Lee Kennard – Oral exam questions
Topics: Differential geometry and algebraic topology
Committee: Wolfgang Ziller (WZ), Chris Croke (CC), and Julius Shaneson (JS)

CC Consider a convex surface in $\mathbb{R}^3$. Must two closed geodesics intersect? (The goal was to prove that they must for strictly convex surfaces but not in general.) Follow up: Does assuming compactness change the answer in either case?

CC Suppose you have a light source shining on a (strictly?, compact? – you decide) convex surface M in $\mathbb{R}^3$; assume the light source is infinitely far away so that the light rays approaching M are parallel. Prove that every closed geodesic on M sees the light, i.e., at some time lies in the closure of the subset of M on which the light is shining. (Useful relevant fact, which I didn't have to prove, is a good exercise nonetheless, and also provides a hint: If one of the (two) regions enclosed by a smooth closed curve on S^2 has area at least 2π, then the curve intersects every closed geodesic.)

WZ If M and N are compact and admit metrics of negative curvature, can $M \times N$? Can the product admit nonpositive curvature metrics? Can it admit a flat metric? (In the last question, we restricted to the case of compact orientable surfaces.)

JS Can a compact orientable surface with negative curvature cover another such surface? If so, when, etc.?

JS Classify and exhibit the S^1-bundles over S^2 up to bundle isomorphism, and classify the total spaces up to homeomorphism. Do the same for $SO(2)$-bundles over S^2 with fiber S^2.

CC Assume M has a metric with $1 \leq \sec \leq 10$. Say what you can about the topology, diameter, injectivity radius, and volume of M. In the cases where you don't have positive results, give counterexamples. What examples do you know? When I mentioned quotients of S^n, $\mathbb{C}P^n$, and $\mathbb{H}P^n$, WZ or JS asked which of these admit nontrivial quotients.

WZ Define critical points of distance functions. Relate the critical points of $x \mapsto d(x,p)$ to the cut locus $\mathrm{Cut}(p)$ of p. Can the set of critical points equal the cut locus? Can it be a proper subset of the cut locus? Can it be empty? By assuming something, can you prove it's not empty? Can it be empty if the cut locus is not empty?