Oral Exam Recap

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Major Area: Combinatorics - Probabilistic
main books: core module - Stanley Vol. 1, Flajolet and Sedgewick; advanced module - Alon and Spencer

Minor Area: PDE's
main book: Evans

Committee: Robin Pemantle (Chairman, combinatorics advanced module), Herbert Wilf (combinatorics core module), Joachim Krieger (PDE's)

Wilf suggested that we hold the exam in his office instead of 4E19. No one had any objections so we moved there. Unfortunately there was no visible clock in his office so I couldn't keep track of time very well, so the times for each "round" of questioning that I give below are just rough estimates. I will try to give a "play-by-play" recap of what transpired in the 2 hours.

They asked me which area I want to start with. I said I have no preference. Pemantle then suggested that Krieger start us off with PDE's.

-KRIEGER (wrote down a BVP on the board): Tell us what you know about this PDE.
Me: It's Burgers' equation. We can use the method of characteristics to get a local solution near the boundary, ...
Eventually, with a few small corrections/pointers from Krieger, I said and wrote enough correct things about Burgers' equation and the method of characteristics to move us onto the next topic.

This part lasted about 15 minutes.

-WILF: Let's talk about posets. What are they? What are some examples?

I gave the definition and a couple of examples.

WILF: What's the Möbius function for a poset? What is the zeta function of a poset? What is the classical (number-theoretic) Möbius function? What's the Möbius function in the case of natural numbers ordered by size?

I gave the definition of the zeta function for a poset, and said the Möbius function is its inverse. Then I gave the Möbius inversion formula in each case, as well as the actual Möbius function for the two specific cases.

WILF: What's a nice combinatorial application of the Möbius inversion formula?
Me: One of the goodies it gives us is the principle of inclusion-exclusion.

WILF: What's the poset in this case? What is its cardinality? What is its Möbius function?
Me: Subsets of $[n]$ ordered by inclusion; 2^n; $(-1)^{|T-S|}$ for $S \subset T$.

Everything Wilf asked thus far can be found in Stanley's book. This part lasted about 30 minutes.

-PEMANTLE: Suppose we have a uniform distribution over the $n!$ permutations of n letters. Let N be a random variable corresponding to the number of fixed points of a random permutation. What is the mean of N? What is its distribution? How might you show that the distribution is correct?

I said that $E(N) = 1$ and gave a simple proof using linearity of expectation. As for the distribution, I was slightly surprised at myself that I didn't already know it, but I made the "educated guess" that it's Poisson with mean 1, since N is nonnegative integer-valued, and I knew that the probability of derangements (permutations with no fixed points) is asymptotically $1/e$. Luckily, this was a correct guess. Then I said perhaps we can use the AGG theorem for Poisson approximation to show it. Unluckily, this was an incorrect guess.

PEMANTLE: (after remarking that Gordon, the second 'G' in AGG, is actually working in finance in Philly) What is the AGG theorem? Give a precise statement.

I gave the theorem, which involved a bunch of definitions. Pemantle informed me that I made a slight error in one of the definitions, which I was able to correct later.

PEMANTLE: Actually, the AGG theorem won't help us here since there are too many complicated dependencies. What might be another way to determine the distribution?

I said we could look at the generating function for permutations on $[n]$ with k fixed points, $1 \leq k \leq n$. By Wilf's generating function version of the inclusion-exclusion principle, $E(x) = N(x-1)$, where $E(x)$ enumerates the number of elements with exactly k properties, $N(x)$ enumerates the number of elements with at least k properties.

PEMANTLE and/or WILF: Give a detailed explanation of what that formula means and how we can apply it here.

I wrote down what I remember from section 4.2 of Wilf's Generatingfunctionology book. In this case, $N_k = \binom{n}{k}(n-k)! = n!/k!$. Wilf/Pemantle gave small corrections/hints along the way.

This part lasted about 40 minutes.

-PEMANTLE: So can you give an example where we can apply the AGG theorem? Perhaps counting triangles in a random graph?

I applied the AGG to find the asymptotic distribution of the number of triangles in a random graph G(n,p). Pemantle actually asked me this problem in one of our meetings 2 months before the exam, so I just gave a slightly cleaner version of the solution I gave back then. While explaining the solution, I also found the small mistake in my statement of the AGG theorem and corrected it.

This part lasted about 15 minutes.

-KRIEGER (wrote down another problem on the board: $\Delta u \geq 0$ on some bounded domain Ω): Is there a maximum principle for such a function u? Me: These are called subharmonic functions. I don't remember if they have a maximum principle... Let me see if I can figure out the answer by looking at the proof of the maximum principle for harmonic functions.

The proof invokes the mean-value property for harmonic functions. They asked me why should there be this mean-value property for harmonic functions, and after some bumbling around at the board (with some helpful hints from Krieger) I eventually arrived at the correct answer. I was internally kicking myself for not having fully internalized the proof of the mean-value property. Anyway, I realized that the proof of the strong maximum principle for harmonic functions can be trivially modified to become a proof of the strong maximum principle for subharmonic functions.

This part lasted about 20 minutes.

In total, the exam lasted about 2 hours. Some suggestions for when you take the exam: be prepared to give precise/technical definitions of things you're talking about, an intuition-based informal definition is often not good enough for people not familiar with the subject. If it's something you're not too sure on, think before you speak instead of quickly venturing a guess that often ends up being wrong. Finally, don't be nervous, everyone makes some mistakes during these oral exams. Your syllabus probably covers a ton of topics and the exam itself will usually only test you on a small fraction of them, so there is some luck involved, as I'm sure you're not equally familiar(unfamiliar?) with all the things you're supposed to know for your orals. In probabilistic terms: almost surely, your level of understanding will not be uniformly distributed over everything on your syllabus.