(x_0) = I_n$. Then $g_{ij} - tv\ell_{ij}$ is singular iff $1/t$ is an eigenvalue of $v \cdot \ell_{ij}$, that is, a t is κ_i^{-1}. So the lemma holds. □

From last time, we saw $HL_p = 2(g_{ij} - tv\ell_{ij})$ so the index at x_0 for L_p is equal to the number of focal points from x_0 to p counted with multiplicity.

Now back to a Lie group G.

Lemma 6.5. *Consider $\mathscr{O}_X$ for $X \in \mathrm{Lie}\, g_0$. $N_X\mathscr{O}_X = \{Y | [X, Y] = 0\}$ and $T_X\mathscr{O}_X = \{[X, Y] | Y \in \mathrm{Lie}\, g_0\}$.*

Proof. Let $X \in \mathrm{Lie}\, g_0$. Then $e^{tX} = T'$ is a torus. The Lie algebra of $C(T')$ is $\{Y \in \mathrm{Lie}\, g_0 | [Y, X] = 0\}$. As $\mathscr{O}_X$ is $G/C(T')$, the normal space to $\mathscr{O}_X$ at X in $\mathrm{Lie}\, g_0$ will be this centralizer's algebra. □

Theorem 6.6. *Choose X_0 such that $\mathscr{O}_{X_0}$ is of maximal dimension and $X_0 \notin \mathscr{O}_X$ The focla points for $\mathscr{O}_X$ between Y and X_0 occur exactly at the places where $\dim \mathscr{O}_{X_0+t(Y-X_0)}$ jumps with multiplicity the difference in dimension.*

7 Lecture

Let G be a compact connected Lie group and $\operatorname{Lie} g_0$ the Lie algebra, with $\operatorname{Lie} g = \operatorname{Lie} g_0 \otimes \mathbb{C}$ with an invariant inner product.

Let $\mathscr{O}_X = G \cdot X \subset \operatorname{Lie} g_0$ it is $G/C(T')$ for some torus. Then $\operatorname{Lie} g_0 = T_X\mathscr{O}_X \oplus N_X\mathscr{O}_X$ which are orthogonal.

Lemma 7.1. *If a line $\ell(t)$ is perpendicular to an orbit $\mathscr{O}_X$ at some point, then it is perpendicular to every orbit it meets.*

Proof. WLOG, assume the point where $\ell(t)$ hits $\mathscr{O}_X$ is X. Write the line as $X+tV$. Then the perpendicularity is equivalent to $V \in N_X\mathscr{O}_X$ implies $[X,V] = 0$. So then $\langle [Y, X+tV], V\rangle = t\langle [Y,V],V\rangle + \langle [Y,X],V\rangle = 0$. □

If T is a maximal torus, then we have the following:

Lemma 7.2. *TFAE*

1. *$X \in \operatorname{Lie} t_0$ is regular*
2. *$\alpha(X) \neq 0$ for all $\alpha \in \Phi$*
3. *$C(X) = \{Y \in \operatorname{Lie} g | [X,Y] = 0\} = \operatorname{Lie} t$*

Let C be a highest Weyl character.

Now $\operatorname{Lie} t_{reg} = \operatorname{Lie} t \setminus \cup \ker \alpha$. Then pick $X_0 \in C \cap \operatorname{Lie} g_0$. In particular, X_0 is regular. Take $L_{X_0} : \operatorname{Lie} g_0 \to [0, \infty)$.

Lemma 7.3. *Critical points of L_{X_0} are $\mathscr{O}_X \cap \operatorname{Lie} t_0$.*

Proof. Let $Y \in \operatorname{Lie} g_0$ be a critical point for L_{X_0}. This holds iff $\bar{Y X_0} \perp \mathscr{O}_X$ iff $\bar{Y X_0} \perp \mathscr{O}_{X_0}$ iff $\langle Y - X_0, X_0\rangle = 0$ iff $[Y, X_0] = 0$, and this implies that $Y \in \operatorname{Lie} t_0$ by the regularity of X_0. □

The critical points of L_{X_0} are the $\mathscr{O}_X \cap \operatorname{Lie} t_0$.

Set $W = N(T)/T$. Then $G/AdG \cong T/W$.

Note now that $\operatorname{Lie} g = \operatorname{Lie} t \oplus \sum_{\alpha\in\Phi} \operatorname{Lie} g_\alpha$. Let $\Phi_X = \{\alpha | \alpha(X) = 0\}$.

7.1 Lefshetz

Theorem 7.4. *If $X^k \subset \mathbb{C}^n$ is a complex analytic manifold and X is closed in $\mathbb{C}^n$, then X^k is homotopic to a k-dimensional CW-complex.*

Theorem 7.5 (Lefshetz Hyperplane Theorem). *Let $V^k \subset \mathbb{CP}^n$ be a complex variety. Let H be a hyplace such that the singularities of V are inside H. Then $H_i(V \cap H; \mathbb{Z}) \to H_i(V; \mathbb{Z})$ is an isomorphism for $i < k-1$ and is surjective for $i = k-1$.*

Proof of second from first:

Look at the LES from the pair $(V, V \cap H)$. $H_{i+1}(V, V \cap H) \to H_i(V \cap H) \to H_i(V) \to H_i(V, V \cap H)$. But $H_i(V, V \cap H) \cong H^{2k-i}(V - V \cap H)$. Now note that $V - (V \cap H) \subset \mathbb{C}^n$ as $\mathbb{C}^n = \mathbb{CP}^n \setminus H$. So as $V - (V \cap H)$ is homotopic to a k-dimensional CW-complex, these cohomology groups are zero for $i < k - 1$.

Proof of first:

Choose $p \in \mathbb{C}^n$, $p \notin X$. Look at L_p. For $q \in X$, the critical points for L_p correspond to q which have focal points and the index is the sum of the multiplicity of focal points on the line qp.

Lemma 7.6. *If $q + tv$ is a focal point for q with multiplicity μ, then $q - tv$ is as well.*

The number of focal points on $q + tv$ can be at most $2k = \dim X$. By the lemma half of them occur for $t < 0$, only leaving at most k. So all that remains is to prove the lemma.

Let $z_1, \dots, z_k$ be complex coordinates at q. The embedding $X \to \mathbb{C}^n$ can be written in these coordinates. Let $v \perp X$ at q. Then $\sum_{\alpha=1}^{n} w_\alpha(z_1, \dots, z_k) v_\alpha$.

This has a Taylor expansion $c + Q(z_1, \dots, z_k) + HOT$ with no linear term because $v \perp X$. Let $z_i = x_i + iy_i$ and set $w \cdot v$ to be the real inner product, that is, the rea part of $\sum w_\alpha \cdot v_\alpha$. In terms fo the x_i and y_i, this Taylor series looks like $c + \sum Q'(x_1, \dots, x_k, y_1, \dots, y_k) + HOT$ where Q' is the second fundamental form in the direction v.

Fact: If λ is an eigenvalue of Q' with multiplicity μ then $-\lambda$ is too.

Proof: i on $\mathbb{C}^n$ is an orthonormal transformation on the real coordinates, and so Q' and $-Q'$ are equivalent by o.n. transformations, and so we are done.