

Orals Transcript for Eugene So

Major Topic: Homotopy Theory
Minor Topic: Complex Algebraic Geometry

Committee Members:
Jonathan Block (Chair)
Tony Pantev
Ron Donagi (Minor)

Enter JB

JB: So how are you feeling?

ES: All right, I guess. A bit nervous.

JB: You shouldn't be. Well, maybe you should be a little. But not too much.

ES: It's just that there's a lot in this subject. It kinda feels like you never know enough, and it makes you feel small, you know?

JB: Yeah. *chuckles* It makes everyone feel that way. Where are Tony and Ron?

Exit JB

Enter RD

RD: How are you doing?

ES: Better than I thought I would be. I'm still alive.

RD: You must have very low standards. Who were the other members of your committee? Tony and Jonathan, right?

Exit RD, Enter TP

ES: Will this be one of those things where only one of you is allowed into the room at a time?

Enter JB and RD

JB: Okay, so this might turn out to be a historic event.

TP: Yes. It could be the first time in Penn's three-hundred-year history for a student to pass his oral examinations in homotopy theory.

RD: Wait, what is all this about? I thought the major topic was algebraic topology?

JB (to RD): Well, this is basically algebraic topology with a different focus. Because when Herman or Julius is on the committee, it usually focuses more on... how should I put this...

TP (to RD): On knots and... low-dimensional topology things.

JB (to RD): Yes, the more... geometric stuff. While this focuses on more of the... algebraic stuff. With categories and things.

TP (to RD): So if Eugene passes, he will be the first student of homotopy theory at the University of Pennsylvania. *chuckles*

JB: So we kind of have a lot riding on you-

ES: So no pressure, right?

Laughter

TP: So what do you want to start with?

ES: Well, I might as well start with the one riding on me.

TP: Okay then, start with one of the problems we gave you.

RD: You gave him the problems in advance?

TP (to RD): Only some of them. The real ones will come later.

Faculty laughter with faint pretend-laughter from myself

ES: Alright, well, I guess I'll start with the cohomology of the loop space of $S^3 \times S^3$. Sorry, it should be $SO(4)$, which $S^3 \times S^3$, or $SU(2) \times SU(2)$ or whatever you want to call it, double covers! Well you see, in general when we have a space X universally covering a space Y , the higher π_n s are equal so when we take the loop space, the loop space of X is homotopy equivalent to one of the components of the loop space of Y . Now that means that the question is reduced to finding the cohomology of $\Omega(S^3 \times S^3)$. Since Ω is right adjoint to suspension, it commutes with limits and thus products and thus we just need to find the cohomology of $\Omega(S^3)$ and proceed by Künneth-

TP: Wait, could you go over-

JB: No, no. I want to see how *fast* he can do it.

Laughter

TP: So could you go over why you are looking at $\Omega(S^3 \times S^3)$?

ES: Well, I had that the loop space of $S^3 \times S^3$ has the same homotopy type as a *component* of the loop space of $SO(4)$.

TP: Right, the component. Continue.

ES: So as I was saying, it's sufficient to calculate the cohomology of $\Omega(S^3)$, which is only \mathbb{Z} on multiples of four? Wait, that can't be right. Well, I guess I can't rely on my memory like I wanted so I'll actually have to calculate it using a spectral sequence. Okay.

I quickly draw the two towers of the spectral sequence and start crossing things out without labeling anything. Jonathan laughs.

ES: Okay, so the cohomology of $\Omega(S^3)$ is \mathbb{Z} on the evens, which means that by Künneth *Künneth formula hand motion* the cohomology of $\Omega(S^3) \times \Omega(S^3)$ is 0 on the odds and \mathbb{Z}^{n+1} on the $2ns$.

TP: Good. How about the next one?

ES: Well, there was the classification of principal $PGL_2(\mathbb{C})$ -bundles over S^2 . Now, principal bundles are classified by maps from the base space to the classifying space of the structure group, so we're looking at $[S^2, BPGL_2(\mathbb{C})] = [S^1, \Omega BPGL_2(\mathbb{C})] = [S^1, PGL_2(\mathbb{C})] = \pi_1(PGL_2(\mathbb{C}))$. Now, $GL_2(\mathbb{C})$ is a circle bundle over $PGL_2(\mathbb{C})$, so we can try to calculate it using the homotopy long exact sequence, but that depends on the map from $\pi_1(S^1)$ to $\pi_1(GL_2(\mathbb{C}))$. So instead we can note that $SL(2, \mathbb{C})$ double-covers $PSL(2, \mathbb{C})$, but $SL(2, \mathbb{C}) = SU(2)$ which is simply-connected, so it's a universal cover, so $\pi_1(PSL(2, \mathbb{C})) = \pi_1(PGL_2(\mathbb{C})) = \mathbb{Z}_2$, so there are two of them.

Then I was supposed to compute the rational homotopy type of the total spaces of the associated S^2 -bundles over S^2 . But I can do that by looking at the rational homotopy type of the spaces directly! The rational homotopy type of S^2 is \mathbb{Z} at 2 and 3 and trivial everywhere else, so I have a long exact sequence

$$\cdots 0 \rightarrow \mathbb{Q} \rightarrow \underbrace{\pi_3(E) \otimes \mathbb{Q}}_{=\mathbb{Q} \oplus \mathbb{Q}} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q} \rightarrow \underbrace{\pi_2(E) \otimes \mathbb{Q}}_{=\mathbb{Q} \oplus \mathbb{Q}} \rightarrow \mathbb{Q} \rightarrow 0 \cdots$$

which has to be filled in the way I have it, and be zero everywhere else. So I don't have to look at the actual bundles, which is good because I don't like looking at actual bundles.

Slight chuckling from JB

TP: So there are two associated S^2 bundles over S^2 . Can you tell me what they are?

ES: Well, there's the trivial $S^2 \times S^2$. Then for no reason at all, I know that the other one is $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$.

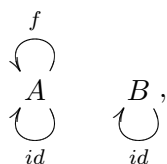
TP: What is that better known as? You should be able to recognize what it is from your minor topic... unless it's not on your syllabus.

RD (to TP): It's not on his syllabus.

TP: Oh, then I'll just tell you. It's the blowup of \mathbb{P}^2 at a point.

ES: That's right, blowups aren't on my minor syllabus, thankfully!

where $f^2 = id$, so when I take the subcategory I get



which gets me $\mathbb{R}P^\infty$ -

TP: So you can say something immediately about that subcategory. It's no longer...

ES: Connected?

TP: Yes. So you have a map from a connected thing to a disconnected thing, so it can't be a homotopy equivalence.

ES: Oh. Right.

TP: So what would $|BD|$ look like? You've already shown that it can be disconnected. What are the components?

ES: Well, for things to be in the same component they have to have invertible maps to each other. Ah, so the components are the isomorphism classes of objects. And for each component you'd have...

Pause

TP: So what are the morphisms between two objects-

ES: The homsets?

TP: Sure, the homsets, whatever. What would it be from an object to itself?

ES: It would be a group. Oh, so you'd have $K(G, 1)$ for each object, so would you have... a product of the corresponding $K(G, 1)$ s?

TP: No. It's not that complicated. Think about what the groups have to be.

Pause

TP: What are we dealing with here? We're dealing with pointed topological spaces, so you have to pick a basepoint in each component.

ES: Okay, so I pick a basepoint. Then I go to another object, go back to itself, then back- Oh, so I have the same group in each object, so the geometric realization of the nerve of the component is just one copy of $K(G, 1)$.

TP: That's right. Now, if we give you a topological space $K(G, 1)$, how do you get G back from the space?

ES: Um... uh...

JB: C'mon, you're blanking! We're not trying to trick you. We just want you to say something basic.

TP: Yes, we just want you to say what it is.

ES: Um... uh... take π_1 ?

TP: Yes! Just take the fundamental group. So, we get a $K(G, 1)$ if we geometrically realize B of a group G . Now, can you give me an example of a category where you don't get a $K(G, 1)$?

ES: Well, it has to have non-invertible morphisms...

TP: Yes, so you know that much.

ES: So you take...

(So here I wanted to say something silly like "That category that which you take the nerve of and geometrically realize to the geometric n -simplex", which I knew was something. What that turned out to be was the poset category of n elements, where objects are subsets of $\{0, 1, \dots, n\}$ and morphisms are set inclusions. But since I didn't have that instead I did)

ES: Uh... buh... bluh...

JB: Well, I liked the example you had before.

ES: What, this?

$$\begin{array}{ccc}
 & f & \\
 \curvearrowright & & \\
 A & \xrightarrow{g} & B, \\
 \curvearrowleft & & \curvearrowleft \\
 id & & id
 \end{array}$$

JB: Yes. What happens to that?

ES: Well, you get \mathbb{RP}^∞ with some things attached... like an infinite-dimensional cone or something... bluh...

JB: Look, *chuckles*. Sorry, you're blanking. Just try it for a 3-simplex.

ES: Like this? *Draws tetrahedron, labels vertices* Well, now I have to figure out how the arrows go...

JB: How about something simpler? How about a 2-simplex?

ES: So you get this *draws a triangle*

TP: Yes, so in the simplex you have a vertex for each of the morphisms, then a vertex for the commutative triangle...

ES: Yes, so I have... *draws a subdivided triangle over triangle in yellow* Ah, yes, you have a barycentric subdivision. So in general you can triangulate any reasonable space and construct a category from the simplex whose nerve geometrically realizes as that space as the nerve will be a barycentric subdivision...

TP and JB nod

TP: Okay, I think we are done for now. Ron?

RD: Yes. *looks up from laptop* If I have a product of S^1 s... When is it a complex manifold?

ES: Uh... buh... there can't be an odd number of them because then the dimension is wrong

TP: That's right.

ES: Uh... buh...

TP: Think of what happens with just two S^1 s.

ES: Oh, then it's \mathbb{C}/Λ , so for the higher dimensions I have a higher-dimensional torus, so I have $\mathbb{C}^n/\mathbb{Z}^n$.

TP: Are you sure about that? Be careful of the dimensions.

ES: Oh right. So for $2n$ of the S^1 s it would be $\mathbb{C}^n/\mathbb{Z}^{2n}$.

RD: Correct. Now, can you tell me the Hodge decomposition?

ES: No matter how long I stand up here I will not be able to do that.

RD: Hmm. Lefschetz decomposition?

ES: Same. (Hodge and Lefschetz decomposition are on my syllabus but along with the rest of the Complex Geometry section I did not study it. As Charlie said afterwards in typical Charlie fashion, "You expected Ron to not ask you simple complex geometry questions? Well, you were *wrong!*")

RD: Hm.

A pause

RD: Okay. Say I have a curve of genus 2. What does the map associated to the canonical divisor map it into?

ES: Well, curves of genus 2 are all hyperelliptic, and the map associated to the canonical divisor is the twofold map into \mathbb{P}^1 .

RD: Okay. What about $2K$?

ES: Uh... buh... Well, the canonical divisor is degree $2(2) - 2 = 2$, so $2K$ isn't very ample, but it's... ample?

TP: What definition of ample are you using?

ES: No wait, I mean it's nonspecial, since $l(K - 2K) = 0$... So by Riemann-Roch, $l(2K) - l(-K) = 4 + 1 - 2 = 3$, so ϕ_{2K} maps the curve into \mathbb{P}^2 . But it's not very ample so it's not an embedding.

TP: What definition of very ample are you using?

ES: The one with the degree? The genus is two so it has to be at least degree five for it to be very ample.

TP: But that is a sufficient condition for it to be very ample, not a necessary one.

ES: Right, right... So I have

$$\begin{array}{ccc} X & \xrightarrow{\phi_{2K}} & \mathbb{P}^2 \\ \phi_K \downarrow 2\times & & \\ \mathbb{P}^1 & & \end{array}$$

...

Okay, so maybe if we look at the line bundles associated to the divisors, so we have the adjunction formula

$$\Lambda^t \mathcal{L}_K = \omega_X = \omega_{\mathbb{P}^2}|_X \otimes \det(\mathcal{N}_X)$$

TP: What- What are you doing?

ES: I don't know! I'm just trying anything I can!

RD: Why don't you try this instead? First of all, write down the definition of $L(2K)$.

ES: $L(2K)$? Just $L(2K)$? Well, that is the set of all meromorphic functions on X such that div of the function is $\geq -2K$...

TP: Are you sure you want him to think about it this way? I think for this problem it's easier if you think about it another way...

RD: No, this is the way they have it in Miranda. Now, what do you know about $L(K)$?

ES: Well, by Riemann-Roch, $l(K) - l(K - K) = 2 + 1 - 2 = 1$, so $l(K) = 1$, wait, that's not right... That can't be right... Wait, $L(0)$ has the constants in it, so $l(K) = 2$, so there's a nonconstant meromorphic function $f \in L(K)$, which means that $1, f, f^2$ generate $L(2K)$.

TP: How do you know that?

ES: Well, $1, f, f^2$ have to be linearly independent, and $l(2K) = 3$, so-

TP: Okay. Now you can write down what ϕ_{2K} does to the coordinates of X explicitly.

ES: Alright. So I have $\phi_{2K}(x) = [1 : f(x) : f(x)^2]...$

TP: So what does that do to the coordinates of the \mathbb{P}^1 at the bottom?

ES: Well, it sends $[1 : f]$ to $[1 : f : f^2]...$

RD (to TP): That's not right... He's not writing down a proper map.

TP (to RD): No, he just hasn't homogenized yet...

TP: Look, if I had the coordinates $[u : w]$ in \mathbb{P}^1 , what would they be sent to in \mathbb{P}^2 ?

ES: Well, they'd be sent to $[u : w : w^2]...$ Oh, right, I'd have to homogenize, so it would be $[u^2 : uw : w^2]$.

TP: *nods*

ES: And that map would be... birational? Bijective?

TP: And how would you check for that?

ES: Well, and I can't believe I'm doing this, but I will check if it is injective directly. If $u^2 = u'^2, uw = u'w', w^2 = w'^2$, then $u = \pm u', w = \pm w'$, and since $uw = u'w'$, it has to be that $u' = u, w' = w$ or $u' = -u, w' = -w$, so you can have two preimages, $[u, w]$ and $[-u, -w]...$

TP: Are those different in \mathbb{P}^1 ?

ES: Oh, no, they aren't. So it's a bijection on the \mathbb{A}^1 patch, since you can go the other way.

TP: And what do you know about those bijections?

ES: So it has to be a homeomorphism of the \mathbb{P}^1 s.

RD: That's right. So the ϕ_{2K} map factors through the ϕ_K map. Now, what can you tell me about the higher multiples of K ?

ES: Well, their degrees are big enough so that they will be very ample, so they will be embeddings.

RD: Okay.

Long pause

TP: Maybe give him a Riemann-Roch problem?

RD: Nah... Tell me, what can you say about the cohomology of line bundles over \mathbb{P}^2 ?

ES: Well, uh, they're all isomorphic to $\mathcal{O}_{\mathbb{P}^2}(d)$ for some integer d . And it is going to take me longer to compute these than they should, but-

RD: Just tell me when they are zero.

ES: Well, they are zero when the cohomology is in the middle, so 1. They are zero at the zeroth cohomology if d is negative. At the 2nd level they are zero if d is either \geq or $> -n - 1$... By Serre duality $H^0(\mathcal{O}_{\mathbb{P}^n}(0)) = H^0(\Omega_{\mathbb{P}^n}^1) = H^{1,0}(\mathbb{P}^n) = H^{n-1,n}(\mathbb{P}^n) = H^n(\Omega_{\mathbb{P}^n}^{n-1}) = H^n(\mathcal{O}_{\mathbb{P}^n}(-n-1))$, so it has to be strictly greater.

RD: Okay.

TP: Do you want to ask him anything else?

RD: No, I think I'm good.

JB: Okay then. Um... Let's see... Do you know how to compute $\pi_4(S^3)$?

ES: Yes I do!

JB: Okay, never mind then.

RD: See, you're not supposed to tell him that you know how to do it!

ES: Well I guess my plan of eating up time by taking a while to solve problems I already know how to do won't work. Also, I don't know how to do $\pi_5(S^3)$ immediately so maybe I could try that?

Nobody takes up the offer

TP: How about this? When are the- First of all, classify S^1 bundles over S^2 .

ES: They're classified by maps $[S^2, BU(1)]$ which is $\pi_1(S^1)$ which is \mathbb{Z} .

TP: Now, when are they boundaries of a manifold?

ES: Well, I could see that by looking at the Stiefel-Whitney numbers, which all have to be zero...

JB: Look, how about the trivial bundle? What is its boundary of?

ES: Bluh...

JB: *chuckles* I know this is homotopy theory, but you should be able to visualize this geometrically.

ES: Oh, that should be the... thickened sphere... with inside and outside identified... so $S^2 \times I$

JB: No, it's the boundary of something, so S^1 is the boundary of D^2 ...

ES: So $S^2 \times D^2$.

JB: Right. Now, what is it for a general S^1 bundle over S^2 ?

ES: My favorite S^1 bundle over S^2 ?

JB: Sure. Pick your *favorite* S^1 bundle over S^2 .

ES: My favorite S^1 bundle over S^2 ... That would be... A complex line bundle over \mathbb{P}^1 ?

TP: Well, that's a \mathbb{C}^* bundle over \mathbb{P}^1 -

ES: Yes, and \mathbb{C}^* is homotopic to S^1 .

TP: Well, yes, but you should think of the fiber S^1 sitting inside of \mathbb{C} instead.

ES: Sitting... inside?

TP: Yes. Sitting inside of \mathbb{C} , so you do something to \mathbb{C} .

ES: Bluh... bluh...

JB: *Chuckles* You're blanking again. We just want you to say something very simple. What do you put on a line bundle?

ES: You put things on line bundles? I thought you just put them in long exact sequences to compute things!

A rimshot roars in the distance and a round of laughter occurs

JB: The answer we were looking for is a Hermitian metric. You can put a Hermitian metric on the line bundle so that the S^1 is the things distance 1 from the origin and is the boundary of the things distance not-1.

ES: Oh, okay.

A pause

TP: So does anyone have any more questions?

JB: Can't think of any.

TP: Ron?

RD: Nah.

TP: All right then. You can wait outside while we discuss the results-

ES: Wait, but nobody asked me anything about Chern classes!

TP: Oh no, you'd better get back in here so we can ask you about it! *Shakes head, shooes me away*

Exit ES

LATER

RD (to TP): Well, your scheme worked.

ES: What scheme?

RD: Well, allow me to congratulate you.

TP: You are the first person in the history of this department to pass an orals in homotopy theory!

ES: Oh, thanks. *shakes everyone's hand*

TP (to the people around me): Make sure to ask him questions about Chern classes since he's dying to answer them!

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So I dunno, I think this is proof that they decide whether or not to pass you before the actual exam because I did pretty badly! For CAG I was all preparing to take on questions about cohomology of complete intersections in \mathbb{P}^n or minimal birational models of singular plane curves, but I did this just by learning to use the tools rather than a good understanding of the basics. I wanted a question about Chern classes because I wanted to show that I could calculate it in both environments, but it never came up.

Also, this transcript was written two weeks after the fact so I can't say it's completely accurate, but it's as close as it can get to what I remember.